

Proof validation in real analysis: inferring and checking warrants

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Abstract

In the study reported here, we investigate the skills needed to validate a proof in real analysis—i.e., to determine whether a proof is valid. We first argue that when one is validating a proof, it is not sufficient to make certain that each statement in the argument is true. One must also check that there is good reason to believe that each statement follows from the preceding statements or from other accepted knowledge, i.e. that there is a valid warrant for making that statement in the context of this argument. We then report an exploratory study in which we investigated the behavior of 13 undergraduates when they were asked to determine whether or not a particular flawed mathematical argument is a valid mathematical proof. The last line of this purported proof was true, but did not follow legitimately from the earlier assertions in the proof. Our findings were that only six of these undergraduates recognized that this argument was invalid and only two did so for legitimate mathematical reasons. On a more positive note, when asked to consider whether the last line of the proof followed from previous assertions, eight students concluded that the statement did not and rejected the proof as invalid.

Key words: counterexamples; implication; limits; logic; proof; real analysis; reasoning; validation; warrants

1. Introduction

1. 1. Research on proof validation

Mathematicians and mathematics educators consider proof validation to be an important mathematical activity. The NCTM (2000) asserts that since validating proofs is an important means for constructing sophisticated mathematical knowledge, all high school students should be able to complete this task by the time they complete 12th grade. Clearly the ability to validate proofs assumes even greater importance for mathematics majors. A primary reason that proofs are presented to undergraduates is to convince and to explain (e.g., Hersh, 1993). If a student cannot reliably determine if an argument constitutes a proof of a theorem, then that argument cannot legitimately explain to them why the theorem is true. Since much of a students' time attending mathematics courses is spent observing the professor as he or she presents theorems and proofs (e.g. Weber, 2004), it follows that if undergraduates cannot validate proofs reliably, then they are unlikely to gain either conviction or understanding from much of the discourse in their advanced mathematical courses (Selden and Selden, 1995).

While there is a consensus on the need for students to validate proofs, there has been limited research on students' ability to do so, and almost no such work on students' validations of proof at the level of line-by-line analysis. Most of the research that has been published has focused on whether students or teachers would accept or reject arguments based on their form (cf., Martin and Harel, 1989; Hoyles, 1997; Segal, 2000). Research of this type has revealed that both students and teachers of mathematics have difficulty in accurately determining whether an argument constitutes a valid proof. For instance, Martin and Harel (1989) and Knuth (2002) asked pre-service elementary and in-

service high school teachers respectively to validate mathematical proofs. Both groups had serious difficulties with this task. In an exploratory study at a more detailed level, Selden and Selden (2003) presented eight undergraduates in a transition proof course with four arguments purporting to prove the statement, “If n^2 is divisible by 3, then n is divisible by 3”. They found that undergraduates’ judgments on the validity of the purported proofs were essentially at chance level, although students’ performance in distinguishing valid from invalid arguments improved dramatically through the reflection and reconsideration engendered by the interview process.

1. 2. Research question addressed in this paper

The purpose of this paper is to further investigate the ways in which students attempt to validate proofs. We do so by reporting an exploratory study in which we presented undergraduates with a flawed argument that purports to prove that the sequence (\sqrt{n}) diverges to infinity and asked them to determine whether or not this argument constituted a proof. The specific questions that guided our analysis were: 1) What features of a proof do undergraduates in a real analysis course attend to when trying to determine whether it is valid? 2) Can these undergraduates accurately determine whether or not a proof is valid? 3) Can leading prompts that focus the undergraduates’ attention on relevant aspects of the proof improve the undergraduates’ performance?

