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A Harish-Chandra Homomorphism for Reductive Group Actions

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Introduction

Consider a semisimple complex Lie algebra $\mathfrak{g}$ and its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. In order to study unitary representations of semisimple Lie groups, Harish-Chandra ([HC1] Part III) established an isomorphism between the center $\mathfrak{Z}(\mathfrak{g})$ of $\mathfrak{U}(\mathfrak{g})$ and the algebra of invariant polynomials $\mathbb{C}[\mathfrak{t}^*]^W$. Here, $\mathfrak{t} \subseteq \mathfrak{g}$ is a Cartan subspace and W is the Weyl group of $\mathfrak{g}$. This is one of the most basic results in representation theory.

Later on ([HC2] Thm. 1), he found a similar isomorphism for a symmetric space $X = G/K$. Here, instead of $\mathfrak{Z}(\mathfrak{g})$, he considered the algebra $\mathcal{D}(X)^G$ of invariant differential operators on X and showed, that it is isomorphic to the ring of invariants of the little Weyl group W_X attached to X. This isomorphism is very important for analyzing the action of G on various function spaces on X. Actually, it is a generalization of the former result if one considers the natural $G \times G$-action on $X = G$.

The proofs of these theorems relied very much on the very special structure theory of symmetric spaces. Therefore, it may be surprising that such an isomorphism can be constructed for every algebraic variety X carrying an action of a connected reductive group G.

Because invariants and centers behave well under field extensions, we assume from now on, that the base field k is algebraically closed of characteristic zero. Let me first explain the case, where X is a smooth, affine G-variety. Here, I obtained the most complete results. This covers all linear actions on a vector space as well as all homogeneous spaces G/H where H is reductive, in particular symmetric varieties.

Consider the algebra of (algebraic) linear differential operators $\mathcal{D}(X)$. We are interested in the subalgebra $\mathcal{D}(X)^G$ of invariant operators and in particular, in its center $\mathfrak{Z}(X)$.

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Our main results are summarized in the following theorem.

Theorem. *Let G be a connected reductive group acting on a smooth affine variety X.*

a) *There is a subspace $\mathfrak{a}_X^* \subseteq \mathfrak{t}^*$, a subgroup $W_X \subseteq W$ and an isomorphism η such that following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{Z}(\mathfrak{g}) & \rightarrow & \mathfrak{Z}(X) \\ \downarrow\sim & & \eta\downarrow\sim \\ k[\mathfrak{t}^*]^W & \stackrel{\text{res}}{\rightarrow} & k[\rho+\mathfrak{a}_X^*]^{W_X} \end{array}$$

Here, ρ is the half sum of positive roots. In particular, $\mathfrak{Z}(X)$ is finitely generated as $\mathfrak{Z}(\mathfrak{g})$-module.

b) *The "little Weyl group" W_X acts on $\mathfrak{a}_X^*$ as a reflection group. In particular, $\mathfrak{Z}(X)$ is a polynomial ring.*

c) *Let $\mathfrak{U}(X)$ be the commutant of $\mathcal{D}(X)^G$ in $\mathcal{D}(X)$. Then there exists $\xi \in \mathfrak{Z}(X)$, $\xi \neq 0$ with $\xi\mathfrak{U}(X) \subseteq \mathfrak{U}(\mathfrak{g})\mathfrak{Z}(X) \subseteq \mathfrak{U}(X)$.*

d) *Let $\mathfrak{C}(X)$ be the commutant of $\mathfrak{Z}(X)$ in $\mathcal{D}(X)$. Then there is a canonical isomorphism*

$$\mathfrak{U}(X) \underset{\mathfrak{Z}(X)}{\otimes} \mathcal{D}(X)^G \xrightarrow{\sim} \mathfrak{C}(X).$$

e) *The algebras $\mathfrak{U}(X)$, $\mathcal{D}(X)^G$, $\mathfrak{C}(X)$ and $\mathcal{D}(X)$ are all free as $\mathfrak{Z}(X)$-modules.*

The last item generalizes results of Kostant [Kos] and Kostant-Rallis [KR].

The description above is upside down. What we actually construct first, is the algebra $\mathfrak{U}(X)$, without reference to $\mathcal{D}(X)^G$. Then, we define $\mathfrak{Z}(X) = \mathfrak{U}(X)^G$. It turns out that $\mathfrak{Z}(X)$ is exactly the center of $\mathfrak{U}(X)$. This approach works in complete generality for any G-variety. In particular a), b) and c) of the theorem above are valid. Only later, we identify $\mathfrak{U}(X)$ as the commutant of $\mathcal{D}(X)^G$ provided X is smooth and affine. Although I have no example, I think, that this is wrong in general. In any case, because W_X occurs also in quite different situations, like equivariant embeddings of X (see [Br2]) or the B-orbit structure of X, I am sure, that $\mathfrak{Z}(X)$ is the "right" center while $\mathcal{D}(X)^G$ may not be the "right" algebra. Perhaps one has to consider more general concepts, like micro-differential operators, instead.

The problem in general is, that $\mathcal{D}(X)^G$ may be too small. But there are cases in which $\mathcal{D}(X)^G$ is expected to be very small, namely, when X is spherical. A G-variety is called spherical if a Borel subgroup has a dense orbit. It turns out, that in this case, $\mathfrak{Z}(X)$ comprises a priori already all invariant differential operators. Hence, $\mathcal{D}(X)^G$ equals $\mathfrak{Z}(X)$ and is therefore a (commutative) polynomial ring. The class of spherical varieties includes all symmetric varieties, as well as many non-affine varieties like G/U where U is maximal unipotent. Thus, this concept unifies, for example, Harish-Chandra modules and highest weight modules, because both are essentially nothing else than H-finite representations of $\mathfrak{g}$, where $X = G/H$ is either symmetric or $X = G/U$.

I have omitted in this paper all applications to representation theory. In the final section we illustrate the theorem only by some examples and by the determination of the center of a classifying ring, i.e., the commutant of a reductive subalgebra in $\mathfrak{U}(\mathfrak{g})$.

Some words to the proof: Because all algebras carry natural filtrations we can consider the associated graded rings. For example, $\operatorname{gr}\mathcal{D}(X)$ is a ring of functions on the cotangent bundle T_X^* and the natural homomorphism $\mathfrak{U}(\mathfrak{g}) \to \mathcal{D}(X)$ corresponds to the moment map $T_X^* \to \mathfrak{g}^*$. The theory on the graded side has already been developed in my paper [Kn1]. In particular, the little Weyl group and the analogues of $\mathfrak{U}(X)$ and $\mathfrak{Z}(X)$ have been constructed. In the present paper we lift these results to the non-graded case. The main results of [Kn1] are reviewed in section 1.

In [Kn1] we constructed a surjective morphism $\Psi : T_X^* \to L_X := \mathfrak{a}_X^*/W_X$. Because W_X is a reflection group, L_X is an affine space. The analogue of $\mathfrak{Z}(X)$ is now the ring R of those functions on T_X^* coming from L_X. For constructing $\mathfrak{Z}(X)$, there are two main difficulties: Given a homogeneous function $f \in R$, why is it a symbol of some differential operator D? This problem it trivial for affine varieties, but even then: Why can one choose D to be in the center of $\mathcal{D}(X)^G$?

Both difficulties are overcome by restricting the attention to the set $\mathfrak{U}(X)$ of those operators having a certain boundary behavior at infinity. More precisely, a differential operator D is in $\mathfrak{U}(X)$ if there is a "good" (so called *pseudo-free*) smooth completion $\overline{X}$ of X, such that D is locally a $\mathcal{O}_{\overline{X}}$-linear combination of differential operators induced by $\mathfrak{U}(\mathfrak{g})$. Later we show, that for $D \in \mathfrak{U}(X)$ to be in the center of $\mathcal{D}(X)^G$ is equivalent to being G-invariant, which solves the second problem (by linear reductivity of G).

But the main advantage of this construction is, that now the morphism which corresponds on the graded side to $\mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(X)$ is *proper*, that means, we have obtained a compactification of the moment map. To this morphism we apply a powerful vanishing theorem of Kollár [Kol], which in turn solves the first problem. The heart of this paper (sections 4 and 5) is devoted to verify the conditions of Kollár's theorem.

The compactified moment map was in the spherical case already well known (e.g. [Abe], [Br2]). The starting point of this paper was a recent proof of the main Vanishing Theorem 4.1 for spherical varieties of a special type by F. Bien and M. Brion. Although, their proof is quite different from mine, it convinced me that something like that is true.

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Notation: All varieties are defined over an algebraically closed field k of characteristic zero. For a smooth variety X let Ω_X be the sheaf of Kähler differentials, $\mathcal{T}_X := \Omega_X^\vee$ the tangent sheaf, $\mathcal{D}_X$ the sheaf of linear differential operators and $\mathcal{D}(X) = \Gamma(\mathcal{D}_X)$ its global sections. We consider $\mathcal{T}_X$ as a subsheaf of $\mathcal{D}_X$. The algebra of regular functions on X is denoted by $k[X]$.

Let G denote a connected linear algebraic group with Lie algebra $\mathfrak{g}$. Except in the sections 2 and 3 we will assume that G is reductive. Let $\mathfrak{U} = \mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(G)$ be the universal enveloping algebra. Let B be a Borel subgroup of G with unipotent radical U and maximal torus T. For an affine G-variety X, let $X/\!\!/G := \operatorname{Spec} k[X]^G$ be the categorical quotient.

In the sequel I will often tacitly use the existence of equivariant resolutions of singularities (see e.g. [AHV]).

1. Review of the moment map

Let X be a smooth G-variety. Then the G-action induces a homomorphism $\mathfrak{g} \to \mathcal{T}_X(X)$ which extends uniquely to a homomorphism

$$\varphi : \mathfrak{U} \to \mathcal{D}(X).$$

Both algebras carry a natural filtration and φ is compatible with them. Thus there is a homomorphism between the associated graded objects:

$$\overline{\varphi} : \operatorname{gr} \mathfrak{U} \longrightarrow \operatorname{gr} \mathcal{D}(X).$$

By the Poincaré-Birkhoff-Witt theorem, $\operatorname{gr} \mathfrak{U}$ is isomorphic to $S^\bullet \mathfrak{g} = k[\mathfrak{g}^*]$ while $\operatorname{gr} \mathcal{D}(X)$ is via the symbol map a subalgebra of $\mathrm{H}^0(S^\bullet \mathcal{T}_X) = k[T_X^*]$ where $\pi : T_X^* \to X$ is the cotangent bundle of X. Thus $\overline{\varphi}$ is induced by the moment map

$$\Phi : T_X^* \longrightarrow \mathfrak{g}^* : \quad \alpha \mapsto \big[\xi \mapsto \alpha(\xi_{\pi(\alpha)})\big].$$

We studied Φ in [Kn1] in some detail. In particular, we considered a canonical factorization

$$T_X^* \longrightarrow M_X \longrightarrow \mathfrak{g}^*$$

of the moment map, where the first map has connected general fibers and the second is finite. The ring of invariants $k[\mathfrak{g}^*]^G$ is by Chevalley's theorem isomorphic to $k[\mathfrak{t}^*]^W$, where $\mathfrak{t} \subseteq \mathfrak{g}$ is a Cartan subspace and W is the Weyl group, or in geometric terms:

$$\mathfrak{t}^*/W \xrightarrow{\sim} \mathfrak{g}^*/\!\!/G.$$

Now, a similar thing happens for M_X: There is a certain subspace $\mathfrak{a}_X^* \subseteq \mathfrak{t}^*$, a subquotient W_X of W and a "Chevalley isomorphism" $\mathfrak{a}_X^*/W_X \cong M_X/\!\!/G$ such that the diagram

$$\begin{array}{ccccccc} M_X & \to & M_X/\!\!/G & \cong & \mathfrak{a}_X^*/W_X & \leftarrow & \mathfrak{a}_X^* \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{g}^* & \to & \mathfrak{g}^*/\!\!/G & \cong & \mathfrak{t}^*/W & \leftarrow & \mathfrak{t}^* \end{array}$$

commutes ([Kn1] p. 12). We denote this common quotient $M_X/\!\!/G = \mathfrak{a}_X^*/W_X$ by L_X and call W_X the *little Weyl group of* X. The principal result is, that it is a finite reflection group ([Kn1] Satz 6.6). In particular, L_X is an affine space and $k[M_X]^G$ is a polynomial ring.

The main purpose of this paper is to carry these results over to the non-commutative case. The direct approach by taking an integral closure doesn't seem to work for $\mathcal{D}(X)$. Instead, we construct an algebra $\mathfrak{U}(X)$, the analogue of M_X, as a set of differential operators with a certain behavior at infinity.

The main tool for studying the variety X was the following degeneration theorem ([Kn1] Satz 2.7):

1.1. Theorem. *Let X be G-variety. Then there exists a $G \times \mathbf{G}_m$-variety Z and a surjective function $\Delta : Z \to \mathbf{A}^1$ with the following properties:*

a) Z and Δ are smooth

b) For all $z \in Z$, $g \in G$ and $t \in \mathbf{G}_m$ we have $\Delta(gz) = \Delta(z)$ and $\Delta(tz) = t\Delta(z)$.

c) The general fiber $X_t := \Delta^{-1}(t)$ $(t \neq 0)$ is G-birational to X.

d) The special fiber X_0 is as a G-variety isomorphic to $V \times G/S$. Here, G acts trivially on V and S is a subgroup of G containing U.

One can show ([Kn1] Satz 2.5) that $S = S_X$ is independent of the choice of Z and Δ. It is called the *horospherical type of* X and $X_h := G/S_X$ is the *horospherical variety attached to* X. The normalizer $P_X := N_G(S_X)$ is a parabolic subgroup and is for example characterized by the fact, that P_X is the smallest parabolic occurring as an isotropy group in any G-variety which is G-birational to X. The quotient $A_X := P_X/S_X$ is a torus. Let $\mathfrak{a}_X := \operatorname{Lie} A_X$ and $T \subseteq P_X$ a maximal torus with Lie algebra $\mathfrak{t}$. Then this yields a projection $T \twoheadrightarrow A_X$ inducing the before mentioned embedding $\mathfrak{a}_X^* \hookrightarrow \mathfrak{t}^*$.

