582 PS5 Solutions

1. We break the calculation into three parts:

$$\mathrm{Var}[X] = \sum_e \mathrm{Var}[X_e] + \sum_{|e\cap f|=1} \mathrm{Cov}[X_e, X_f] + \sum_{e\cap f=\emptyset} \mathrm{Cov}[X_e, X_f].$$

Since $\mathsf{E}X_e = o(1)$, $\mathrm{Var}[X_e] \sim \mathsf{E}X_e$ and

$$\sum_e \mathrm{Var}[X_e] \sim \mathsf{E}[X] \sim \frac{cn}{2e^{2c}}.$$

For $|e \cap f| = 1$, we have

$$\mathrm{Cov}[X_e, X_f] = \mathsf{E}[X_e X_f] - \mathsf{E}X_e\mathsf{E}X_f = -\mathsf{E}X_e\mathsf{E}X_f,$$

since e, f can't *both* be isolated. The number of such terms is $\binom{n}{2}2(n-2) \sim n^3$, so

$$\sum_{|e\cap f|=1} \mathrm{Cov}[X_e, X_f] \sim -n^3(p(1-p)^{2n-4})^2 \sim -n^3\left(\frac{c}{ne^{2c}}\right)^2 = -nc^2e^{-4c}.$$

For $e \cap f = \emptyset$,

$$\begin{aligned}\mathrm{Cov}[X_e, X_f] &= \mathsf{E}[X_e X_f] - \mathsf{E}X_e\mathsf{E}X_f \\ &= p^2(1-p)^{4n-12} - (p(1-p)^{2n-4})^2 \\ &= p^2(1-p)^{4n-12}(1-(1-p)^4)\end{aligned}$$

(The covariance is nonzero because there are 4 edges that affect both X_e and X_f.) Now $1-(1-p)^4 \sim 4p$ (since $p = o(1)$), and the number of e, f with $e \cap f = \emptyset$ is $\binom{n}{2}\binom{n-2}{2} \sim n^4/4$, so

$$\sum_{e\cap f=\emptyset} \mathrm{Cov}[X_e, X_f] \sim \frac{n^4}{4}4\left(\frac{c}{n}\right)^3 e^{-4c} = nc^3e^{-4c}.$$

So, finally,

$$\mathrm{Var}[X] \sim n\left[\frac{1}{2}ce^{-2c} - c^2e^{-4c} + c^3e^{-4c}\right].$$

2. We use the suggested notation and always assume $v, w \in L$. Let X_v be the indicator of the event $\{P_v \subseteq T\}$ and $X = \sum X_v$, the number of $v \in L$

reachable from ρ in T. Then $Q_n = \{X > 0\}$ and (using the 2nd moment method) it's enough to show $\mathrm{Var}(X) = o(\mathsf{E}^2 X)$.

We have $\mathsf{E}X_v = p^n$ (for all v), so $\mathsf{E}X = (np)^n$. Notice that $\mathsf{E}X_vX_w = p^{2n-i(v,w)}$ and that, for a given v and $i \in \{0, \ldots, n\}$,

$$|\{w \in L : i(v,w) = i\}| \le n^{n-i}.$$

(The precise value is $(n-1)n^{n-i-1}$ if $i < n$ and 1 if $i = n$.) Thus we have

$$\begin{aligned} \mathsf{E}X^2 &= \sum_v \sum_w \mathsf{E}X_v \mathsf{E}X_w = \sum_v \sum_w p^{2n-i(v,w)} \\ &\le n^n \sum_{i=0}^{n} n^{n-i} p^{2n-i} = (np)^{2n} \sum_{i=0}^{n} (np)^{-i} \end{aligned}$$

and $\mathrm{Var}(X) \le (np)^{2n} \sum_{i=1}^{n} (np)^{-i} = o((np)^{2n})$, as desired.

