Factors mathematicians profess to consider when presenting pedagogical proofs

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Abstract. This paper concerns proof presentation at the university level. We report on a study in which we observed ten mathematicians constructing or revising proofs for pedagogical purposes. We highlight the factors that they claimed to consider when completing these tasks. We found that intended audience and medium (lecture or textbook) influenced proof presentation. We also found that although mathematicians generally valued pedagogical proofs featuring diagrams and emphasizing main ideas, these mathematicians did not always incorporate these aspects in the proofs they constructed or revised.

Keywords: mathematicians; proof; proof presentation; tertiary education; undergraduate mathematics education

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1. Introduction: importance of proof presentation

Proving is an essential activity in mathematical practice. There is broad consensus amongst mathematics educators that justifying and proving should play a central role in mathematics classrooms (e.g., NCTM, 2000). However, what proving for instructional purposes looks like and how mathematicians engage in this activity remain important open research questions.

This paper concerns proof in advanced university mathematics courses—that is, upper-level proof-oriented courses—in which the predominant way that mathematics is conveyed to students is lecture and text whose content is primarily proofs (e.g., Dreyfus, 1991). In this context, we are interested in the notion of *pedagogical proof*: proofs that transform mathematical knowledge into ways of "representing ideas so that the unknowing can come to know, those without understanding can comprehend and discern, and the unskilled can become adept" (Shulman 1987, p. 7). We are particularly interested in written proof, a prevalent medium in which students encounter proof. We report on an interview-based study of mathematicians who were asked to construct and revise proofs for pedagogical purposes.

Mathematicians are usually responsible for the teaching of advanced university mathematics courses. However, their training focuses on writing proofs for disciplinary, rather than pedagogical, purposes. Shulman (1987) introduced the idea that the knowledge base of teaching is distinguished from that of the disciplinary expert "in the capacity to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students" (p. 15). The cross purposes of discipline and pedagogy (e.g., Ball, 1993) raise the following issues: What conceptions of quality and production of pedagogical proof are held by mathematicians? What support might mathematicians need in producing high quality pedagogical proof? Answers to these questions may be a first step in understanding the potential contrasts between the nature of pedagogical proofs and the nature of proofs produced strictly for advancing the discipline.

The broad issue addressed in this paper is mathematicians' engagement in the process of presenting pedagogical proofs. The specific issues addressed are the factors that mathematicians profess to consider when engaging in this process. Based on think-aloud interviews with ten mathematicians, we report findings on the mathematicians' descriptions of how audience and instructional medium influenced the construction and revision of pedagogical proofs and how mathematicians did (and did not) incorporate diagrams or emphasize main ideas in their proofs. Understanding these phenomena can improve our understanding of mathematicians' pedagogical practice and identify areas of support that mathematicians may need or value in the area of pedagogical proofs.

2. Related Literature

2. 1. Theoretical perspective

Instructors' proof presentation can be viewed as a piece of instructional explanation (Leinhardt, 2001). As such, the proofs that mathematicians present in their classes may support students' acquisition of content, inquiry in the discipline, and perception of disciplinary norms (e.g., Leinhardt, 2001; Larreamendy-Joerns & Muñoz, 2010). Indeed, Dreyfus (1999) hypothesized and Hemmi (2006) documented that students' perceptions of mathematical proof are largely shaped by the proofs they observe in their mathematics lectures.

Balacheff (2004) and Boero, Douek, Morselli, and Pedemonte (2010) argued that there is no shared understanding of what constitutes a proof within the mathematical community. We follow Douek (2009) in distinguishing between argumentation as a form of persuasion and proof as a form of validation while acknowledging that the distinction between the two is not easy to determine. (Other purposes of proof, for example, systemization and discovery, have been indicated by de Villiers (1990), but we do not consider these in this paper.) Mariotti et al. (1997) noted that when judging whether an argument constitutes a proof, one must consider the reference theory in which the proof is situated. In formal mathematics, this pertains to the axiom system of the theory; for example, a proof in Euclidean geometry might not qualify as proof in a non-Euclidean geometry. With pedagogical proofs, the situation is more complex, as one needs to consider what knowledge is familiar to, or within the conceptual grasp of, the classroom community (Stylianides, 2007). One goal of this paper is to investigate how mathematicians profess to conceptualize and make use of students' backgrounds when presenting proofs.