The dimension of A_X is called the *rank* ($\operatorname{rk} X$) of X which can also be characterized as follows: Let $x \in X$ be a general point. Then

$$\operatorname{rk} X = \dim Bx - \dim Ux.$$

The codimension $c(X)$ of a general B-orbit is called the *complexity* of X. It is also the transcendence degree of $k(X)^B$ over k. Thus, the codimension of the general U-orbit is

$c(X) + \mathrm{rk}\, X$. A normal G-variety of complexity zero, i.e. with a dense B-orbit, is called *spherical*.

The general philosophy behind X_h is, that it shares a lot of properties with X, for example:

— The kernel of $\mathfrak{U}(G) \to \mathcal{D}(X)$ is the same for X and X_h ([Kn1] Satz 5.1).

— The closure of the image of the moment map coincides for X and X_h ([Kn1] Satz 5.4).

— The general isotropy group of G on T^*_X is up to conjugacy the same for X and X_h ([Kn1] Satz 8.1).

The last statement is very convenient for the determination of X_h. E.g., it follows that $S^\circ_X = U$ if and only if the general isotropy group of G on T^*_X is finite.

2. The localized moment map

In this and the next section G may be any connected algebraic group. Let X be a smooth G-variety. The smoothness is only convenient and not essential. Recall ([BB]) that $\mathcal{U}_X := \mathcal{O}_X \otimes_k \mathfrak{U}$ carries a canonical algebra structure by setting

$$(f \otimes \xi) \cdot (g \otimes \eta) = fg \otimes \xi\eta + f \cdot \xi g \otimes \eta \qquad (f, g \in \mathcal{O}_X;\ \xi, \eta \in \mathfrak{g}).$$

Then the homomorphism $\mathfrak{U} \to \mathcal{D}(X)$ induces a localized homomorphism between sheaves of algebras $\mathcal{U}_X \to \mathcal{D}_X$. Denote its image by $\mathfrak{U}_X$. It is also a sheaf of algebras on X.

An analogous construction works in the commutative case: The homomorphism $\mathfrak{g} \to \mathcal{T}_X$ induces a homomorphism $\mathcal{O}_X \otimes_k S^\bullet\mathfrak{g} \to S^\bullet\mathcal{T}_X$ between symmetric powers. Its image $\mathcal{S}_X$ is a sheaf of finitely generated graded algebras over X. Set

$$\widetilde{T}_X := \mathrm{Spec}_X\, \mathcal{S}_X.$$

One can construct $\widetilde{T}_X$ also in geometric terms: The relative spectrum over X of $S^\bullet\mathcal{T}_X$ is the cotangent bundle T^*_X while that of $\mathcal{O}_X \otimes_k S^\bullet(\mathfrak{g})$ is $X \times \mathfrak{g}^*$. Thus, $\widetilde{T}_X$ is the closure of the image of the canonical morphism

$$\pi \times \Phi : T^*_X \longrightarrow X \times \mathfrak{g}^*,$$

where $\pi : T^*_X \to X$ is the projection and $\Phi : T^*_X \to \mathfrak{g}^*$ is the moment map. We denote the canonical maps of $\widetilde{T}_X$ to X and $\mathfrak{g}^*$ again by π and Φ. The triple $(\widetilde{T}_X, \pi, \Phi)$ is the localized moment map. It has the following obvious properties:

a) The projection $\pi : \widetilde{T}_X \to X$ is an affine morphism.

b) The map $\Phi : \widetilde{T}_X \to \mathfrak{g}^*$ is proper if X is complete.

The last statement is the crucial property of this construction.

By construction, there is a morphism $\kappa : T^*_X \to \widetilde{T}_X$ which factorizes the moment map:

$$T^*_X \xrightarrow{\kappa} \widetilde{T}_X \xrightarrow{\Phi} \mathfrak{g}^*.$$

The main property of κ is stated in the following lemma.

2.1. Lemma. *The fiber $\kappa^{-1}(y)$ of $y \in \widetilde{T}_X$ is either empty or isomorphic to the conormal space to the orbit Gx in $x = \pi(y)$. In particular, every fiber is irreducible.*

Proof: Let $\lambda := \Phi(y) \in \mathfrak{g}^*$. Then

$$\kappa^{-1}(y) = \{\alpha \in T^*_{X,x} \mid \alpha(\xi x) = \lambda(\xi) \text{ for all } \xi \in \mathfrak{g}\}.$$

Assume there is $\alpha_0 \in T^*_X$ with $y = \kappa(\alpha_0)$. Then any other α is in $\kappa^{-1}(y)$ if and only if $\alpha \in T^*_{X,x}$ and the difference $\beta = \alpha - \alpha_0$ satisfies $\beta(\mathfrak{g}x) = 0$. □

2.2. Corollary. *The morphism $\kappa : T^*_X \to \widetilde{T}_X$ is an isomorphism if and only if G acts transitively on X.*

I don't expect $\widetilde{T}_X$ to have good properties in general, but one case is particularly nice:

2.3. Lemma. *Assume all orbits of X have the same dimension d. Then $\pi : \widetilde{T}_X \to X$ is a vector bundle with fiber $\mathfrak{g}_x^\perp$ over x.*

Proof: The image of T^*_X in $X \times \mathfrak{g}^*$ is

$$Z = \{(x, \lambda) \in X \times \mathfrak{g}^* \mid \lambda(\mathfrak{g}_x) = 0\}.$$

The dimension of the fibers $\mathfrak{g}_x$ of

$$Z' = \{(x, \xi) \in X \times \mathfrak{g} \mid \xi x = 0\} \longrightarrow X$$

is by assumption constant. Therefore it is a vector bundle. This implies that locally around every $x_0 \in X$ there are sections $\xi_i(x)$ of Z' which span $\mathfrak{g}_x$ as a vector space. This implies that Z is locally defined by $\lambda(\xi_i(x)) = 0$. Therefore, Z is closed and coincides with $\widetilde{T}_X$. □

This leads to our main definition:

Definition: A G-variety X is called *pseudo-free* if $\widetilde{T}_X \to X$ is a vector bundle.

Let d be the maximal dimension of an orbit of X and consider the open subset

$$X_m := \{x \in X \mid \dim Gx = d\},$$

the so called generic sheet. Then $\pi : \widetilde{T}_{X_m} \to X_m$ is a vector bundle whose fibers are d-dimensional subspaces of $\mathfrak{g}^*$. This induces a map

$$\sigma_m : X_m \to \mathrm{Gr}_d(\mathfrak{g}^*) : x \mapsto \mathfrak{g}_x^\perp$$

where $\mathrm{Gr}_d(\mathfrak{g}^*)$ denotes the Graßmannian of d-dimensional subspaces of $\mathfrak{g}^*$. There is now an easy criterion for pseudo-freeness:

2.4. Lemma. *X is pseudo-free if and only if $\sigma_m : X_m \to \mathrm{Gr}_d(\mathfrak{g}^*)$ extends to a morphism σ on all of X.*

Proof: Let $\mathrm{Gr} := \mathrm{Gr}_d(\mathfrak{g}^*)$. If $\widetilde{T}_X \to X$ is a vector bundle, then its fibers are d-dimensional subspaces of $\mathfrak{g}^*$. Therefore it induces a morphism $\sigma : X \to \mathrm{Gr}$ extending σ_m.

Assume now that σ_m extends to a morphism $\sigma : X \to \mathrm{Gr}$. Let

$$V := \{(E, \lambda) \in \mathrm{Gr} \times \mathfrak{g}^* \mid \lambda \in E\}$$

be the tautological bundle over Gr. Then $Z := X \times_{\mathrm{Gr}} V$ is a vector bundle over X. On the other hand, Z is a closed subset of $X \times \mathfrak{g}^*$ and the image of T_X^* is dense in Z. Thus $\widetilde{T}_X = Z$ is a vector bundle. □

Thus, the pseudo-freeness of X means that the general subspaces $\mathfrak{g}_x^\perp$ (or equivalently $\mathfrak{g}_x$) degenerate at the boundary to specific limits. If the general isotropy subgroup is finite then we have $\widetilde{T}_X = X \times \mathfrak{g}^*$, and the action is automatically pseudo-free.

2.5. Corollary. *For any smooth G-variety X there exists a smooth pseudo-free G-variety $\widetilde{X}$ together with a projective, birational, equivariant morphism $\widetilde{X} \to X$.*

Proof: Take the closure of X_m in $X \times \mathrm{Gr}_d(\mathfrak{g}^*)$ and choose an equivariant resolution of singularities $\widetilde{X}$. Because $\mathrm{Gr}_d(\mathfrak{g}^*)$ is projective, the morphism $\widetilde{X} \to X$ is projective. □

The link between the commutative and the non-commutative case is provided by filtrations: The sheaf $\mathfrak{U}_X$ carries two filtrations, namely one induced by $\mathfrak{U}^{(n)}$ and the other induced by $\mathcal{D}_X^{(n)}$. Then there is the following commutative diagram of sheaves of graded algebras:

$$\begin{array}{ccccc} \mathrm{gr}_{\mathfrak{U}} \mathfrak{U}_X & & \longrightarrow & & \mathcal{S}_X \\ \downarrow & & & & \downarrow \\ \mathrm{gr}_{\mathcal{D}} \mathfrak{U}_X & \hookrightarrow & \mathrm{gr}\, \mathcal{D}_X & = & S^\bullet \mathcal{T}_X \end{array}$$

The main property of pseudo-freeness is:

2.6. Theorem. *Let X be a smooth and pseudo-free G-variety. Then the $\mathfrak{U}$-filtration and the $\mathcal{D}$-filtration of $\mathfrak{U}_X$ coincide and the canonical homomorphism*

$$\bar{\varphi} : \mathrm{gr}\, \mathfrak{U}_X \longrightarrow \mathcal{S}_X = \pi_* \mathcal{O}_{\widetilde{T}_X}$$

is an isomorphism.

Proof: Denote the n^{th} graded component of $\mathcal{S}_X$ by $\mathcal{S}_X^n$. The fact that $\widetilde{T}_X$ is a vector bundle means that $\mathcal{S}_X^1$ is locally free and $\mathcal{S}_X^n = S^n \mathcal{S}_X^1$. The canonical homomorphism

$\bar{\varphi} : \mathrm{gr}_{\mathfrak{U}} \mathfrak{U}_X \to \mathcal{S}_X$ is by construction an isomorphism in degree 0 and 1. Thus there is a unique homogeneous homomorphism ψ in the reverse direction with $\bar{\varphi} \circ \psi = \mathrm{id}$. Also by construction, $\mathrm{gr}_{\mathfrak{U}} \mathfrak{U}_X$ and $\mathcal{S}_X$ are generated by their elements of degree ≤ 1. Thus $\psi \circ \bar{\varphi}$ is surjective and thus, being a projector, an isomorphism.

Finally, we show that the filtrations on $\mathfrak{U}_X$ coincide. We see from the diagram above, that $\mathrm{gr}_{\mathfrak{U}} \mathfrak{U}_X \to \mathrm{gr}_{\mathcal{D}} \mathfrak{U}_X$ is injective. This implies (where Fil^n denotes the n-th step of a filtration)

$$\mathrm{Fil}^n_{\mathfrak{U}} \mathfrak{U}_X = \mathrm{Fil}^{n+1}_{\mathfrak{U}} \mathfrak{U}_X \cap \mathrm{Fil}^n_{\mathcal{D}} \mathfrak{U}_X = \mathrm{Fil}^{n+2}_{\mathfrak{U}} \mathfrak{U}_X \cap \mathrm{Fil}^n_{\mathcal{D}} \mathfrak{U}_X = \ldots = \mathrm{Fil}^n_{\mathcal{D}} \mathfrak{U}_X. \qquad \square$$

2.7. Corollary. *Let X be smooth and pseudo-free. Then for any $n \geq 0$ there is a short exact sequence*

$$0 \longrightarrow \mathfrak{U}_X^{(n-1)} \longrightarrow \mathfrak{U}_X^{(n)} \longrightarrow S^n \mathcal{S}_X^1 = (\pi_* \mathcal{O}_{\widetilde{T}_X})_n \longrightarrow 0.$$

In particular, the sheaves $\mathfrak{U}_X^{(n)}$ are locally free as left or right $\mathcal{O}_X$-modules.

3. Functoriality

In this section we exhibit some functoriality properties of the sheaves $\mathfrak{U}_X$.

3.1. Lemma. *Let X and Y be smooth G-varieties and $\varphi : X \to Y$ an equivariant generically finite morphism. Then there exists a canonical surjective $\mathcal{U}_X$-module homomorphism $\varphi^\circ : \varphi^* \mathfrak{U}_Y \to \mathfrak{U}_X$. If Y is pseudo-free and φ is also dominant then φ° is an isomorphism.*

Proof: The uniqueness and the surjectivity are clear because $\mathfrak{U}_X$ is generated as $\mathcal{U}_X$-module by $1 = \varphi^\circ(1)$. Let ξ be in the kernel $\mathcal{K}_Y$ of $\mathcal{U}_Y \to \mathfrak{U}_Y$. Then ξ induces the zero-operator on Y and therefore also on $\varphi(X)$ and by generically finiteness on X. Thus, $\varphi^* \mathcal{K}_Y$ is mapped to $\mathcal{K}_X$ which shows the existence of φ°.

