Traditional instruction in advanced mathematics courses:

A case study of one professor's lectures and proofs

in an introductory real analysis course

Keith Weber

Rutgers University

Graduate School of Education

10 Seminary Place

New Brunswick, NJ 08901

USA

E-mail: khweber@rci.rutgers.edu

Much of this research was conducted while I was a faculty member of the department of mathematics and statistics at Murray State University.

Running Head: Traditional instruction in advanced mathematics

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Abstract

It is widely accepted by mathematics educators and mathematicians that most proof-oriented university mathematics courses are taught in a "definition-theorem-proof" format. However, there are relatively few empirical studies on what takes place during this instruction, why this instruction is used, and how it affects students' learning. In this paper, I investigate these issues by examining a case study of one professor using this type of instruction in an introductory real analysis course. I first describe the professor's actions in the classroom and argue that these actions are the result of the professor's beliefs about mathematics, students, and education, as well as his knowledge of the material being covered. I then illustrate how the professor's teaching style influenced the way that his students attempted to learn the material. Finally, I discuss the implications that the reported data have on mathematics education research.

1. Introduction

In the last two decades, there has been a tremendous growth in educational research on advanced mathematics courses, i.e. upper-level proof-oriented mathematics courses at the university level. There have been many studies on individuals' learning and understanding of advanced mathematical concepts (e.g., Davis and Vinner, 1986; Tall, 1991; Pinto and Tall, 1999) and other reports on the nature of the concepts themselves (e.g., Sfard, 1991; Lakoff and Nunez, 2000; Dubinsky and McDonald, 2001). However, there has been comparatively little research on the teaching of advanced mathematical courses. Much of the published research on this topic consists of researchers' suggestions for how the pedagogy in advanced mathematics courses could be improved (e.g., Alibert and Thomas, 1991) and assessments of this novel pedagogy (e.g., Dubinsky et. al., 1994; Cottrill et. al., 1996), while there have been relatively few studies on how advanced mathematical courses are actually taught. The purpose of this paper is to describe the teaching used in one such course.

In this paper, I provide a detailed analysis of one professor's teaching in an introductory real analysis course. This paper has the following specific aims:

1. To describe in detail the teaching styles of this professor using traditional instruction in the advanced mathematical classroom.
2. To offer insight into why the professor taught using the methods that he did.
3. To describe students' resulting learning after receiving this instruction
4. To discuss what consequences the results of this paper might have on the directions of future mathematics education research.

1. 1. Factors influencing the actions of teachers

There is a large body of literature on the teaching of mathematics at the pre-college level. This research indicates that mathematics teachers' practices are dependent upon a multitude of factors, including their understanding of the mathematics that they are covering, their pedagogical content knowledge, their short-term and overarching goals for the course, their beliefs about mathematics, and their beliefs about how students learn (e.g., Schoenfeld, 1999). In short, the actions that a teacher will invoke in the classroom are dependent upon a complex constellation of knowledge, skills, goals, and beliefs. Hence, understanding a teacher's actions in the classroom requires understanding their goals, beliefs, and so on.

1. 2. Research on teaching of advanced mathematics courses

It is widely accepted that advanced mathematics courses are frequently taught in what is colloquially referred to as a "definition-theorem-proof" (DTP) format. Davis and Hersh (1981) assert that "a typical lecture in advanced mathematics … consists entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation" (p. 151). Dreyfus (1991) claims that although mathematics instructors may be aware that new mathematics is created through non-rigorous processes, this "does not usually prevent him or her from almost exclusively teaching the one very convenient and important aspect which has been described above, namely the polished formalism, which so often follows the sequence theorem-proof-application" (p. 27).

While many authors write that mathematics courses are taught in the way that I described above, I am not aware of a precise set of criteria that one can use to define DTP instruction. However, it appears that this traditional paradigm has the following characteristics: The instruction largely consists of the professor lecturing and the students passively taking notes, the material is presented in a strictly logical sequence, the logical nature (e.g., formal definitions, rigorous proofs) of the covered material is given precedent over its intuitive nature, and the main goal of the course is for the students to capable of producing rigorous proofs about the covered mathematical concepts.

