Informal Content and Student Note-Taking in Advanced Mathematics Classes

Tim Fukawa-Connelly
Temple University

Keith Weber
Rutgers University

Juan Pablo Mejía Ramos
Rutgers University

Abstract: This study investigates three hypotheses about proof-based mathematics instruction: (i) that lectures include informal content (ways of thinking and reasoning about advanced mathematics that are not captured by the formal symbolic statements), (ii) that informal content is usually presented orally but not written on the blackboard, and (iii) that students do not record the informal content that is only stated orally but do if it is written on the blackboard. We recorded 11 80-minute mathematics lectures and photographed the notes of 96 students. We found that (i) informal content was common (with, on average, 32 instances per lecture), (ii) most informal content was presented orally, and (iii) typically students recorded written content while not recording oral content in their notes.

Keywords: Lecture; Note-taking; Proof; Undergraduate mathematics education

Collegiate proof-oriented mathematics courses are usually taught by lecture (Fukawa-Connelly, Johnson & Keller, 2016). However, many mathematicians and mathematics educators view this practice as being ineffective for helping students learn advanced mathematics (e.g., Bressoud, 2011; Leron & Dubinsky, 1995), with students typically emerging from advanced mathematics courses with a poor understanding of mathematical content (e.g., Rasmussen & Wawro, in press) and without acquiring the skills, such as proof writing, that these courses are trying to teach (e.g., Stylianides, Stylianides, & Weber, in press).

Recently, we conducted a qualitative case study of what ideas students did and did not understand from a lecture that they attended (Lew, Fukawa-Connelly, Mejia-Ramos, & Weber, 2016). In this case study, we illustrated how students might not recognize the main points that a instructor intended to convey during a lecture, even when these points received explicit emphasis in the lecture. Specifically, we studied a mathematics professor, Dr. A, teaching a real analysis course. We videorecorded Dr. A presenting a ten minute proof of a theorem, showed the videorecording to Dr. A, and asked him to identify the main content that he was conveying to the students. The content that Dr. A identified as important included a method for proving a particular class of statements. We also showed the same videorecording to six students enrolled in Dr. A's course and asked them to describe the mathematical content that they felt Dr. A was trying to convey. The students usually did not identify the content that Dr. A had highlighted as important. For example, Dr. A thrice stated that if one wanted to prove that a sequence converges but could not establish a limit-candidate, that one could try to prove that the sequence is Cauchy. In our interview with him Dr. A identified this methodological heuristic as a key reason for presenting this proof. None of the six students mentioned this when they were asked what Dr. A was trying to convey after watching the videorecording of the proof in its entirety.

We used the research literature on undergraduate mathematics education and note-taking to provide an account for why students might not have learned the main points of Dr. A's lecture.

On the one hand, research in undergraduate mathematics education suggests that, in addition to knowing the formal mathematics, students need access to informal ways of reasoning (e.g., Dennis & Confrey, 1996; Dreyfus, 1991). In the case of Dr. A's lecture, we noted that some of the main points he was trying to convey (like the methodological heuristic mentioned above) represented Dr. A's own informal ways of thinking about converging sequences. As such, we noted that in this case Dr. A's students had been given some access to these informal ways of reasoning in mathematics. On the other hand, the research literature on students' note-taking has found that when students do not record a instructor's points in their notes, they usually forget these points (Kiewra, 1987). Some researchers have estimated that students could recall a point that they did not record in their notes less than five percent of the time (Einstein, Morris, & Smith, 1985).

Thus, Lew et al. (2016) hypothesized that an important reason the students in the case study likely forgot the points that the professor identified as important was because the students did not record these points in their notes. In proposing an account for the students' failure to comprehend the material, we connected the following three observations from the case of Dr. A: 1) The content that Dr. A identified as important was stated orally, but not written on the blackboard; 2) Students' notes usually did not include any of the content that Dr. A presented orally; 3) Consequently, although Dr. A intended to provide students access to the reasoning and heuristics he used to produce the proof, these ideas were not recorded in the students' notes.