If φ is dominant then φ° is generically injective, i.e., its kernel is a torsion module. If Y is pseudo-free then $\varphi^* \mathfrak{U}_Y$ is locally free. Thus, φ° is also injective. $\square$

3.2. Corollary. *Let $\varphi : \widetilde{X} \to X$ be a equivariant proper birational morphism between smooth pseudo-free G-varieties. Then there is a canonical algebra-isomorphism $\mathfrak{U}_X \xrightarrow{\sim} \varphi_* \mathfrak{U}_{\widetilde{X}}$.*

3.3. Lemma. *Let $\varphi : X \to Y$ a dominant equivariant morphism between smooth G-varieties. Assume, Y is pseudo-free. Then there is a canonical surjective $\mathcal{U}_X$-module homomorphism $\varphi_\circ : \mathfrak{U}_X \to \varphi^* \mathfrak{U}_Y$.*

Proof: There is a canonical homomorphism $\mathcal{D}_X \to \varphi^* \mathcal{D}_Y$. Because Y is pseudo-free, $\mathfrak{U}_Y$ hence $\varphi^* \mathfrak{U}_Y$ are locally free. This implies, that $\varphi^* \mathfrak{U}_Y \to \varphi^* \mathcal{D}_Y$ is injective, because the kernel is a torsion module. Hence, $\varphi^* \mathfrak{U}_Y$ is the image of $\mathcal{U}_X$. It follows, that $\mathfrak{U}_X$ is mapped to $\varphi^* \mathfrak{U}_Y$. $\square$

This leads to the following definition:

Definition: Let X be any G-variety and let $\varphi : \widetilde{X} \to X$ be equivariant, birational, proper with $\widetilde{X}$ smooth and pseudo-free. Then let $\bar{\mathfrak{U}}_X := \varphi_*\mathfrak{U}_{\widetilde{X}} \subseteq \mathcal{D}_X$. For any equivariant completion $X \hookrightarrow \overline{X}$ let $\mathfrak{U}(X) := \mathrm{H}^0(\overline{X}, \bar{\mathfrak{U}}_{\overline{X}})$ the algebra of *completely regular* differential operators.

If X is smooth then $\mathfrak{U}_X \subseteq \bar{\mathfrak{U}}_X$ (Lemma 3.1). Corollary 3.2 implies that the definition is independent of the choice of $\widetilde{X}$ and $\overline{X}$. It also implies that $\mathfrak{U}(X)$ is actually an equivariant-birational invariant of X. Note that the notation $\mathfrak{U}(G)$ for the enveloping algebra is consistent with the notation above if G acts on itself by left translation. For in this case, $\bar{\mathfrak{U}}_X$ is just the trivial bundle $\mathcal{U}_X = \mathcal{O}_X \otimes_k \mathfrak{U}(G)$ for any completion X of G. We have the following functoriality properties for $\mathfrak{U}(X)$:

3.4. Corollary. *Let $\varphi : X \to Y$ be an equivariant morphism between normal G-varieties.*

a) If φ is dominant with irreducible general fibers, then it induces an algebra homomorphism $\varphi_\circ : \mathfrak{U}(X) \to \mathfrak{U}(Y)$.

b) If φ is generically injective, then it induces an algebra homomorphism $\varphi^\circ : \mathfrak{U}(Y) \to \mathfrak{U}(X)$.

Proof: We may assume throughout that X and Y are projective and X is pseudo-free.

a) By Lemma 3.3 we have a homomorphism $\varphi_*\mathfrak{U}_X \to \varphi_*\varphi^*\mathfrak{U}_Y = \mathfrak{U}_Y$. Taking global sections yields $\varphi_\circ$.

b) Choose a pseudo-free resolution of singularities $\widetilde{X} \to X$ and let $\widetilde{Y} \subset \widetilde{X}$ be a component of the preimage of Y such that $\widetilde{Y} \to Y$ is dominant. By Zariski's connectedness theorem, all fibers of $\widetilde{Y} \to Y$ are connected. This induces $(\mathfrak{U}_{\widetilde{X}})|_{\widetilde{Y}} \to \mathfrak{U}_{\widetilde{Y}} \subseteq \bar{\mathfrak{U}}_{\widetilde{Y}}$, hence $\varphi^\circ : \mathfrak{U}(X) = \mathfrak{U}(\widetilde{X}) \to \mathfrak{U}(\widetilde{Y}) \to \mathfrak{U}(Y)$. □

Later we will need the following lemma. For a generalization see Corollary 8.2

3.5. Lemma. *Let X be a G-variety which it G-birational to $X_0 \times X_1$ where G acts trivially on X_0. Then there is a canonical isomorphism $\mathfrak{U}(X) \to \mathfrak{U}(X_1)$.*

Proof: We may replace X by $X_0 \times X_1$. By a further birational change we may assume that X_0 and X_1 are both smooth, complete and that X_1 is pseudo-free. But then $\mathfrak{U}_X = p_1^*\mathfrak{U}_{X_1}$ where p_1 is projection on the second factor. This implies $\mathfrak{U}(X) = \mathrm{H}^0(X, p_1^*\mathfrak{U}_{X_1}) = \mathrm{H}^0(X_1, \mathfrak{U}_{X_1}) = \mathfrak{U}(X_1)$. □

4. The vanishing theorem

From now on, let G be a connected reductive group. For a smooth, complete, pseudo-free G-variety X let

$$\widetilde{T}_X \xrightarrow{\overline{\Phi}} M_X \to \mathfrak{g}^*$$

be the Stein factorization. Because $\widetilde{T}_X$ is normal (even smooth), M_X is normal as well. By Lemma 2.1 the general fiber of $T_X^* \to \widetilde{T}_X$ is irreducible. This shows, that M_X coincides with the M_X of [Kn1] §6.

4.1. Theorem. *Let X be a smooth, pseudo-free G-variety and let $\psi : X \to Y$ a G-invariant proper morphism onto a smooth affine variety Y. Assume, that the general fiber of ψ contains a dense G-orbit. Then the following holds for all integers $i > 0$:*

a) $\mathrm{H}^i(X, \mathfrak{U}_X^{(n)}) = 0$ *for all* $n \geq 0$*;*

b) $\mathrm{H}^i(X, \mathcal{S}_X^n) = 0$ *for all* $n \geq 0$*;*

c) $\mathrm{H}^i(\widetilde{T}_X, \mathcal{O}_{\widetilde{T}_X}) = 0$*;*

d) $\mathbf{R}^i\overline{\Phi}'_*\mathcal{O}_{\widetilde{T}_X} = 0$ *where* $\overline{\Phi}'$ *is the composition* $\widetilde{T}_X \to X \times M_X \to Y \times M_X$*.*

The proof of this theorem will occupy this and the next section. First we reduce all assertions to d):

b)⇒a): This follows immediately by induction on n and the long exact sequence associated to the sequence in Corollary 2.7.

c)⇒b): Because $\pi : \widetilde{T}_X \to X$ is an affine morphism, the Leray spectral sequence implies

$$\bigoplus_{n=0}^{\infty} \mathrm{H}^i(X, \mathcal{S}_X^n) = \mathrm{H}^i(X, \pi_*\mathcal{O}_{\widetilde{T}_X}) = \mathrm{H}^i(\widetilde{T}_X, \mathcal{O}_{\widetilde{T}_X}) = 0.$$

d)⇒c): The affinity of $Y \times M_X$ implies $\mathrm{H}^i(\widetilde{T}_X, \mathcal{O}_{\widetilde{T}_X}) = \mathrm{H}^0(Y \times M_X, \mathbf{R}^i\overline{\Phi}'_*\mathcal{O}_{\widetilde{T}_X})$.
Thus everything boils down to show d). For this we use the following theorem of Kollár:

4.2. Theorem. *Let $\pi : X \to Y$ be a projective morphism where X is smooth and Y has rational singularities. Assume that the general fiber F of π is connected with* $\mathrm{H}^i(F, \mathcal{O}_F) = 0$ *for $i > 0$. Then* $\mathbf{R}^i\pi_*\mathcal{O}_X = 0$ *for all $i > 0$.*

Proof: If X and Y are projective then this is (half of) Thm. 7.1 of [Kol]. We describe now how to reduce the general case to that one.

By Hironaka's desingularization theorem one can choose smooth quasi-projective varieties $\widetilde{Y}$, $\widetilde{X}$ and proper morphisms φ, φ' and $\tilde{\pi}$ such that the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\varphi'} & X \\ {\scriptstyle\tilde{\pi}}\downarrow & & {\scriptstyle\pi}\downarrow \\ \widetilde{Y} & \xrightarrow{\varphi} & Y \end{array}$$

is commutative where φ, φ' are birational and $\tilde{\pi}$ has the same general fiber F as π. Assume, that the theorem holds for $\tilde{\pi}$. Then the Leray spectral sequence implies

$$\mathbf{R}^i(\varphi\tilde{\pi})_*\mathcal{O}_{\widetilde{X}} = \mathbf{R}^i\varphi_*\mathcal{O}_{\widetilde{Y}} \quad \text{for } i \geq 0.$$

Because Y and X have rational singularities we have

$$\varphi'_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X \quad \text{and} \quad \mathbf{R}^i\varphi_*\mathcal{O}_{\widetilde{Y}} = \mathbf{R}^i\varphi'_*\mathcal{O}_{\widetilde{X}} = 0 \quad \text{for } i > 0.$$

Thus, the Leray spectral sequence implies for $i > 0$:

$$\mathbf{R}^i\pi_*\mathcal{O}_X = \mathbf{R}^i(\pi\varphi')_*\mathcal{O}_{\widetilde{X}} = \mathbf{R}^i(\varphi\tilde{\pi})_*\mathcal{O}_{\widetilde{X}} = \mathbf{R}^i\varphi_*\mathcal{O}_{\widetilde{Y}} = 0.$$

This reduces the assertion to the case where X and Y are smooth, quasi-projective.

Now again by Hironaka, one can embed Y and X into smooth projective varieties $\overline{Y}$ and $\overline{X}$ respectively, such that π extends to a morphism $\overline{\pi} : \overline{X} \to \overline{Y}$. Now, our theorem is reduced to Kollár's original version. □

We want to apply Kollár's theorem to the morphism $\overline{\Phi}' : \widetilde{T}_X \to Y \times M_X$ which is clearly proper. Thus, we have to show:

a) The variety $Y \times M_X$ has rational singularities.
b) The general fiber of $\overline{\Phi}'$ has vanishing cohomology of the structure sheaf.

a) Because Y is smooth, we need to show that M_X has rational singularities, which is our next lemma.

4.3. Lemma. *Let X be a smooth G-variety. Then M_X has rational singularities. In particular, it is Cohen-Macaulay.*

Proof: Essentially, this has already been proved in [Kn1]: Let $X_h = G/S_X$ be the horospherical variety attached to X. Then, there is an action of the little Weyl group W_X on M_{X_h} with $M_X = M_{X_h}/W_X$ ([Kn1] Satz 6.4). Hence, (see e.g. [Bou]) it suffices to show, that M_{X_h} has rational singularities. But this follows from [Kn1] Lemma 4.1. □

b) Let $X_0 = \psi^{-1}(y)$ be a general fiber of ψ. Then we have $\widetilde{T}_{X_0} \cong (\psi\pi)^{-1}(y) \subseteq \widetilde{T}_X$. Thus, we may replace X by X_0 and therefore assume that X contains a dense orbit.

The higher cohomology of the structure sheaf of a smooth unirational variety vanishes ([Ser] Lemma 1). Thus, the theorem will be proved when we show that the general fiber of $\overline{\Phi}$ is unirational. Because this is a generic property we may replace X by its open orbit and thus $\widetilde{T}_X$ by the cotangent bundle T^*_X. The proof of unirationality in this case will occupy the next section.

5. Unirationality of the fibers of the moment map

The purpose of this section is to prove the following

5.1. Theorem. *Let X be a homogeneous G-variety. Then the general fiber of $\Phi : T_X^* \to M_X$ is unirational.*

Proof: Let $x \in T_X^*$ be a general point with image $x' := \Phi(x) \in M_X$. First, let me note that the proof is very easy if X is spherical: The dimension of $\Phi^{-1}(x')$ is $r = \operatorname{rk} X$ by [Kn1] Satz 7.1. On the other hand, it follows from [Kn1] Satz 8.1 that $G_{x'}$ has an r-dimensional, hence open orbit in $\Phi^{-1}(x')$.

For the general case we need several reduction steps. Remember $L_X = M_X /\!\!/ G$. Then we have the following diagram:

$$\begin{array}{ccccc} T_X^* & \xrightarrow{\Phi} & M_X & \to & \mathfrak{g}^* \\ & {\scriptstyle\Psi}\searrow & \downarrow {\scriptstyle\Pi} & & \downarrow \\ & & L_X & \to & \mathfrak{g}^* /\!\!/ G \end{array}$$

Let $x'' := \Psi(x) = \Pi(x') \in L_X$. By [Kn1] Satz 8.1 the isotropy group G_x is the kernel of the action of $G_{x'}$ on $F := \Phi^{-1}(x')$ and the quotient $G_{x'}/G_x$ is a torus. This implies that F contains a $G_{x'}$-stable open subset of the form $(G_{x'}/G_x) \times F_0$. Thus it suffices to show that F_0 is unirational.

The orbit Gx' is dense in the fiber $\Pi^{-1}(x'')$ ([Kn1] Lemma 7.3). Thus $F_1 := \Psi^{-1}(x'')$ contains the open subsets

$$\Phi^{-1}(Gx') \cong G \overset{G_{x'}}{\times} \Phi^{-1}(x') \supseteq G \overset{G_{x'}}{\times} [G_{x'}/G_x \times F_0] \cong G/G_x \times F_0.$$

In particular, there is a dominant rational morphism $F_1 \to F_0$. Thus, the assertion follows from the next lemma. □

5.2. Lemma. *The general fiber F_1 of $\Psi : T_X^* \to L_X$ is unirational.*

First we will reduce the proof to yet another lemma:

5.3. Lemma. *Let X be any smooth G-variety. Then there exists a B-stable subvariety $Z \subseteq X$ with the following properties:*

a) $k(Z)^B \cong k(X)^B$.

b) All orbits of U in Z have the same dimension. Hence,

$$C_Z := \{\alpha \in T_X^* \mid x := \pi(\alpha) \in Z, \alpha(\mathfrak{u}x) = 0\}.$$

is a vector bundle over Z.

c) *The set $G \cdot C_Z$ is dense in T^*_X.*

Proof of Lemma 5.2: For $\chi \in \mathfrak{t}^* \subseteq \mathfrak{b}^*$ let

$$C_Z(\chi) := \{\alpha \in T^*_X \mid x := \pi(\alpha) \in Z \text{ and } \alpha(\xi x) = \chi(\xi) \text{ for all } \xi \in \mathfrak{b}\} \subseteq C_Z.$$

It is easy to see ([Kn1] Lemma 6.1) that $C_Z(\chi)$ is mapped under

$$C_Z(\chi) \hookrightarrow T^*_X \xrightarrow{\Phi} \mathfrak{g}^* \longrightarrow \mathfrak{g}/\!\!/G = \mathfrak{t}^*/W$$

to one point namely to the image of χ. Recall $x'' \in L_X$ and $F_1 := \Psi^{-1}(x'')$. Hence, $C_Z \cap F_1$ is the finite disjoint union of all $C_Z(\chi)$ where χ runs through all elements of $\mathfrak{t}^*$ which are mapped to x''. Because of property c), there is one χ such that $G \cdot C_Z(\chi)$ is dense in F_1.