3. Let T be random (in the obvious way). Fix $\sigma \in S_n$. Then

$$Fit(T,\sigma) = X = \sum_{i=1}^{\binom{n}{2}} X_i,$$

where the X_i are independent symmetric Bernoullis (which, recall, means $\Pr(X_i = -1) = \Pr(X_i = 1) = 1/2$). We have $E[X] = 0$ and, by Chernoff,

$$\Pr(X \ge n^{3/2}\sqrt{\ln n}) \;\le\; \exp[-(n^{3/2}\sqrt{\ln n}\,)^2/n(n-1)] \;<\; n^{-n}.$$

This is enough, since it gives

$$\Pr(\exists \sigma \in S_n,\ Fit(T,\sigma) \ge n^{3/2}\sqrt{\ln n}\ ) < n! n^{-n} < 1.$$

4. WMA $|S(v)| = 10k$ $\forall v$. For each v, let $\sigma(v)$ be chosen uniformly from $S(v)$, independently of choices at other vertices. For $e = uv \in E$ and $\gamma \in S(u) \cap S(v)$, let $A(e,\gamma) = \{\sigma(u) = \sigma(v) = \gamma\}$, and observe that

$$(e,\gamma) \sim (f,\delta) \quad \Leftrightarrow \quad e \cap f \neq \emptyset$$

defines a dependency graph for the events $A(e,\gamma)$ (why?).

Now $\Pr(A(e,\gamma)) = (10k)^{-2}$. On the other hand, notice that (again with $e = uv$) every $(f,\delta) \sim (e,\gamma)$ satisfies either

$$f = uw \text{ for some } w \in V \text{ and } \delta \in S(u) \cap S(w)$$

or the corresponding statement with u replaced by v. Thus degrees in the above dependency graph are at most $2(10k)k = 20k^2$ (choose u or v, then δ, then w), and the assertion follows using the Local Lemma.

[Remark: This has been proved with $10d$ replaced by $2d$ (Haxell) and by $(1 + o(1))$ (Reed and Sudakov). Reed conjectured that $d + 1$ is enough, but this was disproved by Bohman and Holzmann (but just barely: their examples require lists of size $d + 2$).]

5. Suppose $n = 2k$ and $V = \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$. Let $\sigma(x_1), \ldots, \sigma(x_k)$ be independent symmetric Bernoullis and set $\sigma(y_i) = -\sigma(x_i)$. Then $\sigma(V)$ is automatically 0, and for $H \in \mathcal{H}$, $\sigma(H) \sim S_{m(H)}$, where $m(H) = |\{i : |H \cap \{x_i, y_i\}| = 1\}| \leq t$. Let $A_H = \{|\sigma(H)| > C\sqrt{t \ln t}\}$, with C TBA. Then $\Pr(A_H) < 2t^{-C^2/2}$ (by Chernoff), and the graph on vertex set $\mathcal{H}$ with

$$H \sim H' \Leftrightarrow \text{some } \{x_i, y_i\} \text{ meets both } H \text{ and } H'$$

is a Lovász graph for the A_H's with degrees less than $2t^2$. The statement now follows from the Local Lemma (for any C with $4et^{2-C^2/2} \leq 1$).

6. Modify our proof of Beck-Fiala by setting

$$\mathcal{H}^i = \{A \in \mathcal{H} : |A \cap S^i| \geq t\}.$$

(The condition in our proof was $|A \cap S^i| > t$.)

The proof goes as before *unless* we reach an i for which M^i (the $\mathcal{H}^i \times S^i$ incidence matrix) has all row and column sums exactly t. At this point we can take, for each j in S^i, ε_j to be 1 if $\varepsilon^i_j \geq 0$ and -1 otherwise (and $\varepsilon \equiv \varepsilon^i$ on T^i). This gives discrepancy at most $2t - 3$ on $\mathcal{H} \setminus \mathcal{H}^i$ (this is as in class), and at most t on $\mathcal{H}^i$ (so we used $t \leq 2t - 3$).

7. For the $(n \times n)$ incidence matrix $M = M(\mathcal{H}, V)$ we have $M^t M = qI + J$ and, for any $\varepsilon \in \{\pm 1\}^n$,

$$n\|M\varepsilon\|_\infty^2 \geq (M\varepsilon)^t M\varepsilon = \varepsilon^t(qI + J)\varepsilon = q\varepsilon^t\varepsilon + (\sum \varepsilon_i)^2 \geq qn.$$

(Actually the last inequality is strict, since n is odd.)

Alternate: Suppose $V = R \cup B$ is a coloring achieving discrepancy k, and choose x and y uniformly from R and B respectively. Then

$$\begin{aligned} 1 &\geq \sum_{l \in \mathcal{H}} \Pr(x, y \in l) = \sum_{l \in \mathcal{H}} \frac{|l \cap R||l \cap B|}{|R||B|} \\ &\geq (q^2+q+1) \frac{((q+1-k)/2)((q+1+k)/2)}{((q^2+q+1)/2)^2} = \frac{(q+1)^2 - k^2}{q^2+q+1}, \end{aligned}$$

and the conclusion follows.