Balacheff (1982) stressed that proofs in pedagogical contexts serve multiple purposes, with two of the most important being to convey what proofs are and how proofs can be produced. We argue that these two roles can sometimes create tension for mathematicians when they aim to write pedagogical proofs. To frame this tension, we follow Boero et al.'s (2010) factors that influence how arguments are transformed into proofs. *Epistemic factors* involve logically structuring the proof so that statements within the proof are shown to be necessary deductive consequences from agreed upon assertions (e.g., definitions, axioms, previously established theorems); *teleological factors* reflect the need to reveal how the proof was produced, including the informal representations and non-deductive reasoning that was used in earlier phases of proving; and *communicative factors* consist of using language that reflects the norms of the discipline. As Boero et al. (2010) observed, teaching entails being aware of and applying these factors. This "meta-level" is "not a goal for students, it is a teaching means" (p. 193).

We argue that the teleological goals of a pedagogical proof may be in tension with epistemic and communicative goals. For instance, the need to have each claim in a proof be justified logically may leave the reader confused as to how the claim was derived (e.g., Leron, 1983). Thurston (1994/1998) argued that the formal manner of typical proof presentations masks the mental models used to create the proof. Mathematics educators and mathematicians have argued that proofs should fulfil teleological roles such as providing insight into why theorems are true (e.g., Hanna, 1990; Hersh, 1993); illustrating methods that might be useful for proving other theorems (e.g., Hanna & Barbeau, 2008; Rav, 1999; Weber & Mejia-Ramos, 2011); or conceiving the proving process as a potential cognitive unity among seeing that a theorem is true, building an argument, and constructing a proof (e.g., Pedemonte, 2007). However, satisfying epistemic and communicative needs may prevent teleological roles from being fulfilled. The study reported in this paper investigated how mathematicians profess to manage this tension and how the teleological, epistemic, and communicative roles of their pedagogical proofs depended on instructional medium.

2. 2. Mathematicians’ pedagogical practice and values with respect to proof presentation

Most studies of mathematicians' pedagogical practice for proof in advanced mathematics classes have been based on interviews with mathematicians about their teaching experiences (e.g., Alcock, 2010; Hemmi, 2010; Iannone & Nardi, 2005; Nardi, 2008; Weber, 2012; Yopp, 2011).

Hemmi (2010) described three pedagogical stances that mathematicians adopted with respect to proof: a "progressive" view, in which perceived limitations of students led to avoiding proofs; a "deductive" style, emphasizing logic and rigor for the purpose of learning "real mathematics"; and a "classical" style, in which proofs are presented for beauty but not necessarily for learning (e.g., how to write a proof).

Based on interviews with mathematicians, Iannone and Nardi (2005) discussed the tactics mathematicians used to construct pedagogical proofs, such as highlighting parts of a definition and the implications for using the definition in a proof. These mathematicians also discussed purposes for selecting pedagogical proofs, such as using a proof to teach students how to use examples to develop intuition or translate symbolic statements into verbal statements. Lai, Weber, and Mejia-Ramos (2012) found that mathematicians valued pedagogical proofs that made assumptions and conclusions of the proof explicit, centered important equations to emphasize the main ideas, and did not contain true but irrelevant statements.

Yopp (2011) and Weber (2012) reported substantial variation in the reasons given by professors for presenting proofs to their students. Yopp (2011) found that factors contributing to variation included the type of course (e.g., first year undergraduate, mathematics concentrator, or graduate courses) and the intended audience (applied mathematics or theoretical mathematics concentrators). Weber (2012) and Harel and Sowder (2009) found that mathematicians reported having a limited pedagogical arsenal with which to achieve their pedagogical goals with respect to proof.

Nardi (2008) discussed pedagogical issues that mathematicians raised during focus group interviews in which they analysed selected student proofs from an advanced university mathematics class. The students' proofs at times perplexed these mathematicians, who found the notation or choice of detail seemingly unwarranted and random. These mathematicians suggested that students struggle with the need to "appear" mathematical (adhering to formal mathematical notation) while not fully understanding the need to "be" mathematical (enacting the norms of mathematical reasoning). The mathematicians believed that interaction could help students overcome this tension, such as public negotiation of proofs in tutorials. They also recommended applying expositional standards to instruction more conscientiously.

Alcock (2010) interviewed five mathematicians teaching an introductory proof course. Although these mathematicians valued both syntactic and semantic reasoning (i.e., an emphasis on symbolic logic and calculation versus connecting proofs to other representations such as diagrams and examples; see Weber and Alcock (2004)), these mathematicians lamented that, in practice, their lectures primarily emphasized syntactic reasoning.