Thus, we have to show that $C_Z(\chi)$ is unirational. First observe, that X unirational (having a dense G-orbit) implies $k(Z)^B$ unirational (by a)). Let $Z_0 \subseteq Z$ be a non-empty open subset such that the orbit space Z_0/B exists. Because B is connected and solvable, the fibration $Z_0 \to Z_0/B$ has a rational section ([Bor] 15.12). Thus, Z is unirational as well.

Choose B-semiinvariant rational functions $f_1, \ldots, f_s$ which generate $k(Z)^U$ over $k(Z)^B$. By shrinking Z if necessary, we may assume that all f_i are defined and invertible on Z. Let $\chi_i \in \mathfrak{t}^*$ be the character of f_i, i.e.,

$$\xi f_i(x) = \mathrm{d}f_i(\xi x) = \chi_i(\xi) f_i(x) \text{ for all } \xi \in \mathfrak{t} \text{ and } x \in Z.$$

For every $x \in Z$, these χ_i generate the space $(\mathfrak{b}x/\mathfrak{u}x)^* = (\mathfrak{b}/\mathfrak{b}_x + \mathfrak{u})^* \subseteq \mathfrak{t}^*$ which also contains χ. Hence, there exist $a_i \in k$ such that $\chi = \sum_i a_i \chi_i$. This defines a section

$$\sigma : Z \longrightarrow C_Z(\chi) : x \mapsto \sum_i a_i f_i(x)^{-1} (\mathrm{d}f_i)_x,$$

which induces an isomorphism $C_Z(\chi) \xrightarrow{\sim} C_Z(0) : \alpha \mapsto \alpha - \sigma\pi(\alpha)$. But

$$C_Z(0) = \{\alpha \in T^*_X \mid x := \pi(\alpha) \in Z; \alpha(\mathfrak{b}x) = 0\}$$

is a vector bundle over Z, hence unirational. □

Proof of Lemma 5.3: Let me first indicate another shortcut, this time for quasiaffine varieties: Here, already the open U-sheet in X would do the job. Via the Degeneration Theorem 1.1 this claim can be reduced to X_h which is also quasiaffine. Then an argument as in [Kn1] Korollar 8.2 proves the assertion. Of course, the crucial property is c). It fails badly for a flag variety $X = G/P$ and Z=open B-orbit: Then C_Z lies in the zero-section of T^*_X.

Back to the main proof: We study first the case where X is of the special form $V \times G/S$ where G acts trivially on V and S contains the maximal unipotent subgroup U. Let P be the normalizer of S and L the Levi complement of P containing the maximal torus T. Then we have $R_u S = R_u P =: P_u$ and $L_0 := L \cap S$ is a Levi complement of S and P/S is a torus. Denote the Lie algebras by the corresponding fraktur letters. We will identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of an invariant scalar product. Let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{l}_0$ in $\mathfrak{l}$. Then we have the following decompositions:

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_u = \mathfrak{a} \oplus \mathfrak{l}_0 \oplus \mathfrak{p}_u; \quad \mathfrak{s} = \mathfrak{l}_0 \oplus \mathfrak{p}_u; \quad \mathfrak{s}^\perp = \mathfrak{a} \oplus \mathfrak{p}_u$$

Now we set $Z = V \times Bwx_0$ where w is an element of the Weyl group and $x_0 = eS$ is the canonical base point of G/S. Then conditions a) and b) are obviously satisfied. We show that for a suitable choice of w also c) holds, namely, that $G \cdot C_Z$ is dense in T^*_X. Because of $C_Z = T^*_V \times C_{Bwx_0}$ we may assume that V is a point, i.e., $X = G/S$. For convenience, we shift everything by w, i.e., we look at $\overline{B} := w^{-1}Bw$, $\overline{U} := w^{-1}Uw$ and $\overline{Z} = \overline{B}x_0$. Of course, the definition of $C_{\overline{Z}}$ has to be changed accordingly. Then we have

$$T_{\overline{U}x_0,x_0} = \bar{\mathfrak{u}}x_0 = (\bar{\mathfrak{u}} + \mathfrak{s})/\mathfrak{s} \subseteq \mathfrak{g}/\mathfrak{s} \cong (\mathfrak{s}^\perp)^*.$$

Therefore, the fiber of $C_{\overline{Z}}$ over x_0 is

$$(\bar{\mathfrak{u}} + \mathfrak{s})^\perp = \bar{\mathfrak{u}}^\perp \cap \mathfrak{s}^\perp = \bar{\mathfrak{b}} \cap (\mathfrak{a} \oplus \mathfrak{p}_u) = \mathfrak{a} \oplus (\bar{\mathfrak{b}} \cap \mathfrak{p}_u) \subseteq \mathfrak{a} \oplus \mathfrak{p}_u \cong T^*_{X,x_0}.$$

Let $\mathfrak{p}_u^{\mathfrak{a}}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{p}_u$. Then a general element of $\mathfrak{a} \oplus \mathfrak{p}_u$ can be conjugated by an element of P_u into $\mathfrak{a} \oplus \mathfrak{p}_u^{\mathfrak{a}}$. Now we impose on w the condition

(C) $\qquad \mathfrak{p}_u^{\mathfrak{a}} \subseteq \bar{\mathfrak{b}}$

Then $P_u \cdot [\mathfrak{a} \oplus (\bar{\mathfrak{b}} \cap \mathfrak{p}_u)]$ is dense in $\mathfrak{a} \oplus \mathfrak{p}_u$, hence $G \cdot C_{\overline{Z}}$ is dense in T^*_X. Of course, condition (C) is satisfied for $w = 1$, but later on, we will have less freedom for the choice of w.

Now we reduce the general case to the horospherical one. It is clear that we may replace X by any variety which is G-birational to it. I claim, that there is one with:

i) X is smooth.

ii) X^B is smooth and irreducible with $\dim X^B \geq c(X)$.

iii) The isotropy subgroup of every $x \in X^B$ is P_X and the morphism $G/P \times X^B \to Y := GX^B : (gP, x) \mapsto gx$ is an isomorphism.

First, let me remark that it will become clear from the later discussion, that conditions ii) and iii) mean that $\dim Y$ is actually maximal possible.

A variety X_1 such that $\dim X_1^B \geq c(X)$ exists by [Kn3] Lemma 8.4 (see also [Akh]). In [Kn3] Satz 2.13, I have constructed a projective variety X_2 such that every isotropy group in X_2^B is P_X, the minimal possible group. Let $\overline{X}$ be a desingularization of the closure of the diagonal in $X_1 \times X_2$. Then $\overline{X}$ has already all properties, except that $\overline{X}^B$ may not be smooth and irreducible. Therefore, we can take $X = \overline{X} - GZ$ where Z is a suitable closed subset of $\overline{X}^B$. This proves the claim.

Let $P := P_X$ and P^- the parabolic opposite to P with Levi part $L = P \cap P^-$. Observe, that $A_X = P_X/S_X = L/(L \cap S_X)$ is also a quotient of P^-.

5.4. Lemma. *There is a P^--stable affine open subset X_0 of X with $V := X_0 \cap X^B \neq \varnothing$, and a P^--isomorphism*

$$X_0 \cong P^- \overset{L}{\times} (\mathbf{A}^r \times V) = P_u^- \times \mathbf{A}^r \times V = P_u^- \times \Sigma$$

where L acts by conjugation on P_u^-, trivially on V and on the affine space $\mathbf{A}^r$ by means of characters $\chi_1, \ldots, \chi_r$ which form a basis of $\mathcal{X}(A_X)$. Furthermore, $Y \cap X_0$ corresponds to $P_u^- \times \{0\} \times V$.

Proof: Almost everything follows from the local structure theorem ([BLV] Thm. 1.4, [Kn1] Satz 2.3) applied to (X, Y): There is an L-stable affine subvariety Σ of X with $\Sigma \cap Y \neq \varnothing$ such that $P^- \times^L \Sigma \to X$ is an P^--equivariant isomorphism onto an open subset X_0 of X.

It remains to analyze the structure of Σ. First look at $\Sigma \cap Y$: The map $P^- \times^L (\Sigma \cap Y) \to Y \cong X^B \times G/P$ is an open embedding. It follows, that $\Sigma \cap Y = X_0 \cap X^B =: V$ consists of L-fixed points.

Because of $P = P_X$, the general isotropy group of P^- in X is $L_0 := P^- \cap S_X$ ([Kn1] Satz 2.3). This implies that L_0 is the kernel of the action of L on Σ, i.e., only the torus $A_X = L/L_0 \cong P_X/S_X$ acts effectively on Σ.

Consider now the quotient $\pi_0 : \Sigma \to \Sigma /\!\!/ A_X$. We claim, that V is mapped isomorphically onto $\Sigma /\!\!/ A_X$. It is a closed embedding because $V \subseteq \Sigma^L$. The complexity of X (=minimal codimension of a B^--orbit) is equal to the complexity of Σ with respect to A_X. Hence, $\dim \Sigma /\!\!/ A_X \leq c(X) \leq \dim X^B = \dim V$, which shows the claim.

The last argument also implies that the general fiber of π_0 contains an open A_X-orbit. Thus, the general fiber of π_0 is a smooth affine torus embedding with a fixed point. Hence it is isomorphic to the affine space $\mathbf{A}^r$ with $r = \dim A_X = \operatorname{rk} X$, on which A_X acts by means of characters $\chi_1, \ldots, \chi_r$ which form a basis of the character group of A_X. After shrinking Σ and X_0, we may assume that Σ is isomorphic to $\mathbf{A}^r \times V$ (see e.g. [Bor] 15.12, [Kn4] 6.1). □

Let again $\mathfrak{a}$ be an orthogonal complement of $\mathfrak{l}_0$ in $\mathfrak{l}$. Denote the set of weights of T in $\mathfrak{p}_u^{\mathfrak{a}}$ by Δ_0. It consists of positive roots. The sets Δ_0 and $\{\chi_1, \ldots, \chi_r\}$ lie by construction in the complementary spaces $\mathfrak{a}^\perp$ and $\mathfrak{a}^*$ respectively. Therefore, there is a one parameter subgroup $\lambda : \mathbf{G}_m \hookrightarrow T$ which is strictly positive on $\Delta_0 \cup \{\chi_1, \ldots, \chi_r\}$ and non-zero on every root. We consider the Białynicki-Birula cell of X with respect to λ at V. More specifically, let

$$\begin{aligned}
\overline{B} :=& \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\};\\
\overline{U} :=& \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1\};\\
\overline{Z} :=& \{x \in X \mid \lim_{t\to 0} \lambda(t)x \in V\}.
\end{aligned}$$

Then $\overline{B}$ is a Borel subgroup of G (because λ does not vanish on roots), $\overline{U}$ is its unipotent radical and $\overline{Z}$ is a $\overline{B}$-stable subvariety of X. Furthermore,

$$\Psi : \overline{Z} \to V : x \mapsto \lim_{t\to 0} \lambda(t)x$$

is a $\overline{B}$-invariant morphism, whose fibers are affine spaces. We have $\Sigma \subseteq \overline{Z}$ since λ is positive on the characters χ_i. Thus, Lemma 5.4 implies an $\overline{B} \cap P^-$-isomorphism

$$\overline{Z} \cong (\overline{B} \cap P^-) \overset{\overline{B}\cap L}{\times} (\mathbf{A}^r \times V) \cong (\overline{B} \cap P_u^-) \times \mathbf{A}^r \times V.$$

such that Ψ is just the projection onto the last factor.

Now we show that $\overline{Z}$ satisfies the three conditions a) to c) with respect to $\overline{B}$.

a) The morphism $\Psi : \overline{Z} \to V$ is $\overline{B}$-invariant and $\overline{B}$ (even $\overline{B} \cap P^-$) has a dense orbit in every fiber of Ψ. This and Lemma 5.4 imply $k(X)^B \cong k(V) \cong k(\overline{Z})^{\overline{B}}$.

b) Because of the contracting $\lambda(\mathbf{G}_m)$-action, the lowest dimensional $\overline{U}$-orbits must be those in $Y \cap \overline{Z} \cong (\overline{B} \cap P_u^-) \times V$. Hence all $\overline{U}$-orbits have at least dimension $d := \dim \overline{B} \cap P_u^-$.

The general orbits have largest dimension. Let $x \in \overline{Z}$ be a general point. By $\overline{B} \cap P_u^-$, we can move x into Σ. Then, the $\overline{U}$-invariant map

$$\overline{B}x = \overline{B}/\overline{B}_x \twoheadrightarrow \overline{B}/\overline{B}_x\overline{U} \cong A_X.$$

shows $\dim \overline{U}x \le \dim \overline{B}x - r = d$. Hence, the assertion.

c) For this we use the deformation to the normal bundle of Y (see [Ful] §5.1), i.e., let $\widetilde{X}$ be the result of blowing up $X \times \mathbf{A}^1$ in $Y \times \{0\}$ and removing the proper transform of $X \times \{0\}$. Then the map $\delta : \widetilde{X} \to \mathbf{A}^1$ has the property that the general fiber is isomorphic to X while the zero-fiber $\widetilde{X}_0$ is the normal bundle of Y in X.

Let $x_0 \in V$. Then Lemma 5.4 implies that $G_{x_0} = P$ has a dense orbit in the normal space to Y at x_0 with isotropy group S_X. Thus, the normal bundle $\widetilde{X}_0$ contains $V \times G/S_X$ as an open subset. The proper transform $\widetilde{Z}$ of $\overline{Z} \times \mathbf{A}^1$ intersects this open subset in $V \times \overline{B}x_1$ where $x_1 \in G/S_X$ is the base point. In particular, all $\overline{U}$-orbits in $\widetilde{Z}$ still have dimension d, i.e., it gives rise to a smooth deformation $C_{\widetilde{Z}/\mathbf{A}^1} \to \mathbf{A}^1$ with $C_{\overline{Z}}$ as general fiber and $C_{V \times \overline{B}x_1}$ as special fiber.