Our study took place in a real analysis course, and in such courses the semantic content is central to the discussion of the proofs that are studied. Proofs in real analysis often are presented to demonstrate that mathematical objects have certain properties and to establish relationships between mathematical structures. To understand and validate such proofs we would expect students to focus not only on their logical nature (e.g., does

this proof have a legitimate structure? Was this statement negated correctly?), but also on the plausibility of their semantic content (e.g, is it true that all convergent sequences are bounded? Could a function be differentiable without being continuous?). As we will illustrate in section 2.2, the flaw in the invalid proof used in our study is not in the form of the argument, but in the content. We focus on whether and how individuals identify this type of flaw.

1. 3. Theoretical framework

We use Toulmin's (1969) model of argumentation as a frame for discussing the way in which we believe a line-by-line validation of a proof should proceed (see Weber and Alcock, 2005). In Toulmin's model, an argumentation consists of at least three essential parts, called the core of the argument: the conclusion, the data, and the warrant. When one presents an argument, one is trying to convince an audience of a specific claim. In Toulmin's model, this claim is referred to as the *conclusion*. To support the conclusion, the presenter typically puts forth evidence or *data*. The presenter's explanation for why the data necessitate the conclusion is referred to as a *warrant*. At this stage, the audience can accept the data but reject the explanation that the data establishes the conclusion; in other words, the authority of the warrant can be challenged. If this occurs, the presenter is required to present additional *backing* to justify why the warrant, and therefore the core of the argument, is valid (Stephan and Rasmussen, 2002).

In recent years, several influential mathematics educators have applied Toulmin's model to understand the arguments that students construct (e.g., Krummheuer, 1995; Forman et. al., 1998; Stephan and Rasmussen, 2002; Rasmussen, Stephan, and Allen, 2004). In this paper, we use Toulmin's model in a slightly different way, first in a

normative sense to frame an abstract discussion of what a reader needs to do in order to reliably validate a proof, and second to discuss whether students do this when asked to validate a particular provided proof.

We begin by noting that a mathematical proof consists of a sequence of assertions, each of which can be viewed, in Toulmin's terms, as the conclusion of an argumentation. To determine whether such an assertion is valid, the reader must not only consider the assertion itself, but also the data and warrant used to support this assertion. As proofs would be impossibly long if every logical detail was included (*cf.*, Renz, 1981), the data and the warrant used in mathematical argumentations are often not explicitly stated. When this occurs, the reader must *infer* the data and the warrant used to justify the assertion being made. In order to validate the proof, they must then decide whether all the assertions are supported by valid data and warrants.

To illustrate the importance of inferring warrants, consider the following two mathematical argumentations:

Argumentation 1: Since x^3 and x are continuous functions¹, $x^3 + x$ is a continuous function.

Argumentation 2: Since x^3 and x are continuous functions, $x^3/x = x^2$ is a continuous function.

In our framework, the first argumentation would be an acceptable statement in a proof (assuming that the “data” facts that x^3 and x are both continuous functions are agreed upon or have been established elsewhere in the proof). The conclusion “ $x^3 + x$ is continuous” is based on the implicit warrant “If f and g are continuous functions, then $f+g$ is continuous”. As this warrant is an established theorem in real analysis, the argument in its entirety is valid. The second assertion is not a valid deduction. The data is the

¹ Using “ x^3 ” as shorthand for the function f from the reals to the reals given by $f(x)=x^3$, etc.

same, but the implicit warrant used to establish the conclusion appears to be, “If f and g are continuous functions, then f/g is a continuous function”. While the conclusion that x^2 is a continuous function is true, the implicit warrant used to justify this conclusion is false (e.g., $f(x) = x$ and $g(x) = x^2$ are continuous functions, but $(f/g)(x) = 1/x$ is not). Hence, although the conclusion is correct, the argumentation used to establish the conclusion is flawed. If a proof relied on this assertion in a critical way, that proof should be rejected as invalid (Weber and Alcock, 2005).