Studies of how professors *actually* present proofs to students have been sparse. Jaworski (2002) examined the teaching practice of mathematicians in proof-based mathematics tutorials to conceptualize how teaching knowledge develops. Weber (2004) presented a case study of one mathematics professor who used three proof presentation styles while teaching a real analysis course; these styles were chosen by the professor based upon his beliefs about what his students needed to know and were currently capable of learning. Notably two of these styles—one emphasizing logical structure, and the other emphasizing technique—made little use of informal representations of mathematical concepts, even though the professor averred that this type of understanding was vital to doing mathematics. Nardi, Jaworski, and Hegedus (2005) studied pedagogical awareness, and exemplified its developmental stages with respect to understanding student difficulties, identifying learning goals, identifying instructional strategies, and conducting self-reflection. These findings suggest mathematicians' abilities to produce pedagogical proofs depend on their sensitivity to and accurate understanding of students' difficulties and thinking.

2. 3. Conceptions of proof presentation in mathematics education

In general, mathematics educators have expressed dissatisfaction with how proofs are presented to students, arguing that students learn little from the proofs that they read (e.g., Leron & Dubinsky, 1995; Rowland, 2001) and that the linear formal nature of proof intimidates students (e.g., Hersh, 1993; Leron, 1983; Rowland, 2001) and masks processes used to create proofs (e.g., Leron, 1983, Hemmi, 2008). Consequently, mathematics educators have proposed alternatives such as visual proofs or generic proofs to present mathematics to students (e.g., Hersh, 1993; Leron, 1983; Rowland, 2001). Further, mathematics educators have called for proofs that use diagrams (e.g., Hersh, 1993) or examples (Rowland, 2001) to enhance understanding, as well as proofs that make processes and ideas more transparent (Hemmi, 2008).

These suggestions imply that some mathematics educators believe the proofs that students often observe are not pedagogical proofs in that they do not optimally enhance students' understanding. The extents to which these suggestions would improve student understanding are important open research questions.

3. Methods

3. 1. Rationale for the current study

Research on mathematicians' pedagogical practice with respect to proof largely consists of interview studies that explore mathematicians' beliefs and values at a general level[3], and do not focused specifically on how mathematicians used these values to influence their proof presentation. Research on mathematicians' actual pedagogical practice has consisted of case studies (e.g., Weber, 2004). Consequently it is difficult to notice commonalities in mathematicians' values for proof presentation.

The current study aims to complement the existing literature by exploring the factors mathematicians claim to consider when preparing a pedagogical proof. We observed ten mathematicians constructing one and revising two other pedagogical proofs; the mathematicians were subsequently interviewed about these activities. As Nardi, Jaworski, and Hegedus (2005) and Alcock (2010) argued, it is important to understand the pedagogical values of mathematicians if we are to support mathematicians' pedagogical awareness and pedagogical practice. The tasks in this study required mathematicians to coordinate diagrammatic, symbolic, and verbal representations, an issue that has been flagged as a challenging concern for mathematicians (Nardi, 2008).

Our work complements existing work with interviewing mathematicians in two ways. First, we explored specific processes that mathematicians use to construct or revise proofs.

[3] Nardi (2008) was an exception to this, as mathematicians responded to specific student work and gave their impression of students' conceptions.

Second, giving specific tasks may elicit conceptions of teaching not evinced from general interview questions. This work also complements the small body of work on mathematical practice by allowing us to identify commonalities in mathematicians' perspectives specific to proof presentation.

As with any methodology, the approach used has several limitations. First, we did not explore actual teaching, but rather asked participants to complete tasks related to teaching. Hence while these tasks may have elicited mathematicians' professed values about pedagogical proofs, they did not necessarily tell us how these professors actually teach. Second, participants were asked to "think aloud" in our protocol, which may have led them to be unusually reflective about their behaviour or invent explanations to account for tacit behavior (see Inglis and Alcock (2012) for more detail about this threat to validity). As a result, we view our findings as providing grounded hypotheses about what mathematicians consider as they construct and revise pedagogical proofs. Our analysis may provide insights into what mathematicians profess to value, but there is the possibility that these values would not be evinced if not for our requests for participants to think aloud (the previously cited interview studies have the same limitation). Third, this study involved only a few tasks and hence may elicit only a subset of the considerations that these participants made when constructing or revising pedagogical proofs. For instance, Rowland (2001) has argued that mathematicians should accompany formal demonstrations with example-based arguments. No participant in the study did this but this may have been due to the fact that such a strategy was less applicable to the tasks that we chose. Finally, to avoid misinterpretation, we emphasize the goal of this study is to examine the factors that mathematicians *do* consider when constructing or revising a pedagogical proof, not the factors they *should* consider. The results of this study should not necessarily be viewed as specifying how pedagogical proofs ought to be constructed.