8. For $i \in [n]$, set

$$S_i A = A \setminus \{i\} \quad (A \subseteq [n])$$

and, for $\mathcal{F} \subseteq 2^{[n]}$,

$$S_i \mathcal{F} = \{S_i A : A \in \mathcal{F}\} \cup \{A \in \mathcal{F} : S_i A \in \mathcal{F}\}.$$

Then $|S_i \mathcal{F}| = |\mathcal{F}$ and (*exercise*) $k(S_i \mathcal{F}) \leq k(\mathcal{F})$. This implies WMA $\mathcal{F}$ is an ideal; but for ideals the statement is trivial.

[Remark: The number $k(\mathcal{F})$ is the *Vapnik-Chervonenkis dimension* of $\mathcal{F}$. The (easy but important) inequality given here was proved independently by V and C, Sauer, and Perles and Shelah, all around 1970. You'll sometimes find it referred to as "Sauer's Lemma".]

9. This is an easy consequence of Baranyai's Theorem (similar to the derivation of EKR from Baranyai when $k|n$). Let $\mathcal{F} = \{I \in \binom{[n]}{k} : x_I \geq 0\}$, and suppose $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_m$ is a partition of $\binom{[n]}{k}$ into perfect matchings (existence of which is given by Baranyai). Then $\mathcal{F} \cap \mathcal{F}_i \neq \emptyset \ \forall i$ (why?) implies $|\mathcal{F}| \geq m = \binom{n-1}{k-1}$.

10. Let M be the $(d+1) \times (d+2)$ matrix whose ith column is x_i followed by a 1. Since $d+2 > d+1$, there is some $\alpha \in \mathbf{R}^{d+2} \setminus \{\underline{0}\}$ with $M\alpha = \underline{0}$, i.e.

$$\sum \alpha_i x_i = \underline{0} \quad \text{and} \quad \sum \alpha_i = 0.$$

Let

$$I = \{i \in [d+2] : \alpha_i > 0\}, \quad J = \{i \in [d+2] : \alpha_i < 0\},$$

$$\alpha = \sum_{i \in I} \alpha_i \quad (= -\sum_{i \in J} \alpha_i),$$

and $\lambda_i = |\alpha_i|/\alpha$ for $i \in I \cup J$. Then each of $(\lambda_i : i \in I)$ and $(\lambda_i : i \in J)$ is a set of convex coefficients and (since $\sum \alpha_i x_i = \underline{0}$)

$$\sum_{i \in I} \lambda_i x_i = \sum_{i \in J} \lambda_i x_i.$$

11. (First part from A.R. Calderbank, P. Frankl, R.L. Graham, W.-C.W. Li, L.A. Shepp, *J. Alg. Comb.* **2** (1998), 31-48; this proof and full statement due to A. Blokhuis, *J. Alg. Comb.* **2** (1993), 123-124.)

$\mathbf{Z}_3$: For $y \in X$ set

$$p_y(x) = \prod (x_i - y_i + 1) \in \mathbf{Z}_3[x_1, \ldots, x_n].$$

Then

$$p_y(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \in X \setminus \{y\}, \end{cases}$$

so the p_y's are linearly independent, multilinear polynomials, etc.

For the general statement use

$$p_y(x) = \prod_i \prod_{a \in D} (x_i - y_i + a) \in \{p \in \mathbf{F}_q[x_1, \ldots, x_n] : \deg_i(p) \le d \ \forall i\}.$$

12. For $x \in V = V(\mathcal{H})$, let $N(x) = \bigcup \{A : x \in A \in \mathcal{H}\}$. Choose x with $d_{\mathcal{H}}(x)$ maximum. We'll show that in fact $|\mathcal{H}| \le |N(x)|$ $(\le \Delta(\mathcal{H}))$.

Let G be the bipartite graph on $N(x) \cup \mathcal{H}$ with $v \sim_G A$ iff $v \in A$. It's easy to see that, except in trivial cases, G satisfies the hypotheses of Motzkin's Lemma; so we just need to show that $v \not\sim_G A$ implies $d_G(v) \le d_G(A)$, i.e. $d_{\mathcal{H}}(v) \le |A \cap N(x)|$. But this is easy: if $x \notin A$, then $d_{\mathcal{H}}(v) \le d_{\mathcal{H}}(x) = |A \cap N(x)|$, while if $x \in A$, then $d_{\mathcal{H}}(v) \le |A| = |A \cap N(x)|$.