In this paper we do not directly test our account for student learning, but rather the extent to which the observations we made about one lecture with six students generalize to a larger collection of lectures and students. Specifically, we investigate the extent to which the following phenomena are present in other lectures in advanced mathematics:

(1) Instructors in advanced mathematics discuss informal aspects of mathematics. In particular, these instructors represent mathematical concepts using informal

representations, discuss methods that can be useful for completing other mathematical tasks, model mathematical behavior, and give examples of the concepts.

(2) When instructors discuss informal aspects of mathematics, they usually make their comments orally and do not record them on the blackboard. The blackboard is reserved for formal mathematics, including definitions, theorems, and proofs.

(3) When instructors make these comments orally, students usually do not record these comments in their notes.

Method

Participants

We recruited participants by e-mailing every instructor at three U.S. universities who were teaching a proof-oriented course, asking them if we could record one of their lectures and invite their students to participate via a researcher photographing their notes. Instructors were not told the purpose of the study. The professors were also asked not to let the students know that we would be conducting research on this lecture. Eleven different instructors agreed, and the content of their courses and lectures is summarized in Table 1 below. Each class had between 7 and 30 students enrolled, with a mean of approximately 18 students per class. All instructors gave lectures using a blackboard; no instructor used technology. We did not interview the instructors about their courses so we do not know if there was a textbook assigned to students in every course. However, instructors typically assign textbooks in undergraduate mathematics courses in the United States.

Instructor	Overarching Course-content	Description of content in the lecture we recorded
M1	Set Theory	Transfinite arithmetic, cardinals
M2	Real Analysis	Infinite series, convergence, examples of sequences and series that do and do not converge

M3	Number Theory	Prime Number Theorem, and approximations of the prime number theorem
M4	Linear Algebra	Jordan Canonical Form, T-invariant subspaces
M5	Abstract Algebra	Exam problems, permuations, cycle notation, operations on permutations, order
M6	Number Theory	Reduced residue systems, Euler's theorem, multiplicative functions, the Euler phi function
M7	Geometry	Isometries and similarities
M8	Abstract Algebra	Ideals, Principal ieals, how congruence mod n is similar to congruence in polynomials
M9	Abstract Algebra	Ideals, congruence modulo an ideal, well-defined operations
M10	Real Analysis	Partitions, Riemann integration, Riemann integral
M11	Differential Geometry	Gaussian Curvature, eigenvalues & eignevectors, principal curvature

Table 1. Overview table of instructor, course, and content

Data Collection

For each instructor who agreed to participate, a member of the research team attended a class meeting in which an exam was not given. The researcher audiorecorded the lecture, while transcribing everything that the instructor wrote on the blackboard in the researcher's notes using a LiveScribe pen (i.e., a pen that one can use to simultaneously audio record and take notes so that the timing of the notes is coordinated with the audio recording).

Each lecture was approximately 80 minutes long. At the end of the lecture, the researcher made an announcement to the class inviting students to share their notes with the researcher, even if their notes were not of high quality, or if the student had not taken notes at all. Collectively, 96 students, out of approximately 200, across the 11 lectures agreed and the researcher photographed the notes that the students took for that lecture. If the student had not taken any notes, the researcher simply photographed a blank page from the student's notebook.

Coding the Lectures

Each lecture was transcribed. The first two authors coded the lecture for every time one of the following was presented: *definition, proposition, proof, example, informal representation, mathematical method*, and *modeling mathematical behavior*. We describe and sometimes illustrate each category below. Any disagreements between the two researchers were resolved by discussion.