3. 2. Participants

Ten mathematicians agreed to participate in this study. These mathematicians came from a variety of mathematical subfields, including algebra, analysis, geometric topology, combinatorics, and mathematical physics. We do not have *a priori* reason to believe these participants were more interested or more capable at mathematics teaching than other mathematicians. All mathematicians currently work as members of faculty at a research institution in one of the top 25 mathematics graduate programs in the United States[4]. These participants' teaching experience ranged from 1 to 30 years; we did not notice differences in their behavior as a function of teaching experience, a finding we also observed in the large-scale study reported in Lai, Weber, and Mejia-Ramos (2012). We refer to all participants with pseudonym initials and male pronouns to protect anonymity.

3. 3. Materials and procedures

Participants met individually with the first author for their interview. Each participant was given the Sine Task and the Proof Revision Tasks.

3. 3. 1. Sine Task

Participants were asked to follow a think-aloud procedure (Ericsson & Simon, 1993) while working on the task:

> What is the clearest way to write down the proof of the following problem?
>
> *Show that restrictions of the sine function to intervals of length greater than π cannot be injective.*
>
> Please think about this problem, and then write down a proof that is geared for a sophomore- or junior-level math major, and so that your solution is self-contained.

This task was chosen because its proof construction process involved the coordination of diagrammatic reasoning and algebraic manipulation. We were interested in whether and

[4] According to the U.S. News and World Report 2010 rankings of mathematics graduate schools.

how the diagrammatic and informal reasoning that the participants used was made visible in the pedagogical proofs they produced.

After constructing a proof, participants were interviewed about the process they had just undertaken. We triangulated participants' interview statements and written work with the participants' remarks during the revisions.

We developed our interview protocol based on pilot interviews conducted prior to data collection for this study. We used the pilot interviews to determine questions that would elicit information about the structure, content, and intentions of the interviewees regarding proof construction and revision. Questions in the final protocol included:

- How did your revisions improve the proof?
- Could you describe, in a sentence or two, the main ideas behind your proof?
- Did you try to emphasize the main idea of the proof in any way?
- Is there anything you would like to say about the notation or organization of the proof?

The first question was asked to elicit a general response from the interviewee, and subsequent questions served to follow up on issues concerning structure, content, and intentions.

3. 3. 2. Proof Revision Tasks

Participants were asked to follow a think-aloud procedure while working on the task:

> Consider the following exercise.
>
> > *Prove that a differentiable function from* **R** *to* **R***, with strictly positive first derivative, is injective. Use the Mean Value Theorem in your solution.*
>
> You will be given two solutions of this question to revise. Think about this problem. When you are ready to look at the sample solutions, please let me know.

This portion of the interview, which included the revision of two proofs (A and B, in Figures 1 and 2), was designed to probe the relationship between diagrams, equations, and main ideas. The participants were asked by the interviewer to improve for clarity (as opposed to correctness) for mathematics majors.

*** **Insert Figure 1 and Figure 2 about here** ***

Revision Task A (Figure 1) involved an algebraic argument using values in the domain and range of a function *f*. We included Task A to observe whether and how mathematicians would augment algebraic arguments to make the proofs more comprehensible for their students. In contrast, Revision Task B (Figure 2) featured a geometric argument using the slope of a hypothetical line passing through the graph of *f*. We included Task B to observe whether and how mathematicians would include diagram-based reasoning in their proofs.

After completing both revisions, participants were asked questions similar to those following the Sine Task. Additionally, we examined mathematicians' treatment of algebraic communication:

- Would the exposition of the proof without the slope formula be improved with the inclusion of the slope formula?

3. 4. Analysis

In an earlier paper, we categorized the revisions that mathematicians made in the two proof revision tasks and used these to infer what features of pedagogical proofs that they appeared to value (Lai, Weber, & Mejia-Ramos, 2012). In particular, we examined the way that mathematicians altered the content and structure of the arguments that they read. We then tested the generality of our findings in a separate large scale study with 110 mathematicians. The analysis in this paper uses the original ten mathematicians' interviews to explore the intentions and thinking behind their revisions.

To identify broader issues in the revision process, we adopted the method described in

Boaler, Ball, and Even (2003) of "moving from the particular to the general". We first noted what we considered as key incidents in the participants' work on the tasks, and then we examined corresponding interview reflections for themes in mathematicians' intentions for their revisions, especially regarding communication, main ideas, and diagrammatic and symbolic representation. We hypothesized about what was generic about that incident (i.e., not particular to the specific task or situation). This initial analysis produced four themes:

(a) Participants stressed the crucial role of audience in presenting a proof for pedagogical purposes.

(b) Participants indicated they would design proofs for pedagogical purposes differently depending on whether they were presenting the proofs in a lecture or writing a proof for a textbook.

(c) Participants emphasized the importance of including pictures in proofs for pedagogical purposes; however they seldom included pictures in the proofs they constructed or revised.

(d) Similarly, some participants claimed to value proofs for pedagogical purposes that highlighted the main idea of the proof, but did not always make these ideas explicit in the proofs they produced.