Despite being the dominant mode of teaching advanced mathematics for many decades, the DTP format has been widely maligned by mathematicians and mathematics educators alike (e.g., Kline, 1977; Davis and Hersh, 1981; Leron, 1985; Alibert and Thomas, 1991; Thurston, 1994; Leron and Dubinsky 1995). Both groups argue that DTP instruction intimidates students

(Kline, 1977; Thurston, 1994), gives students a misleading view about the nature of mathematics (Dennis and Confrey, 1996), hides much of the processes that are used in mathematical reasoning (Davis and Hersh, 1981; Dreyfus, 1991), and denies students the opportunity to use their intuition when reasoning about these concepts (Fischbein, 1982; Dreyfus, 1991). Constructivists argue that one reason traditional instruction is ineffective is that simply "telling students about mathematical processes, objects, and relations in not sufficient to induce meaningful learning" (Leron and Dubinsky, 1995). Arguably the most serious criticism against DTP instruction is that students learn far less that we would like when we teach in this way (Leron and Dubinsky, 1995). Perhaps in response to this dissatisfaction with traditional teaching methods, many researchers have offered alternative paradigms for teaching advanced mathematical courses (e.g., Alibert and Thomas, 1991).

In this paper, I analyze one teacher's DTP-style instruction in an introductory real analysis course. The purpose of this paper is neither to praise nor criticize the teacher's pedagogy; likewise, I do not wish to advocate or critique the use of the DTP approach in proof-oriented courses. Rather the purpose of this paper is to present an analysis of this instruction itself. One goal of this article is to offer a more precise description of what occurs in the advanced mathematical classroom. A second goal is to discuss why the professor chose to teach the class in the way that he did.

2. Research context and methods

2. 1. Research context

The research in this paper took place in the context of an introductory real analysis course taught at a regional university in the southern United States. 16 students enrolled in this course; 15 of these students were junior or senior mathematics majors and one student was a graduate mathematics student who had not completed a similar course as an undergraduate. The textbook for the course was Kirkwood's *An Introduction to Real Analysis* (1995) and the course covered chapters 1.1 through 4.1. Topics covered in the course included sets, functions, proof by induction, limits of sequences, the topology of the real line, limits of functions, continuity, and uniform continuity. The course met for 75 minutes twice a week over the course of a 15-week semester.

2. 2. The teacher

The course was taught by Dr. T (a pseudonym). In many respects, Dr. T was highly qualified to teach this type of course. Dr. T had 12 years experience as a faculty member at the university where this study took place, and he had taught this specific course three times before. He had recently been awarded a major university teaching award. His Ph.D was in real analysis and he had published research papers in this field.

2. 3. Data collection

The data from this study was collected from two sources. The first source of data came from my weekly meetings with Dr. T. During these meetings, Dr. T discussed what he hoped to accomplish in the upcoming weeks and he explained why he taught the previous week's lectures in the way that he did. The second source of data came from the classes themselves. I observed and recorded field notes for each of Dr. T's lectures.

2. 4. Data analysis

The data were analyzed using a form of theory construction in the style of Strauss (1987) and Strauss and Corbin (1990). Originally, Dr. T's lectures were to be categorized as 'formal' or 'informal'. However, as the semester progressed, it became apparent that the level of formality in Dr. T's lectures was only of secondary importance. More critical was the way Dr. T described the interaction between formal and informal thought, and the skills that he emphasized when he was presenting proofs. As a result, the categorizations were re-evaluated and revised to 'logico-structural', 'procedural', and 'semantic'. The result of this process is a set of categorizations that are grounded to fit the available data.

3. Results

3. 1. General observations

Dr. T's lectures had many of the characteristics that one would expect from traditional "definition-theorem-proof" instruction. Most of the lectures consisted of Dr. T writing definitions, examples, proofs, and occasionally diagrams on the blackboard and the students studiously copying Dr. T's writing into their notebooks. Students asked questions only infrequently and rarely participated in class discussions. Students' homework assignments and exams asked them to recall definitions, use these definitions to derive relatively straightforward inferences (e.g., give an example of a set that does not include its least upper bound, find the closure of the set {1, 1/2, 1/3, …}), and construct basic proofs.

One feature of Dr. T's instruction that may differ from traditional instruction was the manner in which he presented proofs to the class. Rather than present proofs as a linear set of logical deductions, he continuously strove to illustrate the reasoning behind the proofs so that the students could produce similar proofs themselves (cf., Weber, 2002).

Despite the traditional nature of Dr. T's instruction, his lecture styles were not uniform. In the following sub-section, I will describe three distinct teaching styles that Dr. T used throughout the semester. In the subsequent section, I will discuss why he employed the teaching methods that he did. Before doing so, there are other general characteristics of Dr. T's lectures that are worth noting. Dr. T's lectures were uncommonly well-organized and precise; throughout his lectures, he was never at a loss for words and he rarely made a mistake. He was also very enthusiastic about the course and displayed a genuine concern for the students and their learning. Perhaps as a result of these factors, Dr. T was popular with the students and earned high student evaluations at the end of the course.