Now c) is equivalent to find a point in $G \times C_{\overline{Z}}$ such that $G \times C_{\overline{Z}} \to T^*_X$ is onto at the tangent space of this point. Thus, property c) is an open condition, hence, it suffices to check it for $\widetilde{X}_0$. But now observe, that condition (C) above is fulfilled because we chose λ to be positive on Δ_0. We have already seen that this implies c) for $\widetilde{X}_0$. This finishes the proofs of Lemma 5.3, Lemma 5.2, Theorem 5.1 and Theorem 4.1. □

Remark: The construction of Z as a Białynicki-Birula cell has been inspired by [BL].

6. The algebra of invariant completely regular operators

The main application of the vanishing theorem is:

6.1. Theorem. *Let X be a normal G-variety and $\mathfrak{U}(X)$ its algebra of completely regular differential operators equipped with its natural filtration. Then there are canonical isomorphisms*

$$\operatorname{gr}\mathfrak{U}(X) \xrightarrow{\sim} k[M_X]; \qquad \operatorname{gr}\mathfrak{U}(X)^G \xrightarrow{\sim} k[L_X].$$

Proof: Of course, the second isomorphism follows from the first by linear reductivity of G and $k[L_X] = k[M_X]^G$. We will prove the first isomorphism at the moment only in the case where X contains an open orbit. The proof for the general case will be finished with Corollary 8.2.

We may assume that X is smooth, complete and pseudo-free. Then $\mathfrak{U}(X) = \mathfrak{U}_X(X)$ and $k[M_X] = k[\widetilde{T}_X]$. Hence, the assertion follows from the exact sequence in Corollary 2.7 and the Vanishing Theorem 4.1a, applied to $Y = \operatorname{Spec} k$. □

6.2. Corollary. *Let $\varphi : X \to X'$ be generically étale. Then $\varphi^\circ : \mathfrak{U}(X') \to \mathfrak{U}(X)$ is an isomorphism.*

Proof: The associated graded is an isomorphism by [Kn1] Satz 6.5:3. □

In this section we study first $\mathfrak{U}(X)^G$. Because of its importance we denote it by $\mathfrak{Z}(X)$. In particular, $\mathfrak{Z}(G) = \mathfrak{U}(G)^G$ is the center of the universal enveloping algebra of $\mathfrak{g}$. By Theorem 6.1 we have $\operatorname{gr}\mathfrak{Z}(X) = k[L_X]$.

6.3. Corollary. *Let X be a spherical variety. Then $\mathfrak{Z}(X) = \mathcal{D}(X)^G$, i.e., every G-invariant differential operator on X is completely regular.*

Proof: We may assume that $X = G/H$ is homogeneous. Consider the associated graded algebras:

$$\begin{array}{ccc} \operatorname{gr}\mathfrak{Z}(X) & \hookrightarrow & \operatorname{gr}\mathcal{D}(X)^G \\ \downarrow\sim & & \downarrow \\ k[L_X] & \to & k[T^*_X]^G \end{array}$$

For a spherical variety the bottom arrow is an isomorphism ([Kn1] Satz 7.1). This implies the assertion. □

Now recall the classical Harish-Chandra homomorphism: By the Theorem of Poincaré-Birkhoff-Witt there is a decomposition

$$\mathfrak{U}(G) = \mathfrak{U}(\mathfrak{t}) \oplus (\mathfrak{u}\mathfrak{U}(G) + \mathfrak{U}(G)\mathfrak{u}^-).$$

We can identify $\mathfrak{U}(\mathfrak{t})$ with $k[\mathfrak{t}^*]$. Then Harish-Chandra's Theorem (see [Wal] Thm. 3.2.3) states that

$$p : \mathfrak{Z}(G) \subseteq \mathfrak{U}(G) \overset{\text{proj}}{\twoheadrightarrow} \mathfrak{U}(\mathfrak{t}) = k[\mathfrak{t}^*]$$

induces an isomorphism of $\mathfrak{Z}(G)$ with the Weyl group invariants $k[\mathfrak{t}^*]^{W\bullet}$ where $W\bullet$ indicates the shifted W-action on $\mathfrak{t}^*$:

$$w \bullet \chi = w(\chi + \rho) - \rho \quad \text{with } \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \in \mathfrak{t}^*.$$

Here, Δ^+ is the set of weights in $\mathfrak{u}^-$. Therefore, with the ρ-shift $\tau : \mathfrak{t}^* \to \mathfrak{t}^* : \chi \mapsto \chi - \rho$ we get an isomorphism $\tau^* \circ p : \mathfrak{Z}(G) \to k[\mathfrak{t}^*]^W$.

Now consider a horospherical variety $X_h = G/S_X$. Then $(tS_X)*(gS_X) = gtS_X$ defines a left action of $A_X = P_X/S_X$ on X_h and therefore a homomorphism $\mathfrak{U}(\mathfrak{a}_X) \to \mathcal{D}(X_h)^G$. By Corollary 6.3 and by considering the associated graded algebras we get an isomorphism

$$k[\mathfrak{a}_X^*] \xrightarrow{\sim} \mathcal{D}(X_h)^G = \mathfrak{Z}(X_h).$$

Taking $X_h = G/U$, gives a second homomorphism $\mathfrak{Z}(G) \to \mathfrak{Z}(G/U) \xleftarrow{\sim} k[\mathfrak{t}^*]$. The next lemma is well known (e.g. [BB] Lemme) and claims the compatibility of these two constructions.

6.4. Lemma. *The following diagram commutes:*

$$\begin{array}{ccccc} k[\mathfrak{t}^*]^W & \hookrightarrow & k[\mathfrak{t}^*] & \overset{\text{res}}{\twoheadrightarrow} & k[\rho + \mathfrak{a}_X^*] \\ \uparrow \sim & & \tau^* \uparrow \sim & & \uparrow \sim \\ k[\mathfrak{t}^*]^{W\bullet} & \hookrightarrow & k[\mathfrak{t}^*] & \overset{\text{res}}{\twoheadrightarrow} & k[\mathfrak{a}_X^*] \\ \uparrow \sim & & \downarrow \sim & & \downarrow \sim \\ \mathfrak{Z}(G) & \hookrightarrow & \mathfrak{Z}(G/U) & \twoheadrightarrow & \mathfrak{Z}(X_h) \end{array}$$

Proof: This is only non-trivial for the bottom left square. For $\xi \in \mathfrak{Z}(G)$ let $\xi = p(\xi) + \eta$ with $\eta \in \mathfrak{u}\mathfrak{U}(G) + \mathfrak{U}(G)\mathfrak{u}^-$. Because ξ commutes with $\mathfrak{t}$, we actually have $\eta \in \mathfrak{u}\mathfrak{U}(G)$. Let x_0 be the base point of G/U. Because of G-invariance, we have to show $(\xi f)(x_0) = (p(\xi) * f)(x_0)$ where $f \in k[G/U]$. But that is clear, because $(\mathfrak{u}\mathfrak{U}(G)f)(x_0) = 0$ and because the actions of T on Tx_0 induced by left and right translation on G/U coincide. □

From now on we will identify $\mathfrak{Z}(X_h)$ with $k[\rho + \mathfrak{a}_X^*]$. For any subset $\mathfrak{s}$ of $\mathfrak{t}^*$ let

$$N_W(\mathfrak{s}) := \{w \in W \mid w\mathfrak{s} = \mathfrak{s}\};$$
$$C_W(\mathfrak{s}) := \{w \in W \mid w|_{\mathfrak{s}} = \mathrm{id}_{\mathfrak{s}}\};$$
$$W(\mathfrak{s}) := N_W(\mathfrak{s})/C_W(\mathfrak{s}).$$

We have $C_W(\rho + \mathfrak{a}_X^*) = N_W(\rho + \mathfrak{a}_X^*) \cap C_W(\mathfrak{a}_X^*)$. The isotropy group of ρ is trivial, hence $C_W(\rho + \mathfrak{a}_X^*) = 1$. This yields

$$\begin{array}{ccc} N_W(\rho + \mathfrak{a}_X^*) & \hookrightarrow & W(\mathfrak{a}_X^*) \\ & & \cup\!\mid \\ & & W_X \end{array}$$

6.5. Theorem. *Let X be any G-variety. Then $W_X \hookrightarrow N_W(\rho + \mathfrak{a}_X^*)$ and there is a canonical isomorphism i_h^G such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{Z}(G) & \xrightarrow{\sim} & k[\mathfrak{t}^*]^W \\ \downarrow & & \downarrow \text{res} \\ \mathfrak{Z}(X) & \xrightarrow{i_h^G} & k[\rho + \mathfrak{a}_X^*]^{W_X} \end{array}$$

In particular, $\mathfrak{Z}(X)$ is a polynomial ring.

Proof: Consider a degeneration $\Delta : Z \to \mathbf{A}^1$ as in Theorem 1.1. Then by Lemma 3.5 and Lemma 3.1 we obtain the following homomorphism

$$i_h : \mathfrak{U}(X) = \mathfrak{U}(X \times \mathbf{A}^1) = \mathfrak{U}(Z) \longrightarrow \mathfrak{U}(V \times G/S_X) = \mathfrak{U}(X_h),$$

which induces a homomorphism

$$i_h^G : \mathfrak{Z}(X) \longrightarrow \mathfrak{Z}(X_h) \xrightarrow{\sim} k[\rho + \mathfrak{a}_X^*].$$

The associated graded of i_h corresponds to the surjective finite morphism $M_{X_h} \to M_X$ of [Kn1] Satz 6.4. This shows that i_h and i_h^G are injective. In particular, $\mathfrak{Z}(X)$ is commutative, hence a polynomial ring, and i_h^G is an integral extension. Furthermore, the diagram

$$\begin{array}{ccc} \mathfrak{Z}(G) & \hookrightarrow & k[\mathfrak{t}^*] \\ \downarrow & & \downarrow \\ \mathfrak{Z}(X) & \overset{i_h^G}{\hookrightarrow} & k[\rho + \mathfrak{a}_X^*] \end{array}$$

commutes. With $L_0 := \operatorname{Spec} \mathfrak{Z}(X)$ this translates into

$$\begin{array}{ccc} \rho + \mathfrak{a}_X^* & \hookrightarrow & \mathfrak{t}^* \\ \downarrow & & \downarrow \\ L_0 & \to & \mathfrak{t}^*/W \end{array}$$

Because L_0 is normal, Galois theory shows that there is a subgroup $W_0 \subseteq N_W(\rho + \mathfrak{a}_X^*)$ such that $L_0 = (\rho + \mathfrak{a}_X^*)/W_0$. From $\operatorname{gr} \mathfrak{Z}(X) = k[L_X] = k[\mathfrak{a}_X^*]^{W_X}$ follows $W_0 = W_X$ and hence the assertion.

Finally, the homomorphism i_h^G is independent of the degeneration Z, because it factorizes $k[\mathfrak{t}^*]^W \to k[\rho + \mathfrak{a}_X^*]$ and its associated graded is well defined (see [Kn1] Lemma 6.3). □

Remark: Usually, the ρ-shift is rather annoying. But here, it showed us, that W_X is in a canonical way a subgroup of W and not just a subquotient as in [Kn1].

6.6. Corollary. *Let X be a spherical variety. Then the algebra $\mathcal{D}(X)^G$ of invariant differential operators is a polynomial ring in $\operatorname{rk} X$ generators.*

The homomorphism i_h^G is the Harish-Chandra homomorphism mentioned in the title. There is a little flaw in it: One wants an isomorphism of $\mathfrak{Z}(X)$ with $k[\mathfrak{a}_X^*]^{W_X}$ itself. This is achieved by translation $\mathfrak{a}_X^* \to \rho + \mathfrak{a}_X^* : \chi \mapsto \rho_0 + \chi$ with a fixed point $\rho_0 \in (\rho + \mathfrak{a}_X^*)^{W_X}$. There are two ways to construct ρ_0:

a) Let $N := N_W(\rho + \mathfrak{a}_X^*)$ and take $\rho_0 := \frac{1}{|N|}\sum_{w \in N} w\rho$. Of course, it would suffice to average over W_X but then ρ_0 would depend on W_X and not just on S_X.
b) Choose a W-invariant scalar product on $\mathcal{X}(T) \subseteq \mathfrak{t}^*$. Then take ρ_0 to be the component of ρ which is orthogonal to $\mathfrak{a}_X^*$.

Both constructions are not completely satisfying: The first one has bad functorial properties if one wants to compare different $\mathfrak{a}_X$'s. The second one depends on the choice of a scalar product. The only case when there is no ambiguity in ρ_0 is when $(\mathfrak{a}_X^*)^{W_X} = 0$. This happens e.g. whenever $\operatorname{Aut}^G(X)$ is finite: Indeed, a fixed point in $\mathfrak{a}_X^*$ corresponds to a differential operator of degree one, i.e., to a vector field ξ_*. Because we may assume that X is complete, ξ_* belongs to the Lie algebra of $\operatorname{Aut}^G(X)$. This shows the claim. The condition is in particular satisfied for symmetric varieties. Of course, in this case ρ_0 coincides with the one defined by means of the restricted root system.

7. The full algebra of completely regular operators

Next we study $\mathfrak{U}(X)$. For this we need more results on the associated graded side.

7.1. Lemma. *Let $D_l \subset L_X$ be a prime divisor, D_m its preimage in M_X and D_t a component of its preimage in T_X^*. Then $D_t \to D_m$ is dominant.*

Proof: Same proof as [Kn1] Satz 7.4, except that the divisor D there has to be replaced by one of its components. □

Remark: The morphism $T_X^* \to M_X$ is in general not equidimensional in codimension one, e.g. $G = \mathrm{SL}_2$ and $X = \mathbf{P}^1$.