Once these four themes were identified, we went through the interview data again, flagging for each instance when a participant discussed one of these themes. We used an open coding scheme in the style of Strauss and Corbin (1990) to categorize how participants' comments addressed the themes.

4. Results

In this section, we discuss the ways in which task observations and interviews addressed the themes produced from the initial review of the data. We present tables summarizing the types of comments that participants provided and give representative illustrations of each of these themes. For each illustration, we situate participants' responses by indicating whether comments were given during the task or the open-ended interview phase. If their comments were given during the open-ended interview phase, we indicate whether comments on the issue were given in direct response from a question by the interviewer or were unprompted and offered spontaneously by the participant.

4. 1. The role of audience in proofs for pedagogical purposes

Pedagogical proof depends upon its audience, a point stressed by eight participants without prior prompting by the interviewer. As we noted earlier, the validity of an argument depends upon the reference theory used (Mariotti et al., 1997). In this section, we elucidate how the mathematicians conceptualized this reference theory for pedagogical proofs. We found four ways that audience influenced the mathematicians' work, listed in Table 1.

***** Insert Table 1 About Here *****

Background knowledge, potential difficulties, and familiarity. From an epistemic perspective, a reference theory, comprised of principles and deduction rules for proof, determines when a fact can be stated or an inference made without justification. Not surprisingly, eight participants mentioned that the audience determined what could be assumed and what facts would need to be justified. As PV asserted[5], in response to the interviewers' request to discuss the role of audience for lecture and text:

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[5] Transcripts were lightly edited to improve their readability. We omitted short phrases that did not convey meaning, such as mumbling or repeated phrases. At no point did we add text and we do not believe we altered the meaning of what the participants said.

> But in general, […] you of course don't justify what you think they could justify themselves, and maybe just believe without justification [laughs]. So you justify things that you think your reader won't know and needs justified, and you otherwise just state the ideas.

The importance of audience in deciding the validity of a proof has been noted by Ball and Bass (2000), who conceptualized (mathematically valid) proof for a class as based on facts and procedures accessible to that community; similarly Weber (2008) illustrated how mathematicians' judgments about the validity of a classroom proof may depend on what theorems were previously covered in that course.

However, participants' concern about the background of the audience of the proof went beyond validity. Seven participants identified parts of the proof that, although logically permissible and known to the students, might still confuse the intended audience. When this occurred, they provided additional information to facilitate comprehension. As an example, KT noted—during his think-aloud work on the Sine Task—that the word injective would "bother" students, so he explicitly specified its meaning, between the statement and the proof.

As for the type of proof to present, IR mentioned, within moments of revising Task B, that if possible he would avoid techniques problematic for students; for example, he preferred the (direct) proof in Revision Task B to the contradiction proof in Task A. Five participants mentioned the importance of using ideas familiar to students, especially ideas used in class recently. For example, TR raised unprompted that he "relates things as much as possible to things that are fresh in their minds." He noted that though redundant, he might reiterate previously seen ideas to ensure comprehension. Our interpretation of TR's comments is that whether a fact is in the classroom reference theory may depend on how "fresh" it is. The instructional context mattered to KT as well, who explained that had he been actually teaching this course, "there's some concepts that I know have been covered and emphasized, or de-emphasized, so I'll have a better idea of whether I'm pulling something out of left field for them or not".

These findings illustrate the complexity of considering a reference theory in a pedagogical setting. From a disciplinary perspective, a reference theory includes facts and axioms. From a pedagogical perspective, the goal of a proof is not merely to ensure validity, but also to achieve the epistemic aims of having students be convinced that the theorem being proven is true. Consequently what constitutes the reference theory could depend on shared knowledge of the students and the course context, which may depend on students' conceptual grasp and memory.

Emphasis. Perhaps the most interesting way that audience influenced participants' constructions or revisions concerned what they would emphasize while presenting a proof. In short, the audience influenced the teleological aims of a proof presentation. This was most sharply illustrated by IR's response to the Sine Task. He noted, during the interview, that if the proof were for teaching trigonometry, he would rely on symmetries of the sine and cosine functions; in contrast, if the proof were for more advanced students, he would highlight the use of the fact that every interval of length greater than π contains a turning point (a point at which the derivative changes sign). He even suggested that if the proof were for an honors linear algebra course, he would *avoid* mentioning symmetry.