We developed our coding scheme in an iterative fashion. For the formal mathematics, we chose the terms *definition*, *proposition*, and *proof* from early critiques of mathematics lectures describing them as being entirely composed of that type of content (e.g., Davis & Hersh, 1981) and these being common content in lectures in advanced mathematics (Weber, 2004). From the research on lecture instruction in advanced mathematics, we also took initial codes for *example* (e.g., Fukawa-Connelly & Newton, 2014), *modeling mathematical behavior* (e.g., Fukawa-Connelly, 2012), *mathematical method* (e.g., Weber, 2004), and *informal representation* (e.g., Weber, 2004). Consistent with Davis and Hersh (1981), we refer to instances of definition, proposition, and proofs as formal mathematical content. We refer to instances of example, modeling mathematical behavior, mathematical method, and informal representation as informal mathematical content. We believe that this classification is not only consistent with the existing literature, but that most mathematicians and mathematics educators would categorize these specific types of content as being formal or informal in a similar way. Finally, we observe that this represents only a proper subset of types of formal and informal mathematics.

For each code, we developed an initial description based on the extant literature. When difficult cases arose as we coded, we came to consensus about whether the identified text should be counted as an instance of a particular code. We then revised the description of the code in order to eliminate the ambiguity posed by the instance. After an initial round of coding and code-revision, the first two authors recoded each of the transcripts using the revised coding manual.

We treated the codes as mutually exclusive and, as a result, any specific utterance by the instructor was assigned at most one code.

Definition: A definition is a (set of) sentence(s) in which necessary and sufficient conditions to describe a concept are stated. We coded a (set of) sentence(s) as a definition whenever either of the following occurred:

1. The instructor explicitly labeled the sentence a definition by prefacing the sentence with the word "definition"
2. The instructor wrote or said a term and necessary and sufficient conditions to characterize the construct associated with that term in succession, linked by the verb define ("we define a transposition to be…") or some word or phrase that is synonymous with define such as "we call a transposition…."

Proposition: A proposition is any mathematical statement that has a truth-value. We coded content as a proposition if one of the following two conditions occurred:

1. The instructor explicitly called or labeled the mathematical statement as a theorem, proposition, corollary, or lemma.
2. The instructor made a mathematical statement that was: (a) outside the context of proof, (b) not a claim of the form "object X is an example of concept C", and (c) the statement concerned the new concepts that were being discussed in that lecture.

Proof: A proof is a justification of a proposition (as defined above) that is a coherent set of mathematical statements, consisting of acceptable premises (e.g., definitions, axioms, shared knowledge) and new statements that were deduced logically from previous statements, culminating with the proposition being justified.

Example: An example was coded when the instructor chose a specific mathematical object as a representative of a class of mathematical objects. This includes claims of the form "object O is a member of category C" and instantiating a general claim that applies to a large or infinite class of objects with a particular object.

Informal representation: An informal representation was coded when the instructor's presentation gave meaning to the content beyond what is stated in the formal definition. We coded such content as occurring if one of the following conditions were met:

1. The instructor gave a description of the content in a representation system other than the verbal-symbolic representation in which the definition was stated, including using a diagram or a metaphor.
2. The instructor drew an analogy between the content that was stated and content that the student learned previously in other mathematical settings.
3. The instructor stated the meaning or purpose of a definition, proposition, or proof using colloquial English that was not synonymous with that definition, proposition, or proof.[1]

We illustrate this category in the context of a real analysis course in which the instructor (M2) was beginning to write a proof that the harmonic series diverges. He began by expressing the harmonic series as a histogram (presented in Figure 1 below) and added the following.