7.2. Lemma. *The morphism $\Pi : M_X \to L_X$ is flat with irreducible fibers. Furthermore, Π is smooth in codimension one.*

Proof: That Π has irreducible fibers is [Kn1] Lemma 7.3. Because M_X is Cohen-Macaulay (Lemma 4.3) and L_X is smooth it suffices to check for flatness that Π is equidimensional ([EGA] §15.4.2). Let $X_h = G/S_X$. Then $M_X = M_{X_h}/W_X$ and $L_X = L_{X_h}/W_X$. Thus, we may replace X by X_h. But then $T^*_X \to M_X$ is onto. Thus, the assertion follows from the equidimensionality of $T^*_X \to L_X$ ([Kn1] Satz 6.6c).

It remains to show that Π is smooth in codimension one. Suppose there is an irreducible divisor $D_m \subset M_X$ along which Π is not smooth. Because M_X is normal, the general fibers of Π are normal and hence smooth in codimension one. Thus $D_l := \overline{\Pi(D_m)} \subset L_X$ is a proper subset. By equidimensionality it is a divisor. Because all fibers of Π are irreducible, we have $\Pi^{-1}(D_l) = nD_m$ with multiplicity $n \geq 2$.

Consider $\varphi : \mathfrak{a}^*_X \to \mathfrak{a}^*_X/W_X = L_X$. By [Kn1] Lemma 6.2 there is a morphism $\sigma' : \mathfrak{a}^*_X \to M_X$ with $\varphi = \Pi \circ \sigma'$. This implies that D_l is one of the irreducible divisors of L_X over which φ is ramified. Because W_X is a reflection group it also implies $n = 2$. The variety L_X is an affine space and hence factorial. Thus, D_l is the zero set of an irreducible polynomial $f_0 \in k[L_X]$.

Now we consider everything on T^*_X: By Lemma 7.1, every component of $D'_t := \Psi^{-1}(D_l)$ is mapped dominantly to D_m. That implies that $D_t := \frac{1}{2}D'_t$ is an integral divisor, such that $2D_t$ is the principal divisor defined by the invariant function $f = f_0 \circ \Psi$.

We replace now X by G/H, where H is the component of unity of the isotropy group of a general point of X. This does not change M_X by [Kn1] Satz 6.5:3,4. Then we have $T^*_X = G \times^H \mathfrak{h}^\perp$, and therefore $D_t = G \times^H (D_t \cap \mathfrak{h}^\perp)$. Because the polynomial ring $k[\mathfrak{h}^\perp]$ is factorial, $D_t \cap \mathfrak{h}^\perp$ is the zero set of a polynomial $\bar{h}$ which is unique up to a scalar. Hence, it is H-semiinvariant. But $\bar{h}^2 = 0$ defines $2D_t \cap \mathfrak{h}^\perp$. Therefore, $\bar{h}^2 = af|_{\mathfrak{h}^\perp}$ with $a \in k$. Recall that f is G-invariant and the character group $\mathcal{X}(H)$ is torsion free. This implies that $\bar{h}$ is H-invariant. Hence, $\bar{h}$ defines a G-invariant function h on T^*_X with $h^2 = f = f_0 \circ \Psi$. But this contradicts the fact that $k[L_X]$ is (by definition) integrally closed in $k[T^*_X]$. □

Now we get a refinement of [Kn1] Satz 6.4:

7.3. Corollary. *Let X be a G-variety. Then there is a canonical isomorphism*

$$\gamma : M_{X_h} \xrightarrow{\sim} M_X \times_{L_X} \mathfrak{a}^*_X$$

which induces a W_X-action on M_{X_h} and an isomorphism

$$M_{X_h}/W_X \xrightarrow{\sim} M_X.$$

Proof: In the proof of [Kn1] Satz 6.4 we constructed already γ and showed that it is finite and birational. Thus, we have to show that $M' := M_X \times_{L_X} \mathfrak{a}^*_X$ is normal.

The flatness of $\mathfrak{a}^*_X \to L_X$ (=quotient by a reflection group) implies that $M' \to M_X$ is flat. Thus M' is Cohen-Macaulay because M_X is. Now $M_X \to L_X$ is smooth in codimension one. This implies that M' is smooth in codimension one. Now we conclude with Serre's normality criterion. □

7.4. Lemma. *Let $X = G/S$ be horospherical with $P = N_G(S)$ and $A = P/S$. Then $\mathfrak{U}(X) = \mathcal{D}(X)^A$.*

Proof: We have $M_X = \operatorname{Spec} k[G \times^P \mathfrak{s}^\perp]$ (see [Kn1] 4.1). The torus A acts trivially on $\operatorname{gr} \mathfrak{U}(X) = k[M_X]$, hence on $\mathfrak{U}(X)$. This implies $\mathfrak{U}(X) \subseteq \mathcal{D}(X)^A$. Equality follows from the equality of the associated graded algebras. □

7.5. Corollary. *Let X be any G-variety. Then $\mathfrak{Z}(X)$ is the center of $\mathfrak{U}(X)$. Furthermore, $\mathfrak{U}(X)$ is a free module over $\mathfrak{Z}(X)$.*

Proof: Let Z be the center of $\mathfrak{U}(X)$. It commutes with the image of $\mathfrak{U}(G)$ which implies $Z \subseteq \mathfrak{U}(X)^G = \mathfrak{Z}(X)$. With i_h, we can identify $\mathfrak{U}(X)$ with a subalgebra of $\mathfrak{U}(X_h)$. Then $\mathfrak{Z}(X) \subseteq \mathfrak{Z}(X_h) = \mathfrak{U}(\mathfrak{a}_X)$ commutes with all of $\mathfrak{U}(X_h)$ (Lemma 7.4), hence is contained in Z.

By Lemma 7.2, the morphism $M_X \to L_X$ is flat. This means that $k[M_X]$ is a flat $k[L_X]$-module. Because both rings are positively graded, the former is even a free module ([Bbk] Ch. 2, §11, no. 4, Prop. 7). If we now lift any homogeneous base of $k[M_X]$ to $\mathfrak{U}(X)$ we get a $\mathfrak{Z}(X)$-base. □

The non-commutative counterpart of Corollary 7.3 is part a) of the following theorem.

7.6. Theorem. *Let X be a G-variety.*

a) The homomorphism

$$\mathfrak{U}(X) \underset{\mathfrak{Z}(X)}{\otimes} \mathfrak{Z}(X_h) \xrightarrow{i_h \otimes \mathrm{id}} \mathfrak{U}(X_h)$$

is an isomorphism. It induces a canonical action of W_X on $\mathfrak{U}(X_h)$ and an isomorphism

$$\mathfrak{U}(X) \xrightarrow{\sim} \mathfrak{U}(X_h)^{W_X}.$$

b) There exists $\xi \in \mathfrak{Z}(X)$, $\xi \neq 0$ with $\xi \mathfrak{U}(X) \subseteq \mathfrak{U}(G)\mathfrak{Z}(X) \subseteq \mathfrak{U}(X)$.

Proof: a) Both sides are filtered, and it suffices to show that $i_h \otimes \mathrm{id}$ induces an isomorphism on the graded algebras. Let $\mathfrak{A} := \mathfrak{U}(X) \otimes_{\mathfrak{Z}(X)} \mathfrak{Z}(X_h)$. In view of Corollary 7.3 we have to show that $\operatorname{gr} \mathfrak{A} = k[M_X \times_{L_X} \mathfrak{a}_X^*]$.

Because $k[L_X]$ is the ring of invariants of a finite reflection group, $k[\mathfrak{a}_X^*]$ is a free $k[L_X]$-module. Let $\bar\xi_1, \ldots, \bar\xi_s$ be a basis and let d_i be the degree of $\bar\xi_i$. Now lift $\bar\xi_i$ to $\xi_i \in \mathfrak{Z}(X_h)$. Then the ξ_i form a basis of the $\mathfrak{Z}(X)$-module $\mathfrak{Z}(X_h)$. Thus we have

$$\mathfrak{A}^{(n)} = \bigoplus_{i=1}^{s} \mathfrak{U}(X)^{(n-d_i)} \otimes \xi_i.$$

which implies the assertion:

$$\operatorname{gr}\mathfrak{A} = \bigoplus_{i=1}^{s} \operatorname{gr}\mathfrak{U}(X)\otimes\bar{\xi}_i = k[M_X] \underset{k[L_X]}{\otimes} k[\mathfrak{a}_X^*] = k[M_X \underset{L_X}{\times} \mathfrak{a}_X^*].$$

b) It follows from [Kn1] Satz 4.5 and Lemma 7.4, that there is $\xi \in \mathfrak{Z}(X_h)$ such that $\xi\mathfrak{U}(X_h) \subseteq \mathfrak{U}(G)\mathfrak{Z}(X_h)$. Clearly we may assume that $\xi \in \mathfrak{Z}(X) = \mathfrak{Z}(X_h)^{W_X}$. Then the assertion follows from the fact that $\mathfrak{U}(G)\otimes_k \mathfrak{Z}(X_h) \to \mathfrak{U}(X_h)$ is W_X-equivariant. □

This theorem and Theorem 6.5 show that $\mathfrak{U}(X)$ depends up to unique isomorphism only on X_h and the action of W_X on $\mathfrak{a}_X^*$. Note, that the graded counterpart of b) is wrong in general. Take, for example, $X = G/P$ a flag variety (in particular $\mathfrak{Z}(X) = k$). Then the graded homomorphism comes from the moment map $T_X^* \to \mathfrak{g}^*$, which in general is not birational onto its image. Thus, the assertion in b) is one of the big advantages of the non-commutative theory. There are examples where $\mathfrak{U}_0 := \mathfrak{U}(G)\mathfrak{Z}(X) \neq \mathfrak{U}(X)$. It is an open problem to determine $\mathfrak{U}_0$, or the ideal of ξ's as in b) (the conductor), or just the cases in which equality holds.

7.7. Corollary. *Let φ be a G-automorphism of X. Then $\varphi^\circ : \mathfrak{U}(X) \to \mathfrak{U}(X)$ is the identity.*

Proof: It suffices to prove this for the automorphism induced on $\mathfrak{Z}(X)$ (Theorem 7.6b). Because $k[L_X]$ determines $\mathfrak{Z}(X)$ uniquely (Theorem 6.5), we have to show, that the automorphism on L_X is the identity. But L_X is canonically a quotient of $\mathfrak{a}_X^* = (\operatorname{Lie} P_X/S_X)^*$ and S_X is obviously not affected by φ. □

8. The rigidity theorem

We have seen that $\mathfrak{U}(X)$ depends only on the discrete parameters S_X and W_X. Thus it is not surprising that it is generically rigid if X runs through a smooth family. The next theorem says that this rigidity extends even over the singular fibers.

8.1. Theorem. *Let X be a normal G-variety and $\psi : X \to Y$ a G-invariant proper morphism onto a normal variety Y. Assume that all fibers are connected. Then, there is a unique isomorphism $\psi_\circ : \psi_*\overline{\mathfrak{U}}_X \to \mathcal{O}_Y \otimes_k \mathfrak{U}(X)$ and a non-empty open subset of $Y_0 \subseteq Y$ with: For every $y \in Y_0$ with $\iota : F := \psi^{-1}(y) \hookrightarrow X$ there is a unique isomorphism $\mathfrak{U}(F) \to \mathfrak{U}(X)$ and equality $F_h = X_h$ such that the following diagram commutes:*

$$\begin{array}{ccccc} \psi_*\overline{\mathfrak{U}}_X & \xrightarrow{\iota^\circ} & \mathfrak{U}(F) & \overset{i_h}{\hookrightarrow} & \mathfrak{U}(F_h) \\ \psi_\circ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ \mathcal{O}_Y \otimes_k \mathfrak{U}(X) & \xrightarrow{\varepsilon_y} & \mathfrak{U}(X) & \overset{i_h}{\hookrightarrow} & \mathfrak{U}(X_h) \end{array}$$

Here, ε_y is evaluation at y.

Proof: The sheaf $\psi_*\overline{\mathfrak{U}}_X$ is torsion free as $\mathcal{O}_Y$-module. Hence, $\psi_\circ$ is uniquely determined by its restriction to Y_0 and thus by the commutativity of the diagram. Therefore, we have to prove the existence.

We start with some reductions. Choose resolutions of singularities $\varphi : \widetilde{Y} \to Y$ and $\widetilde{\varphi} : \widetilde{X} \to X$ with a morphism $\widetilde{\psi}$ such that the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\varphi}} & X \\ \downarrow \widetilde{\psi} & & \downarrow \psi \\ \widetilde{Y} & \xrightarrow{\varphi} & Y \end{array}$$

commutes. Because the general fibers of ψ and $\widetilde{\psi}$ are mapped birationally onto each other and because of

$$\psi_*\overline{\mathfrak{U}}_X = \psi_*\widetilde{\varphi}_*\mathfrak{U}_{\widetilde{X}} = \varphi_*\widetilde{\psi}_*\mathfrak{U}_{\widetilde{X}} \xrightarrow{\varphi_*\widetilde{\psi}_\circ} \varphi_*\mathcal{O}_{\widetilde{Y}} \underset{k}{\otimes} \mathfrak{U}(\widetilde{X}) = \mathcal{O}_Y \underset{k}{\otimes} \mathfrak{U}(X)$$

it suffices to construct $\widetilde{\psi}_\circ$, i.e., we may assume X and Y to be smooth. In the same manner we may assume that X is pseudo-free. Finally, we can modify X in such a way that ψ factors in $X \xrightarrow{\psi'} Y' \xrightarrow{\psi''} Y$, such that ψ' is G-invariant and its general fiber contains a dense orbit:

$$\psi_*\mathfrak{U}_X = \psi''_*\psi'_*\mathfrak{U}_X \xrightarrow{\sim} \psi''_*\mathcal{O}_{Y'} \underset{k}{\otimes} \mathfrak{U}(X) = \mathcal{O}_Y \underset{k}{\otimes} \mathfrak{U}(X)$$

By enlarging X and Y we may even both assume to be complete.

Consider now the vector bundle $\pi : \widetilde{T}_X \to X$. By [Kn1] Satz 6.5:4, we have $M_F = M_X$, where F is a general fiber of ψ. It follows, that $\widetilde{T}_X \to Y \times M_X$ is the Stein factorization of $\widetilde{T}_X \to Y \times \mathfrak{g}^*$. Therefore, we obtain $\psi_*\pi_*\mathcal{O}_{\widetilde{T}_X} = \mathcal{O}_Y \otimes_k k[M_X]$. By applying the Vanishing Theorem 4.1 to the exact sequence in Corollary 2.7 we get

$$\operatorname{gr}\psi_*\mathfrak{U}_X \xrightarrow{\sim} \mathcal{O}_Y \underset{k}{\otimes} k[M_X].$$

In particular, $\psi_*\mathfrak{U}_X$ is locally free as $\mathcal{O}_Y$-module.