Dilemmas arising from audience diversity. Finally, we note that five participants indicated the difficulty of preparing proofs for audiences of varying backgrounds, abilities, and ways of thinking. In the context of choosing the level of justification to provide, KM observed in interview, "People's brains absorb details at different rates, so if you tried to give details you'd be boring some people and maybe confusing others". KZ remarked during his interview that different students prefer thinking about mathematics differently, so the same proof might not be best for all students. About his Sine Task proof, he said, "I think the picture would be most helpful because that would be closer to how I think about it. But this depends on whether the student likes to think in terms of pictures or formulas or whatever way".

4. 2. The role of medium in proofs for pedagogical purposes

Without prompting from the interviewer, nine participants discussed differences between pedagogical proofs in textbooks, oral presentation via lecture, and one-on-one interaction with a student. A summary of their viewpoints is presented in Table 2. The distinctions raised by these participants can be interpreted in terms of the tension between epistemic (how knowledge is justified) and teleological (how the proof was constructed) factors, with the former being emphasized in textbook proofs and the latter being highlighted in lecture.

***** Enter Table 2 About Here *****

Rigor, completeness, details, and the purpose of lectures. Many participants used the interactive nature of lectures when they justified differences in written and oral proofs.

Five participants argued that it was desirable to leave some logical details out of proofs presented in lectures, as the lecturer could "read" his audience. For example, when TR was asked whether his proof presentation would have differed if it were for lecture, he stated:

> I think a lecture is so much more flexible because you can talk and watch the student's faces and see if they're getting it or they're not getting it. Whereas a textbook, you put it out there and then you have absolutely no control over how anyone responds or understands it, so yeah, you want to be super careful what you're writing in a textbook.

Furthermore, as cited by two participants, due to limitations of time, student interest, and students' abilities to comprehend arguments quickly, it is not always possible to cover every detail – as illustrated by the following unprompted distinction between lecture and textbook by KM:

> KM: [In a lecture] the point of the instructor is that the instructor can give a big picture view of things that is harder to get from a textbook, than details.
> I: So there might be a difference between the sort of arguments that you might present as an instructor versus as an author of a textbook?
> KM: I think so. I think, well, if you're writing a textbook, one of your goals is to have someone be able to read your book and understand it in complete detail, what you're talking about. Whereas for many things, this is impossible in a classroom setting because you don't have time to go into all of the details.

Three participants claimed that the purpose of lectures is to give a high-level outline, or the big idea, of a proof – rather than present complete, rigorous proofs.

As illustrated by TR's excerpt, five participants felt that textbooks, in contrast to lecture, should be formal and contain all relevant details as textbooks are not interactive as lectures are.

Diagrams. In lieu of complete logical justifications, five participants suggested using informal arguments, with diagrams, in lectures. For instance, BT described how he would establish that every interval of length greater than π must contain some $k\pi/2$, where k is an odd integer. As opposed to his written proof:

> What I would do in class, actually, but this is again, not a rigorous proof, is that I would write an interval of length slightly greater than π, and I would imagine sliding it around, and seeing that as you can't leave one multiple of $\pi/2$ without having another one come in because it's π.

The dynamic nature of a lecture, including gesturing, could aid diagram interpretation in ways that linear textual presentation cannot.

4. 3. The inclusion of diagrams in proofs for pedagogical purposes

As discussed in Section 3, Revision Task B presents a geometric argument using a line. We used Task B to examine whether participants would include a diagram representing this line and their reasons for doing so.

Inclusion of diagrams. When completing the task, only one participant, IR, added a diagram. One other, TR, added a diagram when discussing revisions with the interviewer. The remaining eight participants were asked if including a diagram would improve the pedagogical quality of the proof. Seven of these eight participants agreed that it would. Some participants mentioned that if the proof implicitly referenced a diagram, then this diagram should be displayed to the reader. For instance, KT noted during his interview:

> KT: In the second proof, it's requiring a leap of imagination. For this proof to work for the student reading it, they have to have a picture of a line connecting two points with positive slope. They have to be able to read this proof and have that picture in their head.
> I: Do you think it would be improved with a picture?
> KT: Yes. Oh, yes.

Only one participant, KZ, claimed that including a diagram would not improve the pedagogical quality of Revision Task B; he feared that students would find the proof unnecessary upon seeing the diagram.

Role of diagrams in the Sine Task. Participants' use of diagrams was similar for the Sine Task. All but one participant immediately began the Sine Task by sketching a graph of the sine function. For many participants, this diagram played a pivotal role in their proof generation. However, only one participant (TR) included a diagram in the proof they were asked to prepare for students. We did not explicitly ask participants whether a diagram should have been included in their Sine Task proofs, but two participants, when reflecting on the proofs that they produced, lamented that they did not include a diagram. For instance, when looking for weaknesses in his proof, KT said, unprompted, "Well for one thing, there's no pictures". As a consequence, the final pedagogical proofs that the participants constructed masked the potential cognitive unity of the proof construction process. The diagrams that convinced participants that the claim was true and helped generate the proofs were not included in the final product.