3. 2. Teaching styles

3. 2. 1. A logico-structural teaching style- the case of sets and functions

Typical demonstration of a proof using a logico-structural style

In the second week of the course, Dr. T asked the class to consider the following statement:

$$f(A \cup B) \subseteq f(A) \cup f(B)$$

He presented its proof in the following manner:

Dr. T: So we are asked to prove things about f(A), f(B), and f(A∪B). When we are asked to write a proof about a group of objects, it is always helpful to have a clear understanding of precisely what these objects mean. So I'm going write down what these things actually mean. I'm going to write it over here as scratch work.

[On the right side of the board, Dr. T writes "Scratch work".]

Dr. T: From our definitions, f(A) is the set of elements mapped to by an element of A.

[Underneath 'scratch work', Dr. T writes $f(A) = \{f(x) | x \in A\}$]

Dr. T: And likewise, f(B) is the set of elements mapped to by an element of B.

[Writes $f(B) = \{f(x) | x \in B\}$]

Dr. T: And we also have this.

[Writes $f(A \cup B) = \{f(x) | x \in A \cup B\}$]

Dr. T: OK, to prove this statement, we always need to start with y as a member of f(A∪B) and we need to show that this y is a member of f(A) union f(B).

[On let left side of the board, Dr. T writes "Proof". Below this, he writes "Assume $y \in f(A \cup B)$". Near the bottom of this board, he writes "$y \in f(A) \cup f(B)$". The written work on the board thus far is presented in Figure 1].

***** Insert Figure 1 About Here *****

Dr. T: What does it mean to be in there [alluding to $f(A) \cup f(B)$]? To answer that, I have to decipher y is in $f(A) \cup f(B)$. This means that either there is an x in A such that y is equal to f(x) or there is an x in B such that y equals f(x).

[Above the bottom line in the proof, Dr. T writes “Either there is x ∈ A such that y = f(x) or there is an x ∈ B such that y = f(x).]

Dr. T: So this is what we need to show. It is always helpful to know exactly what you need to show. Sometimes, once you figure out what you need to show, you’ve already done most of the work for the proof. Let’s look at our assumption, y is a member of f(A∪B). Let’s decipher this to see what it really means. y is a member of f(A∪B) means there is an x in A∪B so that f(x) = y.

[Below the line “Assume y ∈ f(A∪B)”, Dr. T writes “Let x ∈ A∪B so that y = f(x)”].

Dr. T: So if x is in A∪B, then x is in A or x is in B. So this is what we needed to show. We’ve shown that either there is an x in A such that y = f(x) or there is an x in B such that y = f(x).

[In the middle of the proof, Dr. T writes “Since x ∈ A∪B, x ∈ A or x ∈ B”. The completed proof that Dr. P produced is presented in Figure 2].

***** Insert Figure 2 About Here *****

Dr. T: So let’s look at we’ve shown that if we have y ∈ f(A∪B), then there is an x in A∪B so that f(x) = y. Since x is in A∪B, x is a member of A or a member of B. So either there is an x in A so that f(x) = y or there is an x in B so that f(x) = y. If the x is in A, y is in f(A). If it is in B, y is in f(B). So y is in f(A) ∪ f(B). So we’ve shown [pointing to the top of the proof], if y is a member of f(A∪B), then y is a member of f(A) ∪ f(B) [now pointing to the bottom of the board]. This is exactly what we need to do to show that f(A∪B) is a subset of f(A) ∪ f(B).

[At the very bottom of the board, Dr. T writes “Since if y ∈ f(A∪B), then y ∈ f(A) ∪ f(B), f(A∪B) ⊆ f(A) ∪ f(B).”].

Short-term goals of a logico-structural style

The day before Dr. T presented the proof described above, I met with him to discuss his goals for his upcoming lectures. In the lecture prior to this meeting, Dr. T had his class complete a quiz testing their basic logical skills. (On the quiz were questions like “Negate the following sentence: Every car in the parking lot is red"). Dr. T expressed dissatisfaction at students' performance on this quiz and wondered how they would be able to construct proofs if their logical skills were so weak. He stated, "I first want students to understand the logic of proofs before we prove anything meaningful". Later he added, "I would like the students to all have a common core of experience of proving things when it is fairly easy. I don't want proofs to discourage them. I want them to be comfortable with proofs before they get lost."

Characteristics of a logico-structural style

During the first lectures of the course, Dr. T continuously stressed the importance of carefully using the definitions to understand how to begin and conclude a proof. For instance, one of his in-class comments was "A guiding principle when writing these proofs is to write down what we have and where we are headed. Many of these proofs are really just a matter of following through the definitions until we reach the conclusion." Later, he told the class, "Now it is certainly not the case that you can do every proof in this course just by writing down the definitions and following them through until we reach the conclusion. However, it certainly is the case that doing this can take you a long way on many of the problems. And it is certainly the case that if you do not know where you are starting and where you are going, then you probably will not produce a correct proof".