Now choose a non-empty open subset $Y_0 \subseteq Y$ such that for every $y \in Y_0$ with fiber $F := \psi^{-1}(y)$ the following are satisfied:

— F is smooth.

— F is pseudo-free; this is satisfied whenever F contains an element of the generic sheet.

— $F_h = X_h$; this is possible by the definition of X_h ([Kn1] Korollar 2.4).

— $W_F = W_X$; this is possible because W_X depends only on the general orbit ([Kn1] Satz 6.5:4).

The algebra $\mathfrak{U}_0 := \mathfrak{U}(F)$ depends only on F_h and W_F and is hence independent of $y \in Y_0$. Furthermore, for any $y \in Y_0$ and $n \in \mathbb{N}$ we have by Theorem 4.1

$$\mathrm{H}^1(F, \mathfrak{U}_X^{(n)}|_F) = \mathrm{H}^1(F, \mathfrak{U}_F^{(n)}) = 0.$$

This implies ([Har] Ch. III, Thm. 12.11) an isomorphism

$$(\psi_* \mathfrak{U}_X)_y \xrightarrow{\sim} \mathrm{H}^0(F, \mathfrak{U}_F) = \mathfrak{U}_0.$$

Thus, we get a unique homomorphism

$$\psi_\circ^0 : \psi_* \mathfrak{U}_X|_{Y_0} \longrightarrow \mathcal{O}_{Y_0} \underset{k}{\otimes} \mathfrak{U}_0$$

such that the diagram in the theorem commutes with $\mathfrak{U}(X)$ replaced by $\mathfrak{U}_0$. We have to show that $\psi_\circ^0$ extends to all of Y. Because it is a map between locally free sheaves it suffices to extend it in codimension one, i.e., we may assume that Y is the spectrum of a discrete valuation ring R with maximal ideal tR and $Y_0 = \operatorname{Spec} R[t^{-1}]$. Thus $\psi_* \mathfrak{U}_X$ is a subalgebra of $R[t^{-1}] \otimes_k \mathfrak{U}_0$.

The algebra $\operatorname{gr} \psi_* \mathfrak{U}_X = R \otimes k[M_X]$ is finitely generated as $R \otimes_k k[\mathfrak{g}^*]$-module. This implies that $\psi_* \mathfrak{U}_X$ is a finitely generated $R \otimes_k \mathfrak{U}(G)$-module. Because the latter algebra is clearly mapped into $R \otimes_k \mathfrak{U}_0$, there is an $n \in \mathbb{N}$ with $\psi_* \mathfrak{U}_X \subseteq t^{-n} R \otimes_k \mathfrak{U}_0$. It is easy to see, that this implies $\psi_* \mathfrak{U}_X \subseteq R \otimes_k \mathfrak{U}_0$, i.e., $\psi_\circ^0$ extends to Y. Because $\psi_\circ$ induces an isomorphism on the associated graded algebras it is itself an isomorphism.

Finally, assume Y to be complete again. Then,

$$\mathfrak{U}(X) = \mathrm{H}^0(Y, \psi_* \mathfrak{U}_X) = \mathrm{H}^0(Y, \mathcal{O}_Y \underset{k}{\otimes} \mathfrak{U}_0) = \mathfrak{U}_0.$$

This proves the theorem. □

The following corollary finishes the proof of Theorem 6.1 and therefore of all following theorems which have so far only been proved in the quasihomogeneous case.

8.2. Corollary. *Let X be a normal G-variety. Then there is a G-stable non-empty open subset $X_0 \subseteq X$ such that for every $x \in X_0$ the restriction $\mathfrak{U}(X) \to \mathfrak{U}(Gx)$ is an isomorphism.*

Proof: First, shrink X such that an orbit space $Y = X/G$ exists. Then embed X into $\overline{X}$, such that the orbit map extends to a proper morphism $\psi : \overline{X} \to Y$. Then apply Theorem 8.1 and $X_0 = X \cap \psi^{-1}(Y_0)$ has the required properties. □

9. Invariant differential operators

In this section we study the algebra $\mathcal{D}(X)^G$ of invariant (global) differential operators on X.

9.1. Theorem. *Let X be any smooth G-variety. Then the commutant of $\mathfrak{U}(X)$ in $\mathcal{D}(X)$ is $\mathcal{D}(X)^G$.*

Proof: Let $\mathcal{C}$ be the commutant. To commute with the image of $\mathfrak{U}(G)$ means precisely to be invariant. Thus, $\mathcal{C} \subseteq \mathcal{D}(X)^G$.

For the converse, we have to show that every $D \in \mathcal{D}(X)^G$ commutes with $\mathfrak{U}(X)$. By Theorem 7.6b, it suffices to prove that it commutes with $\mathfrak{Z}(X)$. Suppose, that it doesn't. Then, the derivation

$$\delta : \mathfrak{Z}(X) \longrightarrow \mathcal{D}(X) : \xi \mapsto [D, \xi] = D\xi - \xi D.$$

is non-trivial. Let $m \in \mathbb{Z}$ be the smallest number such that

$$\delta(\mathfrak{Z}(X)^{(n)}) \subseteq \mathcal{D}(X)^{(n+m)} \quad \text{for all } n \in \mathbb{N}.$$

Obviously, $m \leq \operatorname{ord} D - 1$, hence m exists. Then, δ induces a non-trivial derivation $\bar{\delta} : k[L_X] \to k[T^*_X]$ of degree m. By definition, δ vanishes on the image of $\mathfrak{Z}(G)$. This implies, that $\bar{\delta}$ vanishes on the image R of $k[\mathfrak{t}^*]^W$ in $k[L_X]$. But $k[L_X]$ is algebraic over R, which implies $\bar{\delta} \equiv 0$ contradicting the non-triviality of $\bar{\delta}$. □

9.2. Corollary. $\mathfrak{Z}(X)$ *is contained in the center of* $\mathcal{D}(X)^G$.

It would be nice to have equality. For this, one has to construct for every D not in $\mathfrak{Z}(X)$ a D' which does not commute with D. I can solve this problem only in two cases: X is spherical, when no such D exists (Corollary 6.6) or X is smooth and affine. In the latter case, let me first explain the graded situation: The cotangent bundle T^*_X carries a natural symplectic structure, which induces on $k[T^*_X]$ a Poisson product

$$\{f_1, f_2\} := \omega(\mathrm{d}f_1, \mathrm{d}f_2),$$

where ω is the symplectic form on the cotangent spaces of T^*_X.

It is related to differential operators as follows: Let $D_1, D_2 \in \mathcal{D}(X)$ with principal symbols $\overline{D}_1, \overline{D}_2 \in k[T^*_X]$. Then we have the relation $\{\overline{D}_1, \overline{D}_2\} = \overline{[D_1, D_2]}$.

We say that $f_1, f_2 \in k[T^*_X]$ commute, if $\{f_1, f_2\} = 0$. Then we can speak about the center of $k[T^*_X]^G$, of commutants and so on. From now on, we consider $k[M_X]$ and $k[L_X]$ as subalgebras of $k[T^*_X]$. The field of rational functions on T^*_X is denoted by $k(T^*_X)$.

9.3. Lemma. *Let X be a smooth affine G-variety. Then the general G-orbits in T^*_X and M_X are closed. This implies that $k(T^*_X)^G$ is the quotient field of $k[T^*_X]^G$.*

Proof: The first assertion holds also for quasiaffine varieties. The second is then a formal consequence for reductive group actions on affine varieties. Let $\widetilde{M}_X \subseteq \mathfrak{g}^*$ be the closure of $\operatorname{Im}\Phi$ where $\Phi : T^*_X \to \mathfrak{g}^*$ is the moment map. The general isotropy group in T^*_X is reductive ([Kn1] Korollar 8.2). By [Kn1] Satz 8.1 the same holds for $\overline{M}_X$. Because $\overline{M}_X \subseteq \mathfrak{g}^*$ the general orbit is closed in $\overline{M}_X$. Choose $y \in \overline{M}_X$ general. For all $x \in T^*_X$ we have $\mathfrak{g}x = [\ker(\mathrm{d}\Phi)_x]^\perp$ (see e.g. [Kn1] p. 21) Because $\Phi^{-1}(y)$ is smooth, the orbit dimension in $\Phi^{-1}(Gy)$ is constant. Hence, the orbits are closed. □

9.4. Theorem. *Let X be a smooth, affine G-variety. Then*

a) The algebras $k[M_X]$ and $k[T^*_X]^G$ are the commutants of each other inside $k[T^*_X]$.

b) Their intersection $k[L_X]$ is the center of both $k[M_X]$ and $k[T^*_X]^G$.

c) The algebras $k[T^*_X]$, $k[T^*_X]^G$ and $k[M_X]$ are free $k[L_X]$-modules.

Proof: a) is proved in [Kn1] Satz 7.6 (see also the first remark after it), provided $k(T^*_X)^G$ is the quotient field of $k[T^*_X]^G$. Part b) is obvious from a). All algebras are flat $k[L_X]$-modules ($k[T^*_X]$: [Kn1] Satz 6.6c; $k[T^*_X]^G$ as a direct summand of $k[T^*_X]$; $k[M_X]$: Lemma 7.2), hence free ([Bbk] Ch. 2, §11, no. 4, Prop. 7). □

Here is the non-commutative analogue:

9.5. Theorem. *Let X be a smooth, affine G-variety. Then*

a) The algebras $\mathfrak{U}(X)$ and $\mathcal{D}(X)^G$ are the commutants of each other inside $\mathcal{D}(X)$.

b) Their intersection $\mathfrak{Z}(X)$ is the center of both $\mathfrak{U}(X)$ and $\mathcal{D}(X)^G$.

c) The algebras $\mathcal{D}(X)$, $\mathcal{D}(X)^G$ and $\mathfrak{U}(X)$ are free $\mathfrak{Z}(X)$-modules (the first one either as left or as right module).

Proof: The associated graded of the various objects are:

$$\operatorname{gr}\mathcal{D}(X) = k[T^*_X];\ \operatorname{gr}\mathcal{D}(X)^G = k[T^*_X]^G;\ \operatorname{gr}\mathfrak{U}(X) = k[M_X];\ \operatorname{gr}\mathfrak{Z}(X) = k[L_X].$$

The first equality uses that X is affine, the second that G is linearly reductive and the last two are Theorem 6.1. Thus c) follows from Theorem 9.4, while b) is a consequence of a). It remains to show that $\mathfrak{U}(X)$ equals the commutant C of $\mathcal{D}(X)^G$. But this follows from $\mathfrak{U}(X) \subseteq C$, $\operatorname{gr} C \subseteq k[M_X]$ (by Theorem 9.4) and $\operatorname{gr}\mathfrak{U}(X) = k[M_X]$. □

The statements c) generalize results of Kostant [Kos] for $\mathfrak{U}(G)$ and Kostant-Rallis [KR] on $\mathcal{D}(X)$ for X symmetric. Of course, we used [Kos] to derive our theorems.

9.6. Corollary. *Let X be a smooth affine G-variety. Then the center of $\mathcal{D}(X)^G$ is a polynomial ring in* $\operatorname{rk} X$ *generators.*

The following lemma shows that there are no relations between $\mathfrak{U}(X)$ and $\mathcal{D}(X)^G$ except the obvious ones:

9.7. Lemma. *Let X be any smooth G-variety. Then, the homomorphisms*

$$\mathfrak{U}(X) \underset{\mathfrak{Z}(X)}{\otimes} \mathcal{D}(X)^G \to \mathcal{D}(X) \qquad \textit{and} \qquad k[M_X] \underset{k[L_X]}{\otimes} k[T_X^*]^G \to k[T_X^*]$$

are injective.

Proof: Because $\operatorname{gr}\mathfrak{U}(X)$ is a free $\operatorname{gr}\mathfrak{Z}(X)$-module, the same argument as in the proof of Theorem 7.6 shows, that the associated graded of the left hand side is the tensor product of the associated graded rings. Hence, it suffices to prove the second assertion.

Let $T^0 \subseteq T_X^*$ be open, non-empty, G-stable such that the orbit space $Y = T^0/G$ exists and is affine. Because $k[M_X]$ is a flat $k[L_X]$-module, it suffices to show that $k[M_X \times_{L_X} Y] \to k[T^0]$ is injective. Now, $Z := M_X \times_{L_X} Y$ is flat over Y with reduced and irreducible general fibers. Hence, Z is reduced, irreducible and we have to prove that $T^0 \to Z$ is dominant. Let $x \in T^0$ be general with image $y \in M_X$. The fiber F of $T^0 \to Z$ containing x is $\Phi^{-1}(y) \cap Gx = G_y x$. Therefore by [Kn1] Satz 8.1, $\dim F = \operatorname{rk} X = \dim L_X$. This implies that the image of T^0 in Z has the same dimension as Z. □

Of course, the image of the homomorphisms above is contained in the commutant $\mathfrak{C}(X)$ of $\mathfrak{Z}(X)$ in $\mathcal{D}(X)$ respectively in the commutant $C(X)$ of $k[L_X]$ in $k[T_X^*]$. Let $C_X := \operatorname{Spec} C(X)$.

9.8. Theorem. *Let X be a smooth affine G-variety. Then:*

a) The homomorphism $\mathfrak{U}(X) \otimes_{\mathfrak{Z}(X)} \mathcal{D}(X)^G \to \mathfrak{C}(X)$ is an isomorphism.

b) The morphism $M_X \times_{L_X} (T_X^* /\!\!/ G) \leftarrow C_X$ is an isomorphism.

Proof: Of course, we have $\operatorname{gr}\mathfrak{C}(X) \subseteq C(X)$. Therefore, it suffices to prove b).