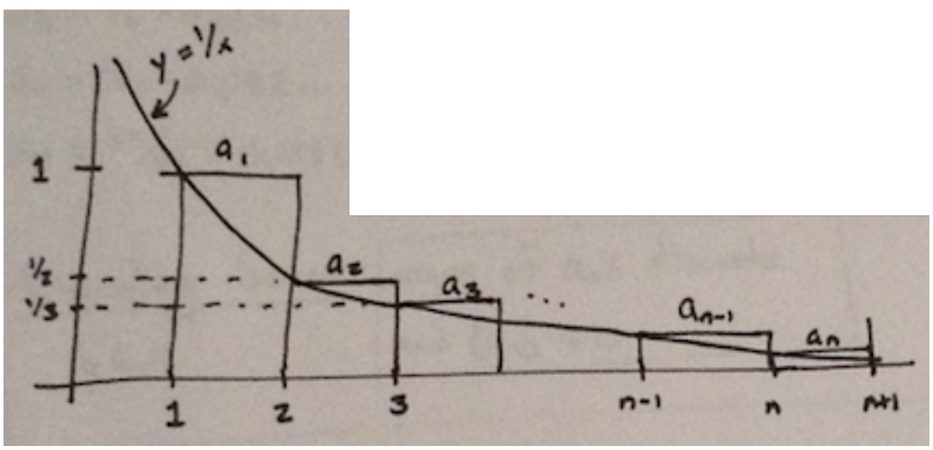

Figure 1 A copy of the histogram the instructor drew

[1] Although there is a continuum of formality, we aimed to be conservative and consistent in our coding of informal content. For instance, although describing a limit as "we can make the values of $f(x)$ arbitrarily close to L (as close as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a" (Stewart, 2012, p. 50) departs from entirely formal language, this would not have been coded as an informal mathematical representation because the provided description was sufficiently synonymous with the formal statement.

> M2: So I'm going to draw a histogram. And I'm not going to go very far with this nonsense, but the histogram is a-sub-1 is 1, a-sub-2 is ½, a-sub-3, is there, now the point that you should notice is that the subscript goes to the right of this. So to the right of this is a-sub-n minus 1 and the right of this is a-sub-n. Now if I connect these, these vertices, I get y equals 1 over x.

This satisfied criterion 1. M2 is expressing the concept of harmonic series, which was originally (and is conventionally) represented as $\sum_{k=1}^{\infty}\frac{1}{k}$, visually as the area of an infinite histogram. A second illustration occurred in the context of M7's geometry classroom in which the notion of homothety (i.e., a function on a space that dilates distances with respect to a point).

> M7: Oh and by the way, so we said if we have a homothety it sends a triangle to a similar triangle. We said if we have similar triangles, we can map one to the other using a homothety plus possibly an isometry to get things in the right place, so now we can redefine… Just as we redefined congruence we can redefine similarity [...] We said isometries are the tool for congruence, for proving congruences, homotheties are going to be the tool for proving similarities, and in fact there's this theorem and it is 5.3.25, that goes, check it if you want, if… Sorry, how do I want to phrase it? If two triangles are similar and not congruent, then…

This transcript satisfied criterion 2. M7 is relating the relationship between homothety and similarity to the relationship between congruence and rotation and translation.

Mathematical method: By mathematical method, we mean any instance in which a instructor described a non-algorithmic approach to accomplish a general mathematical task or the instructor states conditions under which a particular technique is likely to be useful. To recognize an instance of mathematical method, we looked for instances in the text where:

1. The instructor provided a general guideline for completing a task (such as writing a proof) that (a) could be applied to other tasks beyond the particular task at hand and (b) was not logically necessary for performing the task at hand.
2. The instructor described a trick or heuristic for accomplishing a mathematical task.

3. The instructor described specific conditions under which a mathematical concept or method was either likely to be useful or likely not to be useful.

We illustrate this category with two examples. In an abstract algebra class, the instructor (M5) was addressing how one would prove the following claim that was written on the blackboard:

> *Let I = {f in C| f(2) = 0}. Claim* $C/_I \cong R$
>
> M5: What we wanted to do was show that [this] is isomorphic to the real numbers using the first isomorphism theorem. Remember, when you see this type of statement, you always want to think about constructing a homomorphism from C into the real numbers.