The proofs that Dr. T presented about set theoretic topics were similar to the one described earlier in this section. Dr. T would start out by writing the definitions of the terms in the statement to be proven on the side of the board as scratch work. Next, he would list his assumptions at the top of the blackboard and stating his desired conclusion at the bottom. He would then draw inferences primarily by unpacking the definitions of the concepts involved, proceeding down from his assumptions and up from his conclusions. This process would continue until he met in middle and had constructed a proof. Afterward, Dr. T would go through his written work linearly, explaining why his proof was logically sound.

There are other characteristics of this instruction that are worth noting. Diagrams rarely accompanied Dr. T's proofs when teaching in a logico-structural style; he did not produce a set theoretic diagram until his third lecture on the topic and in this case, he produced only a single diagram. He did not produce any diagrams at all when using this lecture style to discuss the axioms of the real line, the completeness axiom, or the Archimedean property of real numbers. Finally, the semantic meaning of the concepts and proofs were not discussed. Concepts were not covered beyond stating their definitions, and the summaries that he gave of his completed proofs were solely used to establish the proofs' logical veracity.

3. 2. 2. A procedural teaching style- the case of limits of sequences

Typical demonstration of a proof using a procedural style

In the fifth week of the course, Dr. T asked the class to consider the following statement:

$$\lim_{n\to\infty} (n+1)/n = 1$$

He presented the proof in the following manner:

[Dr. T begins his proof on the left side of the board by writing: "Let $\varepsilon > 0$. Let N =". Below this, Dr. T writes "If $n > N$, then $|(n + 1)/n - 1|$". At this point, he leaves and extended gap and writes "$< \varepsilon$." Below this, he writes "Since $n > N$ implies that $|(n+1)/n - 1| < \varepsilon$, $\lim_{n\to\infty} (n+1)/n = 1$." His work to this point is presented in Figure 3.]

*** **Insert Figure 3 About Here** ***

Dr. T: So this proof has the usual structure. We start with an ε greater than zero. We need to find an N that makes this inequality true. We'll find this N by scratch work. We need to make this inequality less than ε.

[On the left side of the board, Dr. T writes "Scratch work". Under this he writes "$|(n+1)/n - 1|$", leaves an extended gap, and then writes "$< \varepsilon$".]

Dr. T: So what does this simplify to? The left hand part simplifies to 1 minus 1 over n, so we have…

[Next to “$|(n+1)/n - 1|$”, Dr. T writes “$= |1 + 1/n - 1| = |1/n|$”].

Dr. T: The absolute value of $1/n$ is just going to be $1/n$, since we are only dealing with the positive integers, so we have this.

[Dr. T writes “$= 1/n$”.]

Dr. T: Since we are assuming n is greater than N, we know that $1/n$ is less than $1/N$.

[Dr. T writes “$< 1/N$”. His written work until this point is presented in Figure 4.]

***** Insert Figure 4 About Here *****

Dr. T: So we need to show that this is less than ε. We know that $1/N$ will be less than ε when N is greater than $1/\varepsilon$. How can we do that? [posing question to the class]. How do we know that we can always choose an integer N to make N larger than any value that we’d like? [After five second pause] What property do we have that says no matter what value that we have, we can always choose *an integer* larger than it? [A student suggests the Archimedean property] Yes, the Archimedean property. I can always choose an integer N bigger than anything. So I will let N be an integer greater than $1/\varepsilon$.

[Returning to the proof on the left side of the board, Dr. T completes the following line. “Let N be an integer greater than $1/\varepsilon$". Dr. T then completes the missing gap between "$|(n+1)/n - 1|$" and “$< \varepsilon$” using the work on the right side of the board. Dr. T's finished proof is presented in Figure 5].

***** Insert Figure 5 About Here *****

Dr. T: So for any epsilon [underlines "Let $\varepsilon > 0$"], we can find an N [underlines “Let N be an integer greater than $1/\varepsilon$"] so that when little n is bigger than N [underlines “If $n > N$"], this absolute value is less than epsilon. So this shows that the limit of n plus 1 over n is 1.

Short-term goals of a procedural lecture style

During our interview prior to teaching students how to write proofs about limits of sequences, Dr. T explained to me, "One of the biggest things that I've heard in the past is 'how did you ever think to do that?'. I want to be very clear in my work so that students could perform the steps themselves". Later in the interview, Dr. T indicated that he would try to illustrate a set of techniques and heuristics that a student could use to prove theorems about limits. In his words, "I would like for the students to have a *mathematical toolbox*. If they know the limit of something exists, they should immediately think of ways to make the desired quantities small" (italics are my emphasis).