Summary of diagram use. Diagrams played an important role for some participants' proof generations. However participants generally did not include diagrams in the proofs they constructed and revised for pedagogical purposes. As can be seen from the above episodes, epistemic, communicative, and teleological factors can be in tension. The use of a diagram can be convincing, but may not use theory in a strictly deductive way. In lecture, teleological obligations may be privileged over epistemic and communicative obligations because of potential interactivity. However, textbooks, as static references, may need to adhere more to epistemic and communicative obligations. The communicative needs for lectures and textbooks, and consequently their use of diagrams, may differ.

4. 4. Bringing out the main idea in proofs for pedagogical purposes

After completing the Sine Task, participants were asked about the main idea and the way it was communicated. Six of the mathematicians used extrema to characterize their main idea; PV used "points of symmetry" of the sine function; TL used "the Intermediate Value Theorem in the right way"; and KT and KZ described the main idea as finding repeated values.

Three participants, when asked, felt they had brought out the main idea. Four participants believed their proofs did not bring out the main idea, but should have. For instance, KT commented, "No, and you're right. For pedagogical purposes, that's probably the most important thing to do". KZ noted that conciseness led him to ignore the main idea, although this was not desirable: "One of the things I usually try to do is not write too much, but in this case, it might help with the intuition".

In summary, most participants valued proofs that emphasized the main idea, but some participants did not do this in their proof construction for the Sine Task. The participants who brought out the main idea did so using the introductory sentence or calculations. We interpret these results as indicating that the mathematicians valued the teleological goals of proofs, including using diagrams and emphasizing the main ideas of the proof so students could understand how the proofs were constructed. However, they did not include all these features in the pedagogical proofs that they constructed or revised.

5. Discussion

We discuss the implications of our findings with respect to existing and potential research on pedagogical proof.

5. 1. Audience and the quality of proofs for pedagogical purposes

One result from this study included the role of audience in determining which deductions to and methods to use. The study corroborated two findings discussed in the literature. First, as discussed in Section 4.1, the role of the audience's background knowledge in determining the validity or accessibility of a proof has previously been proposed (e.g., Ball & Bass, 2000; Weber, 2008). Second, the findings that the participants attended to familiarity of techniques and challenges of ideas to students can broadly be viewed as displaying sensitivity to students' reasoning, something important to pedagogical development (e.g., Jaworski, 2002; Nardi, Jaworski, & Hegedus, 2005). As the mathematicians in Iannone and Nardi (2005) noted, using simpler, more familiar ideas as a platform for introducing more complex ideas may have cognitive and affective value.

These findings illustrate the complexity of evaluating the quality of a proof for pedagogical purposes as the reference theory might be ambiguous. It may not be enough to specify the course and audience for a proof (e.g., is this a good pedagogical proof for junior mathematics majors in an introductory abstract algebra course?); one might also need to know what instruction preceded that proof and what specific ideas the students found comfortable or difficult. Consequently, being an effective instructor may require not only understanding the difficulties and ways of reasoning of students—something that can perhaps to some extent be conveyed to mathematicians (Alcock & Simpson, 2009)—but also a sensitivity to students' recent mathematical experiences.

One new finding from this study is that the emphasis given to certain aspects of the proof depends upon the learning goals for the population being taught. As Yopp (2011) and Weber (2012) illustrated, mathematics professors do not generally present proofs to their students with the sole aim of convincing them that some theorem is true. Rather they hope students gain some other insights, such as explanation or access to proving methods. Some participants' comments highlight the fact that these insights depend upon who is being taught, even when presenting the same proof. As Nardi, Jaworski, and Hegedus (2005) argued, coordinating instructional strategies with goals is a signal of increased sophistication in pedagogical awareness.

5. 2. Medium and the quality of proofs for pedagogical purposes

The results from this study illustrate that some mathematicians conceptualize pedagogical proofs in textbooks and lectures to serve different purposes, so different standards should be used to evaluate their quality. In particular, as opposed to textbooks, for oral proofs given in lecture, they feel it is permitted or even beneficial to omit details and draw inferences from diagrams. In other words, the epistemic requirements of written proofs are relaxed for proofs given in lectures. The distinction between proof as text and proof as (oral) argument was suggested briefly in Nardi and Iannone (2006), when they noted that textual proofs that students read are “almost dead” and “part of the script” (p. 1809). The current study illustrates specific differences between text and oral argument and highlights the fact that these differences are largely due to different mediums' affordances. This suggests that some mathematics professors' proofs in lecture are less formal than the proofs that they expect students to write.