We conjecture that one’s line-by-line validation of a proof might proceed like this. Each deduction in the proof is an argumentation whose conclusion is the new statement being asserted. The reader of the proof identifies the data and the warrant used in this argumentation, inferring them if necessary. Assuming that the reader finds the data to be sound, there are three judgments that he or she can make. If the warrant for the argumentation is socially agreed upon in the given context, this statement is accepted as valid. If the warrant is false, this line and the entire proof are declared to be invalid. If the warrant of an argumentation is plausible, but not socially agreed upon in the given context, backing for this warrant is required and the proof is said to have a “gap” in it (Weber and Alcock, 2005). To illustrate the difference between the first and third possibilities, note that the truth of a warrant does not guarantee its socially agreed status – in the context of a real analysis course, for example, a student may not be allowed to use the result about sums of continuous functions without including a proof of this result.

2. Research context and methods

2. 1. Research context

The results reported here were obtained from a larger study investigating students' understandings of definition and proof in real analysis. Eighteen student volunteers taken from two first-term, first-year introductory real analysis courses at a British university participated in this study (see Alcock and Simpson, 2001). Each of these students had earned an A in their A-level mathematics course and strong grades in their other courses. In other words, all were good students and each demonstrated a high degree of mathematical competence before taking this course. These students were interviewed by the first author of this paper, in pairs, five times throughout the ten-week term. The data reported here were obtained from a task-based interview given to these students during the fifth week of the course.

2. 2. Materials

Each pair of students was presented with the following mathematical argument and was asked to determine whether or not the argument was a valid proof².

Theorem: $(\sqrt{n}) \rightarrow \infty$ as $n \rightarrow \infty$.

Proof: We know that $a < b \Rightarrow a^m < b^m$.

So $a < b \Rightarrow \sqrt{a} < \sqrt{b}$.

$n < n + 1$ so $\sqrt{n} < \sqrt{n + 1}$ for all n .

So $(\sqrt{n}) \rightarrow \infty$ as $n \rightarrow \infty$ as required.

The first three lines of the proof contain minor errors (e.g., $a < b$ only implies $a^m < b^m$ if a and b are both positive or appropriate restrictions are placed on m , the scope of the variable n was not properly defined), but with such modifications do serve as a correct demonstration that $\sqrt{n} < \sqrt{n + 1}$ for every positive integer n . The fundamental

² It was standard in the students' real analysis course for the notation (a_n) to indicate a sequence.

flaw with this argument occurs in the last line, which asserts the statement to be proven. The author appears to be implying that since (\sqrt{n}) is an increasing sequence, it diverges to infinity. However, it is not the case that all increasing sequences diverge. Note that in our terms, the data and conclusion of this argument are both true, but the warrant for this argument is false and hence this argument is invalid.

2. 3. Procedure

Each pair of students was presented with the argument previously described, along with a request to, “Check the proof and make corrections to it where appropriate”. At first, the students were encouraged to confer with each other and to use the paper on which the question appeared for writing anything that might be useful. During this time the interviewer answered clarification questions, but deflected content-based questions back to the students. After the students made modifications to the proof, the interviewer asked whether they would accept the proof given these modifications. If the participants had not modified the argumentation formed by the third and fourth lines, the interviewer then focused the students’ attention on these lines, and asked whether the fourth necessarily followed from the third.

3. Results

We examined the responses of thirteen of the eighteen participants of this study. The other five students were not considered because their partners dominated this part of the interview session so that it was not possible to follow their thinking from their comments. All of these thirteen participants began by making minor modifications to the first three lines of the argument. Our analysis does not focus on these modifications, but

rather on their behavior when they read the last line in this proof. Our concern was whether the students would accept this line as a valid conclusion, given their earlier suggested modifications. Eleven of the students' responses in this regard could be sorted into three categories, each of which is described below. The remaining two students are briefly discussed at the end of this section.