Characteristics of a procedural lecture style

Most proofs that Dr. T introduced were similar to the one that I presented above. Dr. T would start each proof by writing an incomplete argument designed to illustrate the proof's general structure. He often would remark about how one should always start that type of proof in the way that he did. He would say the proof would be complete if he could fill in the gaps missing from his argument and then describe his thinking as he attempted to fill in those gaps, writing his work on the board off to the side as scratch work. During his deliberation, he would stress the techniques and heuristics he was using, with comments such as, "A lot of learning how to write these proofs are learning techniques that let us give ground" and "We can give more ground than we need. We do not need to find the smallest N that will work, we just need to find an N". Finally, after doing the appropriate scratch work, Dr. T would complete the proof.

The logical validity of a proof was only discussed after the proof was completed. The semantic nature of the proof usually was not discussed at all. For instance, in the proof above, Dr. T did not mention that as n became large, (n+1)/n would be a number very close to 1. Diagrams were not employed often during Dr. T's proofs. During the three 75 minute lectures discussing limits, Dr. T only drew four diagrams. Finally, when Dr. T introduced a concept, he would give the definition of the concept and then immediately write proofs using the concept. He made little effort to describe intuitively what the concept meant. For example, when introducing the concept of limit of a sequence, Dr. T gave the definition, proved that limits of particular sequences were unique, and then verified the limits of certain sequences (like the proof above). However, Dr. T did not, for instance, describe sequences as "tending toward their limit" or draw a representative diagram of a sequence and its limit.

3. 2. 3. A semantic teaching style- the case of the topological concept of interior point

A description of interior point

During the tenth week of class, Dr. T gave the following lecture.

[Dr. T writes on the board "Def: Let A ⊆ **R**" and draws an arbitrary connected, convex set to represent A.]

Dr. T: We are going to introduce a new definition. Let me tell you what we're looking for. Look at this set A. Now A is a set that has some of its border, but not all of its border. For an example in the reals, consider the half-open interval [0, 1).

[Dr. T writes on the right side of the board, "[0, 1)" and then draws a diagram of the set using typical notation. His written work thus far is presented in Figure 6.]

*** **Insert Figure 6 About Here** ***

Dr. T: When I talk about the interior of A, I want to talk about the points that are really inside A, not the points that are on the border and fringes of A. If you are inside the set, you can go a little bit in each direction and still be inside the set. One half is inside [0, 1) because if you go a little bit to the left or right of one half, you are still inside [0, 1). 0 is not. If you go a little bit to the left, you leave [0, 1). The precise definition of interior point is, if there exists a δ greater than 0 such that all of $(a - \delta, a + \delta)$ is a subset of A, then a is an interior point of A.

[Next to "Def: Let A ⊆ **R**", Dr. T writes "if $\exists\delta > 0$ such that $(a - \delta, a + \delta) \subseteq A$, then a is an interior point of A"]

Dr. T: So you can still go δ in either direction and still be inside the set. The interior of A is just defined to be all of the interior points of A.

[Underneath the first definition, Dr. T writes "$\text{int}(A) = \{x \in A | x \text{ is an interior point of } A\}$".]

Dr. T: Let's look at an example letting A be [0, 1) along with the single point 3 and the open interval between 4 and 5. So we have this.

[On the right side of the board, Dr. T writes "$A = [0, 1) \cup \{3\} \cup (4, 5)$ and then draws a diagram of this set on the real line. Dr. T's written work to this point is presented in Figure 7].

*** **Insert Figure 7 About Here** ***

Dr. T: The interior of this set here would be the open interval 0, 1 and the open interval 4, 5. Although 0 is in A, 0 here 'doesn't count', because no matter how small I go out, some things are in A, but some things are not. Also, 3 'doesn't count'. No matter how far I go out around 3, nothing will be in A except 3 itself.

[Dr. T draws little intervals around the points 0 and 3 in his diagram of the set A. He goes on to define boundary points and boundaries of sets in a similar ways and proceeds to prove theorems about interiors and boundaries of sets].

Short-term goals of a semantic teaching style

During our interview prior to the lecture described above, Dr. T explained that his goals were for students to have rich imagery that they could associate with the concepts being taught. In his words, "The definitions and the theorems are not so important here. The students can look them up... I want students to have a sense so that something occurs in their mind when they hear 'interior point'. They should be able to draw a picture illustrating 'interior point' just like that".