If this hypothesis is correct, this can highlight a potential source of students' misconceptions about proof. Iannone and Nardi (2005) argued that learning to cope with proof can profitably be viewed as learning a new genre of speech or writing that students learn from interacting with mathematicians. As Balacheff (1982) noted, proofs presented in lectures may be particularly important in this regard, as these proofs reveal to student what proofs are and how they are written (see also Dreyfus, 1991 and Hemmi, 2006). If professors present proofs that they would find unacceptable for their students to hand in, particularly presentations that make use of informal dynamic references to diagrams that would not easily translate to precise arguments written in mathematical notation, the incongruity is likely to mislead and frustrate some students. Discerning amongst norms for different genres while learning content is complex, especially given students' struggles with adopting the norms of mathematical communication and reasoning (Nardi, 2008). We suggest more research is needed on the differences between the genres of proof in mathematics lectures (where teleological aspects of proof are valued) and the proofs expected of students in their problem sets (where epistemic and communicative aspects of proof are valued) is needed. As the mathematicians in Nardi's (2008) study suggested, explicit negotiation with students on what is expected from them on their homework assignments (and why this is expected) and why these standards are sometimes relaxed in lectures for pedagogical purposes may ameliorate some students' difficulties.

5. 3. Diagrams, main ideas, and proofs for pedagogical purposes

The data illustrate how, in pedagogical proofs, participants largely professed to value the teleological features of diagrams and highlighting main ideas; however, they did not always include these aspects in their proof constructions and revisions. This finding compares with results of other empirical studies, as discussed in Section 2, with Alcock (2010) and

Weber (2004) noting that professors sometimes presented highly formal arguments emphasizing logic and deduction – in other words, emphasizing epistemic and communicative factors – even though these professors recognized the importance of students seeing the informal side of mathematical proof. Students might benefit if the cognitive unity between the generation of the proof and the formal product were more transparent when possible (Hemmi, 2008).

One plausible hypothesis to account for this type of data is that mathematicians' instruction is shaped by their beliefs, goals, knowledge, and action plans (Schoenfeld, 1998). Some participants in this study professed the belief that pedagogical proofs incorporate diagrams and highlight main ideas, but this is not sufficient to ensure that the proofs that these professors present will have these properties. These beliefs must also be activated when mathematicians are designing instruction or presenting proofs, and they must be accompanied with corresponding action plans—the knowledge of how proofs can be made to incorporate these features (see Schoenfeld, 1998). As we did not observe the participants' actual teaching, we do not have the data to determine if this hypothesis is correct. However, we do note that previous research has illustrated that mathematicians were surprised to observe that they taught in a manner that contradicted their values (Weber, 2004, 2012).

Much of research on proof presentation has highlighted the importance of explanation, informal reasoning, and diagrams in proofs for pedagogical purposes (e.g., Alibert & Thomas, 1991; Hanna, 1990; Hersh, 1993); such research has aimed to shape mathematics educators and mathematicians' beliefs about what constitutes pedagogical proof. However, studies with mathematicians (e.g., Weber, 2012; Weber & Mejia-Ramos, 2011; Yopp, 2011) revealed that many mathematicians already possess the beliefs that these mathematics educators find desirable. As Nardi, Jaworski, and Hegedus (2005) discussed, supporting mathematicians' pedagogical practice depends on developing pedagogical awareness of aims, strategies, and thinking about these. More examination is needed on ways to enable professors to enhance their pedagogical awareness and incorporate their beliefs into instruction.

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Fig. 1 Proof used in Revision Task A. Algebraic errors in this proof were intentional mistakes that the mathematicians were meant to comment upon.

Please see attached LaTeX2e-generated files for better quality image.

Fig. 2 Proof used in Revision Task B.

Please see attached LaTeX2e-generated files for better quality image.

Table 1 Mathematicians' views of how audience influences the presentation of pedagogical proofs. Note that all participants commented on the role of audience.

Category	Participants
What background knowledge can be assumed and what needs to be justified	BT, CY, FR, KM, KT, PV, TL, TR
What actions need to be taken to avoid potential student difficulties	BT, FR, IR, KM, KT, KZ, TR
Use techniques that students are familiar or comfortable with	FR, IR, KT, TR, KM
Audience affects what mathematical ideas are emphasized by the proof	CY, IR, KM

Table 2 Mathematicians' views of pedagogical proofs, depending on the medium (textbook versus lecture).

Category	Participants
Textbook proofs should be rigorous and complete	CY, IR, KM, PV, TR
It is permissible and desirable for proofs in lectures to omit details	BT, KM, KZ, PV, TR
Proofs in lectures should incorporate pictures	BT, CY, IR, KM, KT
Proofs in lectures should highlight main ideas	IR, KM, KZ