3. 1. Students who rejected the proof because of an invalid warrant

Three students argued that the last line of the proof did not follow from the previous work because the warrant used to establish this claim was invalid. We illustrate this by considering the response of Greg. In examining the last line, Greg infers the warrant as asserting that increasing sequences diverge to infinity. When asked to focus on what is wrong with the proof, Greg replies:

G: Well, they're, what they've done is they've gone straight from the fact, that...erm...square root of n , is smaller than square root of n plus 1...so that...the series [*sic.*] is increasing...and they've presumed that that, from there, that it goes to infinity.

I: Yes.

G: Whereas there's no...you can have series which are always increasing, and which don't go to infinity.

I: Right.

G: So you need to prove that it's not one of them, basically, as well.

When asked whether he can produce an example of a sequence that is increasing but does not tend to infinity, he immediately does so.

G: Well, an example would be, 1 minus 1 over n [$1 - \frac{1}{n}$]. I think because 1 over n is always, going to zero, it's decreasing to zero so 1 minus 1 over n , is increasing to 1.

The responses of Adam and Jenny were similar. In terms of our framework, they both inferred warrants that were being used implicitly to justify the last line of the proof and cited counterexamples to show these warrants were not valid. (Unlike Greg, they cited these counterexamples without prompting). Jenny mentioned that the first three lines of the proof would also apply to the sequence $(n/n+1)$, but that this converges to 1. Adam cited the sequence $(0, 1, 0, 2, 0, 3, \dots)$ as an example of an unbounded sequence that does not tend to infinity. He appeared to have inferred that the warrant used to establish the final line was, “If a sequence is not bounded above, then it tends to infinity”. Of course, this is not an appropriate warrant given the previous assertions in the proof, and illustrates how inferring warrants in a presented proof can be a non-trivial task. Greg and Jenny were the only students in this study to reject the proof as invalid for fully legitimate mathematical reasons.

3. 2. Students who rejected the proof because definitions were not employed

Three students criticized the argument because standard definitions for divergence were not used, but did not cite a problem with the last line of the proof. We illustrate this with Fred. After reading the proof, Fred first notes that the standard definition was not used. While his statement of this definition contains errors (he fails to correctly quantify his variables and seems to use n to represent both an index of a sequence terms and the numerical value of that term), he is rejecting the proof on the basis of its absence.

F: No hang on, they need, do they not need to prove, that...if you're going to prove that something tends to infinity then you need to say, you know there exists a... n so that...there, there exists a number r , so that n is greater than r for all... n greater than a certain big N , yes? In which case, we'd have to prove that. Yes? Does that, make sense?

After this, Fred spends some time struggling with the content of the proof. When asked about whether the presented argument proves that (\sqrt{n}) tends to infinity, he seems uncertain but does not pronounce it invalid.

F: Y-es. I think it does.

I: Mm... [Pause] Right...

F: Yes, I guess it does.

However, when the interviewer points out that the third line of the proof asserts that the sequence is increasing, Fred realizes that this does not necessarily imply that the sequence diverges.

I: Does that mean, that necessarily, that the sequence root n tends to infinity? Just because each term's greater than the previous one?

F: No, not necessarily because it could, converge to some, limit. For all we know. It could get, ever increasingly closer to, erm...42.

I: Mm. I think so. So, are we allowed to go from the third line to the last line then in that case?

F: No, I'm, not sure that we are. I don't like it.

Two other students, Cary and Kate, gave reasons for rejecting the proof that were similar to Fred's. Comments illustrating this (taken from their separate interviews) are given below:

C: Well, I wouldn't do it that, well I wouldn't do it that way, and I don't know whether that, erm, stands as a, a criticism for him. I would, prefer, to use the definition of tending to infinity in that, proof there.

K: If you're being picky, that is... it's got no, definition to say that that is, tending to infinity.

When the interviewer turned their attention to the last two lines of the proof, both of these students also recognized that increasing sequences do not necessarily diverge and proclaimed the proof to be invalid.