Characteristics of a semantic teaching style

Topological concepts were introduced similar to the way that Dr. T introduced the concept of interior point. Dr. T would first give an intuitive description of the idea that the concept was trying to capture, usually using a two-dimensional diagram. (Several times, Dr. T explained to the class that although the real line had one dimension, he preferred to use two dimensional diagrams because they were "richer" and were "better to show what was really going on"). He would then give the definition and explain how the definition explicitly related to the diagram. After this, he would apply the definition to several example problems to show that the definition yielded the results that one would intuitively expect.

Prior to presenting a proof, Dr. T would hand out a copy of the completed proof; he would ask the students not to take notes but instead to try to understand his work. When Dr. T presented proofs, he would first draw a picture illustrating the plausibility of the statement to be proven. A rigorous proof based on the diagram would follow later. Some details in the proof were omitted from Dr. T's classroom presentation, but perfectly rigorous and complete proofs were always given to the students in their hand-outs.

3. 3. Discussion of Dr. T's instruction

In recent years, there has been research on the cognitive skills necessary for individuals to construct proofs in advanced mathematics. Although Dr. T was generally unaware of this literature, his lecture styles appear to be designed to teach students some of these cognitive skills. At an elementary level, a student needs syntactic skills, i.e., an ability to unpack and logically manipulate definitions, to construct proofs (cf., Hart, 1994; Selden and Selden, 1995). Dr. T's logico-structural lecture style primarily stressed these proof-writing skills. While syntactic skills are important, it has also been demonstrated that these skills alone are not sufficient to construct proofs in advanced mathematical domains (Weber, 2001, in press). Students also require *strategic knowledge*, or "heuristic guidelines that they can use to recall [mathematical] actions that are likely to be useful or to choose which action to apply among several alternatives". (Weber, 2001). Dr. T's procedural lecture style appears to be centered on explicitly teaching students this strategic knowledge. In an influential paper, Tall and Vinner (1981) distinguish between one's knowledge of a concept's definition and their image of the concept. Tall and Vinner claim that one's concept image, i.e. their total cognitive structure associated with that concept, is crucially important for how one will reason both formally and intuitively about that concept. Building on this notion, Weber and Alcock (in press) have argued that students in advanced mathematics courses need to build useful concept images and to have strong explicit links between concept images and concept definitions if they are to construct proofs effectively. Dr. T's semantic instruction appears to be centered on leading students to do this.

4. Why did Dr. T teach in the way that he did?

During my final interview with Dr. T, I described to him the three lecture styles that I reported in the previous section and then asked him to comment on them. He initially acknowledged that my descriptions were "basically correct" and then became reflective on why he progressed from a logico-structural lecture style to a procedural lecture style and ultimately to a semantic-lecture style.

Dr. T explained that he started the course with a focus on logic since "it is very difficult for students to follow the course if they do not have a basic understanding of proof. In the first few weeks, I want students to have a solid base for what a proof is, and how to prove things by looking at the hypotheses and the conclusions". Dr. T also indicated that sets and functions were topics that were particularly amenable to his logico-structural lectures. "The ideas in these proofs [about sets and functions] are divorced from other intuitive ideas in mathematics... one can go from place to place in these proofs just by following his nose".

In describing his procedural lecture style, Dr. T stressed the importance of explaining his work in a detailed way so that students can replicate it. "When I first taught this course, students would come up to me and say, 'I can follow your proofs but I can't write them.' and 'how did you ever think to do that?' These students would often become so frustrated that they would stop working on the material. Stating step-by-step descriptions and hints gives the students a crutch to write proofs, so they won't be scared off and give up on the course". Further, Dr. T noted that the crutch he provides is often sufficient to obtain proof-writing competence in some aspects of analysis, by saying, "In analysis, one can often prove something by following inequalities by rote". Dr. T also stated an additional benefit to stressing heuristics in proofs about limits. "After doing a number of proofs, proving by following inequalities becomes second nature. You know, these techniques are going to come up again and again in analysis, even if the students go on to graduate school. To be strong in analysis, you need to be able to handle inequalities just like that".

Finally, Dr. T progresses to a semantic lecture style because, at some point, rote strategies become inadequate. "Ultimately students need to see what's really going on in the definitions and the inside of proofs... it's a natural progression of the difficulty of the material in this course. Early in the course, you could get by with certain tricks and skills. At the end of the course, you cannot and you need to really understand what's going on". However, Dr. T believes his previous lectures laid the groundwork for students to acquire this understanding. "In the beginning of the course, students could only understand the topics as a string of words. Now, they can begin to understand what these words really mean".