The critical line of this transcript is "Remember when you see this type of statement, you always want to think about constructing a homomorphism from C into the real numbers". We interpreted "this type of statement" meaning a statement of the form $C/_I \cong R$, where I could be any ideal. Here M5 is providing a general guideline ("think about constructing a homomorphism from C to the real numbers") beyond this particular problem (indicated explicitly by "always" that this can be applied for any ideal I), although this is not strictly logically necessary (i.e., you can prove a quotient ring is isomorphic to another ring without invoking the first isomorphism theorem; it's just usually easier and more efficient to use the first isomorphism theorem). A second example occurred in the linear algebra lecture in which the instructor (M4) described a method for showing that two square matrices are similar:

> M4: Two matrices, square matrices, with all eigenvalues in F, the field, are similar if and only if they have the same Jordan Canonical form. [W*rites* Corollary: Two square matrices with all eigenvalues in F are similar ⟺ they have same JC form] If you're ever presented with two square matrices and you wanna know if they're similar, that's a hard question to answer unless you take this approach. Figure out their Jordan canonical forms and see if they're the same.

The first sentence was coded as a proposition because the instructor specifically called it a corollary in the written text. The remainder of this transcript satisfies criterion 2 and 3. The informal statement that M4 has given gives two additional pieces of information beyond the formal statement. First, the instructor has presented it as a means to accomplish a specific task, and, second, noted that otherwise the task would be difficult. The formal statement makes no mention of when it would likely be useful to apply this statement.

Modeling Mathematical Behavior: We defined an instructor as modeling mathematical behavior if he or she:

1. Made an aesthetic appraisal of a mathematical contribution that went beyond describing a statement as true or false and arguments as valid or invalid. This includes describing definitions or notational choices as "good" or proofs as "parsimonious" or "elegant".
2. Asked a question or began an investigation with a suggestion or an explicit indication that the question or investigation was a natural question to ask in a mathematical context.
3. Described the motivation for things other than solving a mathematical problem or completing a mathematical task, including choosing notation, the names of concepts, which examples to study,
4. Described mathematical habits or dispositions that are desirable and what habits and dispositions are not productive

We illustrate this with an example from M9's abstract algebra class in which he was discussing elements of the ring of polynomials:

> M9: Now, where do you expect to find the something, where do the representatives come from? They come from F square bracket x, and so you say to yourself, ah, this is in F square bracket x, but, it appears only to be a field element. That's the place where you want to exercise your mathematical judgment and, indeed, call upon notational conventions, and so you say, "ah" that field element, how can I interpret that as a polynomial? It is the constant polynomial that has that value.

This transcript satisfies criterion 4. M9 described a habit of "exercise[ing] your mathematical judgment" in interpreting notation in helpful and sensible ways.

Coding Whether Content Was Written

In deciding whether a instructor recorded a piece of content on the blackboard or if a student recorded this content in their notes, we used a generous coding scheme. If the instructor or student wrote any aspect of this content or anything that referred to this content, this was coded as written. There is an important caveat. For content to be coded as written by the instructor or recorded in students' notes, what was written must contain or refer to the reason that we coded the content the way that we did. We illustrate this with M4's description of a theorem about Jordan canonical forms.

> M4: Two matrices, square matrices, with all eigenvalues in F, the field, are similar if and only if they have the same Jordan Canonical form. [W*rites* Corollary: Two square matrices with all eigenvalues in F are similar ⇔ they have same JC form] If you're ever presented with two square matrices and you want to know if they're similar, that's a hard question to answer unless you take this approach. Figure out their Jordan canonical forms and see if they're the same.

As described above, the first sentence was coded as a proposition while the second two sentences were coded as an instance of mathematical method. Because the instructor wrote the corollary on the board, we coded that as a written proposition. We coded the method as being only orally presented because nothing in the written text described either the difficulty of other approaches or when the fact stated in the corollary might be useful. Similarly, if students only wrote the corollary in their notes, but nothing indicating when it would be useful to apply the corollary, the corollary would be judged to be present in their notes, but the method would not.