In our framework, we would argue that these students rejected the proof because they found the data unacceptable. They seemed to believe that only data employing the definition of divergence to infinity could be used to support the claim that a sequence has this property. It is not surprising that some students hold this view. In real analysis courses, it is frequently stressed to students that their proofs must be based on definitions (in part to address students' tendencies to present non-deductive arguments). In fact, disciplining oneself to reason from a concept's definition requires a sophisticated understanding of advanced mathematics that many beginning analysis students lack (*cf.*, Alcock and Simpson, 2002). Hence one could argue that Fred, Cary, and Kate showed considerable mathematical maturity in this respect. Still, it is not the case that every proof must incorporate definitions – once theorems have been established, these may be used instead. For instance, showing that a sequence is increasing and unbounded is sufficient to show that it diverges to infinity.

3. 3. Students who accepted the proof as valid

Five students initially accepted the argument as a valid proof. We first illustrate this by describing the behavior of Steve and Tom. As the following excerpt shows, Steve and Tom initially struggle as to whether the fourth line of the proof is valid, before ultimately deciding that the proof is acceptable.

S: Yes but is this a good enough proof to show that that-

T: Yes...well...

S: Has it got anything to do with it at all?

T: Well...I think it is because, because erm... [*Pause*]

S: Oh I, yes, I suppose, if this bit's right, then,

T: Yes,

I: Can we clarify what we're talking about sorry, for the tape?

T: Right,

S: So, if the line n is less than n plus 1...so...root n 's less than the root of n plus 1 for all n , then...the final line, root n tends to infinity as n tends to infinity, is right.

T: Yes. Definitely. I like it.

At this point, the interviewer focuses Steve and Tom's attention on what is being claimed in the fourth line of the proof. This leads Steve and Tom to change their minds about the validity of the proof.

I: Line 3 basically tells you that, the sequence...that the sequence is increasing, yes?

T: Yes.

I: The sequence root n .

T: Yes.

I: Steve?

S: Mm...

I: Yes?

S: Yes.

I: Does that prove that, the sequence tends to infinity?

T: Well, I was thinking about this, and...and obviously it could, it could tend to a limit couldn't it?

I: Right.

T: But...if you had like...so it's, you could, it doesn't have to be plus 1 it could be n , plus, n again, or it could be n times n . And if it was n times n , then you'd get n squared, the square root of n squared which is basically, n ...

I: Right. [Pause]

I: My question really is does this person's proof, prove that result?

T: [Pause] No.

S: No, it proves that it's increasing,

T: But it doesn't prove it tends to a limit.

S: But not that it's going, it doesn't prove that it's going to infinity.

Three other students responded similarly to Steve and Tom. Once they made minor modifications to the first three lines of the proof, they decided that this showed that $\sqrt{n} < \sqrt{n+1}$ for every natural number n . After this point, they were prepared to accept the proof is valid, apparently without considering why the fourth line would follow from the third. However, when the interviewer turned the students' attention to the last two

lines of the proof, two of the students reconsidered and immediately rejected their proof as invalid. The response of one of these students, Vic, is given below:

V: Well because, that just says that, that just says that that one's bigger than the other one. But you don't know how much bigger, it gets. So it could just, sort of, the difference might get smaller and smaller and smaller, to, like a converging sequence or something. But, what I'd do would probably say that erm, square root of infinity, is infinity.

Vic clearly sees that the third line of the proof does not imply the fourth line. In an important sense, one can infer a deeper understanding from Vic's comments than if he had simply constructed an increasing, convergent sequence. Providing a counterexample is sufficient to show *that* the implicit warrant is false; this is all that needs to be done to show that the proof is invalid. Vic's explanation goes further, and offers an insightful explanation for *why* the warrant fails. Of course, the last line of this excerpt suggests that Vic still might not know what it would take to construct a valid proof of his own.