It is clear that Dr. T's instruction is based on a large number of complimentary beliefs about students, learning, and mathematics. I do not (and cannot) give a comprehensive list of all of Dr. T's relevant beliefs. However, I do list some of Dr. T's most important beliefs below.

- If students find analysis too difficult, they will become frustrated and give up on the course.
- Students need to have an elementary understanding of logic to follow an advanced mathematics course. An understanding of logic and advanced mathematical concepts cannot co-emerge.
- There are basic symbolic skills (e.g., proof techniques, working with inequalities) that students need to master before tackling tougher problems.
- Students cannot intuitively understand advanced mathematical concepts without sufficient experience working with these concepts at a symbolic level.

Dr. T's instructional plans were also influenced by his knowledge of the mathematical topics that were being covered. Here, I am not speaking of his *mathematical* knowledge of these topics, per se. Rather, I am speaking of his knowledge of the skills and understanding that students require in order to produce competent performance at proof-writing. He observes that in writing set-theoretic proofs, one needs only to write the hypotheses and conclusions, and then "follow his nose", since these topics are relatively divorced from intuitive mathematics. Proofs about limits can be done if one masters manipulating inequalities. However, reasoning about more advanced concepts, such as the topology of the real line, require something more.

5. Students' learning outcomes

Research has shown that students learn about advanced mathematical concepts in at least three qualitatively different ways. Pinto and Tall (1999) distinguish between *natural learners* and *formal learners*. Natural learners use their pre-existing intuitive understanding of mathematical concepts to give meaning to the concept's definition and associated formal work; formal learners build their concept intuition by examining the logical entailments of a concept's definition. Weber (2003) describes *procedural learners* as learners who first master flexible procedures to write a class of proofs and only later reflect on these procedures to give meaning to the techniques and concepts that they are studying. When procedural learners first learn to write a class of proofs, they do not understand why these arguments are valid proofs, but rather simply produce these arguments to receive credits on assignments. Research has demonstrated that students may be successful or unsuccessful taking any of these three learning approaches (Pinto and Tall, 1999; Weber, 2003). The issue that I will address in this section is what effect Dr. T's pedagogy had on students' approaches to learning the material.

As part of a larger study, I met with six students in Dr. T's analysis class every other week. There were eight such meetings in the 15-week course. The purpose of these meetings was to examine the stages students progressed through as they learned about the mathematical concepts and proof techniques covered in the course. During these meetings, I asked students to give an intuitive description of the concepts covered in the course as well as to state the concept's formal definition (in whatever order they liked). Next, students were asked to write basic proofs about the concept and then asked to explain why their arguments constituted valid proofs.

I examined students' learning approaches in the initial interviews after they learned about the following topics: a function evaluated over a set (f(A)), a topic taught in a logico-structural style; limits of sequences, which was taught in a procedural style; and topological closure, which was taught in a semantic style. I coded students' learning approach using the following scheme:

Natural learning approach- A student was said to take a natural learning approach to learn about a topic if that student could give an intuitive description of that topic and used this intuitive description to guide his or her formal thought (e.g., used his or her intuitive description to help recall or reproduce a definition or to construct a proof).

Formal learning approach- A student was said to take a formal learning approach if that student had little initial intuition about the concept with which he or she was working and the student's intuition was not used when reasoning about this topic. Students taking a formal learning approach could logically justify why their proofs were valid.

Rote/procedural learning approach- A student was said to take a rote or procedural learning approach to a topic if he or she constructed proofs by applying procedures that the professor taught them. When asked why their arguments constituted proofs, students taking a rote or procedural learning approach were not able to give an answer with respect to the formal theory.

In each case examined below, every student clearly fell into one and only one of these categories. The approach that students took toward learning the concepts of f(A), limits, and topological closure is presented in Table 1. One of the six students took a natural learning approach to learning all three of these topics. However, the other five students used different learning approaches depending upon the topic.

***** Insert Table 1 About Here *****

When learning about functions evaluated over sets, five of the students approached the proofs in the manner that Dr. T presented, by clearly formulating the hypotheses and conclusions of the statement to be proven and unwrapping definitions. All students had great difficulty with proofs that did not simply involve obvious inferences; for instance, at no point in the course could any of the six students prove that $f(A)/f(B) \subseteq f(A/B)$.

When proving statements about limits, five students directly used Dr. T's methods of first writing out a proof framework and then doing scratch work to complete the relevant details. Three students (initially) did not understand why their work was logically valid. When these students were asked why their arguments qualified as a proof, they either simply gave a chronological account of their work (cf., Dreyfus, 1999) or were unable to provide a response at all. They did not offer a logical or intuitive explanation for why their arguments were correct.