Results

Table 2 presents the number of instances of each category, the percentage of instances that were written on the blackboard or only presented orally, and the percentage of possible instances that these comments appeared in students' notes (for example, there could have been up to 411 total recorded instances of Oral Method collectively in student notes, only 5 instances were recorded). This was calculated by taking the number of student notes from a lecture and multiplying by the number of instances of that type of content in the lecture and summing across lectures.

		Instances in all lectures	Recorded in students' notes
Method		Total: 65	
	Oral	51 (78% of Method instances)	1% (5 out of 411)
	Written	14 (21%)	82% (89 out of 109)
Informal representation		Total: 157	
	Oral	114 (73%)	2% (20 out of 935)
	Written	43 (27%)	64% (219 out of 343)
Model math behavior		Total: 69	
	Oral	67 (97%)	1% (3 out of 538)
	Written	2 (3%)	100% (10 out of 10)
Examples		Total: 65	
	Oral	10 (15%)	3% (3 out of 89)
	Written	55 (85%)	79% (371 out of 472)
Definitions		Total: 20	
	Oral	1 (5%)	0% (0 out of 9)
	Written	19 (95%)	86% (127 out of 148)
Propositions		Total: 59	
	Oral	2 (3%)	0% (0 out of 20)
	Written	57 (97%)	86% (452 out of 525)
Proofs		Total: 31	
	Oral	1 (3%)	0% (0 out of 5)
	Written	30 (97%)	83% (238 out of 286)

Table 2. Summary of content and recording in notes

These data largely confirm the three hypotheses that we test in the paper. First, there were 356 instances of mathematicians presenting mathematical methods, conceptual content, modeling mathematical behaviors, and examples across the 11 lectures, or over 32 instances per lecture on average, and each instructor presented at least 14 instances of informal types of content.

Second, for method, informal representation, and modeled mathematical behaviors, most of these comments were made orally and not written on the blackboard. For the three non-example categories, each instructor presented at least 75% of them orally. The presentation of examples was an exception. Examples usually were written on the blackboard; we believe that this is because this allowed the mathematics instructors to perform formal calculations and derivations with the examples. Third, when instructors presented their comments orally, these comments rarely were recorded in students' notes. For the three non-example categories, for each lecture, the oral comments were collectively recorded less than 3.2% of the time. However, if instructors wrote their comments on the blackboard, they usually were recorded in students' notes. When formal content (definitions, propositions, and proofs) was not written on the blackboard, the students did not record it. This suggests that what students record in their notes is determined primarily by the mode of presentation rather than the type of content being presented.

Discussion

Lew et al. (2016) reported the case of a lecture that mathematicians perceived to be of high quality, but in which students failed to grasp the main ideas the instructor intended to convey. In that report, we provided theoretical explanations for why students might not understand this kind of lecture. One of these possible explanations was based on the literature on note taking and a number of observations we had made in that particular lecture. On the one hand, studies on note taking have found that students recall lecture content mainly if that content is recorded in their notes. On the other hand, in that case study we noticed that students only recorded in their notes what was written on the blackboard, while the instructor only stated orally the main ideas of the lecture. In the current study we have confirmed that these observations generalize to a larger set of mathematics lectures, and that these student and instructor behaviors are not idiosyncratic to that one observed lecture. Thus, although our explanation remains one of many other possible explanations for why students might not understand mathematics lectures, these new findings offer support for the generality and validity of the observations substantiating that explanation, namely:

(1) When mathematics instructors present formal mathematics in their advanced mathematics lectures, they usually write this formal mathematics on the blackboard.

(2) Instructors present informal content, including examples, informal representations, mathematical methods, and modeling mathematical behavior during their advanced mathematics lectures, at least some of the time.

(3) Instructors usually write examples while stating aloud any informal representations or mathematical methods, and orally modeling mathematical behaviors.

(4) Regardless of type of content, students usually record what is written on the blackboard in their notes, but not what is only stated orally.