In analyzing the data, we considered the possibility that the students who accepted the proof did so due to an inadequate knowledge of the properties of sequences. Students' misconceptions about limits are well documented (e.g., Davis and Vinner, 1986; Cornu, 1991), and it may have been the case that these students believed that all increasing sequences diverged. However, in this situation, this does not appear to be the cause of their difficulties. From their comments, it appears that they focused exclusively on the data and conclusions employed in the argumentation of the fourth line of the proof and did not consider what warrant was used at all. Once they were induced to consider this implicit warrant, they did both demonstrate appropriate knowledge and use it to conclude that the warrant was invalid.

3. 4. Remaining students

There were two other students who could not be coded into the three groups described above. One student, Dean, believed that the proof was basically correct, but found it vague. He indicated that he felt somewhat uncomfortable with the argument, but could not articulate why. At this point, the interviewer focused Dean's attention on the implicit warrant being used in the fourth line of the proof, but this discussion did not resolve Dean's difficulties. The other student, Wendy, initially concluded that the since the sequence is strictly increasing, the proof is valid. Even when the interviewer provided a counterexample to this warrant in the form of the sequence $(1 - \frac{1}{n})$, she still did not reject the proof as it applied to the sequence (\sqrt{n}) .

4. Discussion

4. 1. Summary of results

We now return to the three research questions that guided our analysis. Our first question asks what features of a proof do students attend to while they are determining whether it is valid. Fred, Cary, and Kate rejected the proof because an appropriate definition was not used, and indeed, one might infer that at this stage they would likely reject any proof in real analysis that did not employ a definition. Due to their immediate rejection of the proof, they did not consider data or warrants in the remainder of the proof. Other students' behaviors varied, but many seemed to focus on whether the assertions made were true, rather than considering whether they were substantiated. What is most disappointing is that only three students spontaneously inferred a warrant used to establish the fourth line of the proof. Failure to consider the warrants used in a proof will not only cause students to be unable to validate proofs reliably, but, as we

argued in the introduction, can also prevent them from gaining conviction and understanding from proofs presented in their classrooms.

Our second question asks how successful students are in their proof validations. Our results suggest that many of the students in our study could perform this task competently, but did not do so without prompting. Only six of the thirteen students initially rejected the proof as invalid, and only two of the students did so for legitimate mathematical reasons. However, addressing our third question, leading prompts from the interviewer resulted in a total of ten of the thirteen students rejecting the proof as invalid due to a false warrant. Our results are consistent with Selden and Selden's (2003) finding that students' performance at such tasks can improve considerably when they are encouraged to reflect upon their critique of a proof. This suggests that the ability to validate proofs may be in many students' zone of proximal development and that students' abilities in this regard might improve substantially with relatively little instruction.

4. 2. Pedagogical considerations

Selden and Selden (2003) observe that the skills needed to validate proofs often receive little attention in the mathematics classroom. As a result, students' conceptions of reading, understanding, and validating a proof may drastically differ from a mathematician's. When this occurs, what students learn from observing proofs in their advanced mathematics classroom may not be what the instructor intends. In our experience, it seems that the issue of warrants is not discussed in advanced mathematics courses. For instance, most textbooks for proof-oriented mathematics courses stress logical standards for what constitutes a valid mathematical deduction (rules of inference

such as modus ponens, tautologies, etc.), while the issue of warrants is rarely discussed at all. Our observations of the teaching of advanced mathematics courses (e.g., Weber, 2004) also support the notion that warrants are not commonly discussed in proof-oriented university mathematics courses. We speculate that one reason that warrants are not discussed is that mathematicians fear this may confuse students; students may fail to see the difference between evaluating the logical *truth* of an implication and using a warrant to determine whether a statement is *legitimate* to assert in a proof. We concur that it will be difficult for students to coordinate these two notions. However, we nonetheless recommend that explicit classroom activities be devoted to inferring and evaluating the often implicit warrants used in mathematical proofs. Although it will necessarily demand time and energy to engage with and resolve any confusion this generates, we would argue that omitting this discussion deprives students of an opportunity to develop a skill that could greatly enhance their understanding of the proofs they encounter in all of their future mathematics courses.

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