When initially reasoning about topological concepts, all students would use their intuitive understanding of these topics, but would not use definitions or proofs to justify their work. For instance, when asked what the closure of the set $(0, 1) \cup (1, 2)$, all students would quickly answer $[0, 2]$, but would not justify why their answer was correct. When the students were asked a question that they could not address via intuitive means (e.g., find the closure of $\mathbf{Q}$ in the reals), they did not know how to proceed. When specifically told to use the definition to address these questions, students were often able to answer them by constructing valid arguments.

6. Discussion

The work presented in this paper focused on a case study of a single professor teaching analysis. Although many of the aspects of the teaching reported in this paper were surely unique to Dr. T, I believe that three general conclusions can be extracted from my data. I discuss these conclusions in the remainder of this paper.

First, although the instructor taught in a traditional definition-theorem-proof (DTP) format, his instruction varied widely depending on the topic that he taught. It appears that DTP instruction is not a single teaching paradigm, but rather a diverse collection of pedagogical techniques sharing some core features. This observation has an important consequence for educational research. As Clarke (2002) observes, "a study of learning in classroom settings would be incomplete without the simultaneous documentation of the social and cultural practices in which the learner participated, the instructional materials, and … the teacher's actions that preceded and followed the learning under investigation". Indeed, as the data in section 5 suggest, the lecture styles of Dr. T appeared to have a direct effect on the way some students attempted to learn the material. The consequence for educational research is that when examining the learning of students in their advanced mathematics courses, it may be insufficient to simply state that they were taught in a traditional format. Lecture styles that fall under the umbrella of traditional DTP instruction may vary widely and lead to drastically different learning on the part of the students.

The second issue concerns perceptions of why mathematics professors use the DTP paradigm for instruction. Although not based on empirical data, some researchers have proposed reasons for why professors use the DTP format that paint the typical mathematics professor in an unflattering light. Kline (1977) argues that because of pressures to publish, mathematicians have neither the time nor the desire to teach advanced mathematics courses well. Thus, in a viscous cycle, these mathematicians resort to teaching their courses in the dry manner in which they were taught. Davis and Hersh (1981) suggest that some professors use DTP instruction because they have the desire to appear brilliant; by presenting mathematical theory as a polished product, students will be impressed by these professor's deductive abilities. They also suggest that other professors are forced to teaching straight from the textbook because they do not really understand the subject that they are teaching. Leron and Dubinsky (1995) believe that professors may teach the way they do because these professors believe their instruction will be ineffective no matter what they do; advanced mathematical concepts are simply too difficult for most students to understand in a semester-long course. In short, some conjecture that professors teach using a DTP format because they are disinterested, arrogant, following custom, or insecure about their understanding of the mathematics involved. While I concede that this might be true in some cases, I think it would be unfair to say that this was the case with Dr. T. Dr. T was both a dedicated and respected teacher whose pedagogy was based on a good deal of thought. I would suspect that many professors use a DTP format for, what are at least to them, very good reasons. It is probably a mistake to dismiss their choice of instruction based on the broad negative characteristics listed above.

Finally, as discussed in section 4, Dr. T's instruction was based on a coherent belief structure that was consistent with his experiences as a teacher and his knowledge of analysis. I believe that this has an important consequence for research in collegiate mathematics education. As mentioned in the introduction, most mathematics educators and many mathematicians agree that DTP instruction is ineffective and alternative forms of pedagogy are sorely needed. Fortunately, in recent years, several such approaches have been suggested (e.g., Alibert and Thomas, 1991; Leron and Dubinsky, 1995; Cottrill et. al., 1996; Alcock and Simpson, 2001). However, merely disseminating this research to mathematicians does not ensure that this will improve the way that they teach. If this research is to have a significant impact on the way collegiate courses are taught, mathematics professors must *choose* to employ these methods and they must employ these methods *effectively*. Instructors will most likely choose to teach in a manner that is consistent with their goals and beliefs. Further, if a teacher tries to use a pedagogical technique that is at variance with his or her goals and beliefs, that instruction most likely will not be effective (cf., Aguirre and Speer, 1996).

Leading professors to improve their teaching of advanced mathematics courses requires leading these professors to adjust their goals for the course and their beliefs about mathematics education. In Dr. T's case, his beliefs were coherent and stable, and hence would likely not be changed easily. It follows from basic constructivist principles that simply telling professors the beliefs that mathematics educators would like them to have would most likely do little good. Rather, perhaps the best way for mathematics educators to meaningfully change the way that mathematics professors teach is for both groups to engage in a mutual negotiation about goals for advanced mathematics courses and appropriate beliefs about mathematics education.

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