This study did not allow us to determine the importance that the instructors placed on the formal and informal mathematics. However, prior studies with mathematicians indicate that they believe it important to help students develop intuitive understandings, proof-writing skills, and modeling mathematical behavior (e.g., Alcock, 2010; Fukawa-Connelly, 2012; Lew et al., 2016; Weber, 2004, 2012). This study suggests that these important ideas are typically presented orally and are not usually recorded in students' notes, which is one reason that they may not be learned or are quickly forgotten. We note that the distinction we have made between formal and informal was in response to the literature. We do not make any claims about the quality of the exposure to the informal content that the instructors offered to the students; we only note the presence of it.

Although we attempted to test the generality of phenomena observed in a case study (Lew et al., 2016), our sample of 11 mathematicians remains modest and the mathematicians who agreed to participate may represent a biased sample. Given the large effects that we observed, we hypothesize that the trends that we observed would still be present with a larger sample, but more research is needed to confirm this. Moreover, there are two unknowns that this study did not address but that are opportunities for future research. We do not know why mathematics instructors usually did not write their informal mathematics on the blackboard or why students did not record their instructors' oral comments. More research on both groups' rationality for these behaviors would be an important topic for future research.

References

Alcock, L. (2010). Mathematicians' perspectives on the teaching and learning of proof. In F. Hitt, D. Holton, & P. Thompson (Eds.), *Research in Collegiate Mathematics Education VII* (pp. 63-92). Providence, RI: American Mathematical Society. https://doi.org/10.1090/cbmath/016/03

Artemeva, N. & Fox, J. (2011). The writing's on the board: the global and the local in teaching undergraduate mathematics through chalk talk. *Written Communication,* 28, 345-379. https://doi: 10.1177/0741088311419630

Bressoud, D. (2011). The worst way to teach. From D. Bressoud's MAA blog: *Launchings*: . Last downloaded June 28, 2016.

Davis, P. J. and Hersh, R. (1981). *The mathematical experience*. New York: Viking Penguin Inc.

Dennis, D. & Confrey, J. (1996). The creation of continuous exponents: A study of the method and epistemology of John Wallis*. Research in Collegiate Mathematics Education* 2, 33-60. https://doi.org/10.1090/cbmath/006/02

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.) *Advanced Mathematical Thinking.* (pp. 25-41). Kluwer: Dordrecht.

Einstein, G. O., Morris, J., & Smith, S. (1985). Notetaking, individual differences, and memory for lecture information. *Journal of Educational Psychology, 77,* 522-532. https://doi.org/10.1037/0022-0663.77.5.522

Fukawa-Connelly, T., Johnson, E., & Keller, R. (2016). Can Math Education Research Improve the Teaching of Abstract Algebra?. *Notices of the American Mathematical Society*, 63, 276-281. https://doi.org/10.1090/noti1339

Kiewra, K. A. (1987). Notetaking and review: The research and its implications. *Instructional Science, 16,* 233-249. https://doi.org/10.1007/BF00120252

Leron, U., & Dubinsky, E. (1995). An abstract algebra story. *American Mathematical Monthly*, *102*, 227-242. https://doi.org/10.2307/2975010

Lew, K., Fukawa-Connelly, T., Mejía-Ramos, J.P., & Weber, K. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education,* 47, 162-198. https://doi.org/10.5951/jresematheduc.47.2.0162

Rasmussen, C. & Wawro, M. (in press). Post-calculus research in undergraduate mathematics education. In J. Cai (Ed). *Compendium for research in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.

Stewart, J. (2012). *Calculus* (7th ed). Brooks Cole: Boston, MA.

Stylianides, G., Stylianides, A., & Weber, K. (in press) Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed). *Compendium for research in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.

Weber, K. (2004). Traditional instruction in advanced mathematics classrooms: A case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior*, *23*, 115-133. https://doi.org/10.1016/j.jmathb.2004.03.001

Weber, K. (2012). Mathematicians' perspectives on their pedagogical practice with respect to proof. *International Journal of Mathematics Education in Science and Technology*, *43*(4), 463-475. https://doi.org/10.1080/0020739X.2011.622803