For any $f \in k[T_X^*]$ let H_f be the corresponding Hamiltonian vector field. By definition of $C(X)$ we have $\mathrm{d}f(H_h) = \{f, h\} = 0$ for every $f \in C(X)$ and $h \in k[L_X]$. For every $x \in T_X^*$ this implies that $(\mathrm{d}f)_x$ vanishes on the subspace of the tangent space at x, which is spanned by the H_h with $h \in k[L_X]$. Generically, this subspace has dimension $r = \dim L_X$, which implies $\dim C_X \le \dim T_X^* - r$.

Let again $Z := M_X \times_{L_X} (T_X^* /\!\!/ G)$. The argument in the proof of Lemma 9.7 shows that the general fiber of $T_X^* \to Z$ equals $G_{\Phi(x)} x$ which has dimension r and is connected. Therefore, the factorization $T_X^* \to C_X \to Z$ implies that $C_X \to Z$ is birational. The assertion follows now from the following lemma (see e.g. [Lun] Lemme 1.8). □

9.9. Lemma. *The variety $Z := M_X \times_{L_X} (T^*_X /\!\!/ G)$ is normal and $T^*_X \to Z$ is surjective in codimension one.*

Proof: Because $M_X \to L_X$ is smooth in codimension one, also Z is smooth in codimension one. Let $Q := T^*_X /\!\!/ G$. Both M_X (Lemma 4.3) and Q (Hochster-Roberts, see [Bou]) are Cohen-Macaulay. Because $k[L_X]$ is a polynomial ring and $M_X \to L_X$ is flat, Z is defined inside $M_X \times Q$ by a regular sequence. Hence, Z is Cohen-Macaulay and therefore normal by Serre's criterion.

It remains to show that $T^*_X \to Z$ is surjective in codimension one. Let $D_z \subset Z$ be a prime divisor whose general point is not in the image of T^*_X. Every fiber of $M_X \to L_X$ contains a dense G-orbit and the general fiber is an orbit (Lemma 9.3). Hence, the same is true for $Z \to Q$. Thus, because D_z is G-invariant, there is a divisor $D_q \subseteq Q$ with $D_z = M_X \times_{L_X} D_q$. But $T^*_X \to Q$ is a quotient morphism and therefore surjective. Hence, there is a prime divisor $D_t \subset T^*_X$ which is mapped onto D_q. Because $D_t \to D_z$ is not dominant, the fibers of $D_z \to D_q$ cannot be orbits. This implies that the closure D_l of the image of D_q in L_X is a proper subset. By flatness it is a divisor. Let $D_m \subset M_X$ be the preimage of D_l. Every fiber of $D_m \to D_l$ contains a dense orbit. Let $D^0_m \subseteq D_m$ be the union of these. Because the projection $D_t \to D_m$ is dominant by Lemma 7.1, the preimage D^0_t of D^0_m is non-empty. The set $D^0_z := D^0_m \times_{D_l} D_q$ is of codimension one in Z, hence open in D_z. But $D^0_t \to D_q$ is dominant and every fiber of $D^0_z \to D_q$ is an orbit. This implies that $D_t \to D_z$ is dominant, in contradiction to the choice of D_z. $\square$

The following diagrams summarize the results of this section:

$$
\begin{array}{ccccc}
 & & \mathcal{D}(X) & & \\
 & & \uparrow & & \\
 & & \mathfrak{C}(X) & & \\
 & \nearrow & & \nwarrow & \\
\mathfrak{U}(X) & & & & \mathcal{D}(X)^G \\
 & \nwarrow & & \nearrow & \\
 & & \mathfrak{Z}(X) & & \\
 & & \uparrow & & \\
 & & k & &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & T^*_X & & \\
 & & \downarrow r & & \\
 & & C_X & & \\
 & \swarrow 2c & & 2u \searrow & \\
M_X & & & & T^*_X /\!\!/ G \\
 & \searrow 2u & & 2c \swarrow & \\
 & & L_X & & \\
 & & \downarrow r & & \\
 & & * & &
\end{array}
$$

— Each variety on the right hand side is the spectrum of the associated graded of the corresponding algebra on the left hand side.

— Each algebra is the commutant of the algebra opposite to it in the diagram.

— All homomorphism from $\mathfrak{Z}(X)$ and all morphisms to L_X are flat.

— The numbers at the arrows in the right diagram indicate relative dimensions. Here, $r = \operatorname{rk} X$, $c = c(X)$, $u = \max_{x \in X} \dim Ux$ with the relation $\dim X = r + c + u$.

10. Examples

It is not easy to work out examples because the calculations soon become very involved. Also, it is not easy to find "interesting" examples. In the vast majority of all cases, the little Weyl group W_X coincides with the normalizer $N_W(\rho+\mathfrak{a}_X^*)$. That means, that every differential operator in $\mathfrak{Z}(X)$ is a quotient of operators induced by $\mathfrak{Z}(G)$. This happens for example for all symmetric varieties. Also the opposite case, when W_X is trivial, is not so interesting, because then X is a horospherical variety ([Kn1] Satz 9.1) and $\mathfrak{Z}(X)$ is generated by the vectorfields induced by the right A_X-action.

There is a classification of affine homogeneous spherical varieties ([Krä], [Mik], [Br1]). The most interesting cases out of that list are $G = \mathrm{SO}_{2n+1}$ and $H = \mathrm{GL}_n \subset G$ embedded as the Levi subgroup of the last maximal parabolic subgroup. We consider $X = G/H$. Let $V = k^n$ be the standard representation of H. Then we have as an H-representation

$$\mathfrak{h}^\perp = V \oplus V^* \oplus \bigwedge^2 V \oplus \bigwedge^2 V^*.$$

An easy calculation, based on induction on n, shows that the general isotropy group of H in $\mathfrak{h}^\perp$ and hence of G in T_X^* is trivial. Thus, [Kn1] Satz 8.1 implies $S_X = U$ and $\mathfrak{a}_X^* = \mathfrak{t}^*$. It follows, that the general isotropy group of B on X is trivial. A quick dimension calculation shows that X is spherical. This implies $k[L_X] = k[\mathfrak{h}^\perp]^H$. With the help of the tables in [Sch] one figures out that this ring is generated by functions of degree

$$\begin{array}{ll} 2,2,4,4,\ldots,2m,2m & \text{for } n=2m \\ 2,2,4,4,\ldots,2m,2m,2m+2 & \text{for } n=2m+1 \end{array}$$

The only reflection subgroup of $W = \mathsf{BC}_n$ having these degrees is

$$W_X = \begin{cases} \mathsf{BC}_m \oplus \mathsf{BC}_m & \text{for } n=2m; \\ \mathsf{BC}_m \oplus \mathsf{BC}_{m+1} & \text{for } n=2m+1; \end{cases}$$

There is only one ambiguity for $n = 5$ when also $\mathsf{A}_1 \oplus \mathsf{D}_4$ is possible. To rule this out and to determine the embedding $W_X \hookrightarrow W$, I will use another tool: There is a W_X-stable lattice $\Gamma_X \subseteq \mathfrak{a}_X^*$ (see [Br2] or [Kn4] 4.2), namely the group of characters χ_f of B-semiinvariant rational functions f on X (see [Kn4] §2, last formula). Because X is affine, homogeneous the algebraic Peter-Weyl theorem implies that Γ_X is also generated by the set of weights χ such that the corresponding irreducible module M_χ contains a non-zero H-fixed vector.

Returning to our example, let $\varepsilon_1, \ldots, \varepsilon_n$ be the canonical generators of the weight lattice Γ of SO_{2n+1}. Then one checks $\Gamma_X = \Gamma$, hence no progress. But H is of index 2 in its normalizer N. An easy calculation shows

$$\Gamma_{G/N} = \{\textstyle\sum a_i\varepsilon_i \in \Gamma \mid \sum_{i\equiv n(2)} a_i \text{ is even}\}.$$

This implies that $W_{G/N} = W_X$ is a subgroup and hence equal to the group claimed above, and that the embedding is given by the decomposition

$$\Gamma = \langle \varepsilon_i \mid i \text{ odd} \rangle \oplus \langle \varepsilon_i \mid i \text{ even} \rangle.$$

Finally, the ring extension $\mathfrak{Z}(G) \hookrightarrow \mathfrak{Z}(X)$ has degree $\binom{n}{m}$.

Irreducible representations which are spherical as G-varieties have been classified by Kac [Kac]. These representations are called multiplicity-free. The ring of invariant differential operators has been worked out by Howe-Umeda [HU]. The table is an extract and a reinterpretation of their results. Only representations containing the scalars are considered.

	X		$\operatorname{rk} X$	$\mathfrak{a}_X^* \subseteq \mathfrak{t}^*$	W_X
i)	$\mathrm{GL}_m \otimes \mathrm{GL}_n$	$m \geq n \geq 1$	n	$\varepsilon_1 + \varepsilon_1', \ldots, \varepsilon_n + \varepsilon_n'$	A_{n-1}
ii)	$S^2\,\mathrm{GL}_n$	$n \geq 1$	n	$\varepsilon_1, \ldots, \varepsilon_n$	A_{n-1}
iii)	$\Lambda^2\,\mathrm{GL}_n$	$n \geq 2$	$r := \left[\frac{n}{2}\right]$	$\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \ldots, \varepsilon_{2r-1} + \varepsilon_{2r}$	A_{r-1}
iv)	$\mathrm{SO}_n \otimes \mathrm{GL}_1$	$n \geq 3$	2	$\varepsilon_1, \varepsilon'$	A_1
v)	$\mathrm{Sp}_{2n} \otimes \mathrm{GL}_1$	$n \geq 1$	1	$\varepsilon_1 + \varepsilon'$	1
vi)	$\mathrm{Sp}_{2n} \otimes \mathrm{GL}_2$	$n \geq 2$	3	$\varepsilon_1 + \varepsilon_1', \varepsilon_2 + \varepsilon_2', \varepsilon_1 + \varepsilon_2$	$\mathsf{A}_1 \oplus \mathsf{A}_1$
vii)	$\mathrm{Sp}_{2n} \otimes \mathrm{GL}_3$	$n \geq 3$	6	$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1', \varepsilon_2', \varepsilon_3'$	$\mathsf{A}_3 \oplus \mathsf{A}_2 \subset \mathsf{BC}_3 \oplus \mathsf{A}_2$
viii)	$\mathrm{Sp}_4 \otimes \mathrm{GL}_m$	$m = 3$	5	$\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2', \varepsilon_3'$	$\mathsf{BC}_2 \oplus \mathsf{A}_2$
		$m \geq 4$	6	$\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2', \varepsilon_3', \varepsilon_4'$	$\mathsf{BC}_2 \oplus \mathsf{A}_3$
ix)	$\mathrm{Spin}_7 \otimes \mathrm{GL}_1$		2	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon'$	A_1
x)	$\mathrm{Spin}_{10} \otimes \mathrm{GL}_1$		2	$\varepsilon_1 + 2\varepsilon', \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5$	A_1
xi)	$\mathrm{Spin}_9 \otimes \mathrm{GL}_1$		3	$\varepsilon_1, \varepsilon_2 + \varepsilon_3 + \varepsilon_4, \varepsilon'$	$\mathsf{A}_1 \oplus \mathsf{A}_1$
xii)	$\mathrm{G}_2 \otimes \mathrm{GL}_1$		2	ω_1, ε'	A_1
xiii)	$\mathrm{E}_6 \otimes \mathrm{GL}_1$		3	$\omega_1, \omega_6, \varepsilon'$	A_2

Under "$\mathfrak{a}_X^* \subseteq \mathfrak{t}^*$" we indicated a basis. For the classical groups, ε_i means a weight in the defining representation. A prime (e.g. ε_1') refers to the second factor. In case xii) and xiii) ω_i denotes a highest weight in the lowest dimensional representations. There is always a W_X-fixed vector in $\mathfrak{a}_X^*$ corresponding to the Euler vectorfield. Case vii) is the only one in which W_X is smaller than the normalizer of $\rho + \mathfrak{a}_X^*$: It is of index 2. More precisely, the generators of $\mathcal{D}(X)^G$ have degrees 2, 3, 4, 1, 2, 3 while those coming from $\mathfrak{U}(G)$ are of degree 2, 4, 6, 1, 2, 3.

As a last example, I consider the commutant of a reductive subalgebra in an enveloping algebra.

10.1. Theorem. *Let $H \subseteq G$ be a connected reductive subgroup with Lie algebra $\mathfrak{h}$. Assume that $\mathfrak{h}$ does not contain any non-trivial ideal of $\mathfrak{g}$. Let $\mathfrak{U}(\mathfrak{h})''$ be the second commutant of $\mathfrak{U}(\mathfrak{h})$ in $\mathfrak{U}(\mathfrak{g})$. Then*

$$\mathfrak{Z}(\mathfrak{g}) \underset{k}{\otimes} \mathfrak{U}(\mathfrak{h}) \longrightarrow \mathfrak{U}(\mathfrak{h})''$$

is an isomorphism. In particular, the center of the first commutant $\mathfrak{U}(\mathfrak{g})^{\mathfrak{h}}$ (the "classifying ring") equals $\mathfrak{Z}(\mathfrak{g}) \otimes_k \mathfrak{Z}(\mathfrak{h})$.

Proof: Let $\overline{G} := G \times H$ with Lie algebra $\bar{\mathfrak{g}}$ and let $\mathfrak{t}_G$, $\mathfrak{t}_H$, $\bar{\mathfrak{t}} = \mathfrak{t}_G \oplus \mathfrak{t}_H$ be Cartan subalgebras of $\mathfrak{g}$, $\mathfrak{h}$, $\bar{\mathfrak{g}}$ respectively with Weyl groups W_G, W_H, $\overline{W} = W_G \times W_H$.

Then $\overline{G}$ acts on $X := G$ where the first factor acts by left and the second factor by right translations. Now we carry out the previous constructions (M_X, L_X, W_X,...) with respect to $\overline{G}$ instead of G.

In the proof of [Kn2] Satz 2.1, I have shown, that the general isotropy group of $\overline{G}$ in T^*_X is finite. By [Kn1] 8.2, this implies $\operatorname{rk} X = \operatorname{rk} \overline{G}$, hence $\mathfrak{a}^*_X = \bar{\mathfrak{t}}^*$. Next I claim $W_X = \overline{W}$. This means, that the general fibers of

$$\Psi : T^*_X \longrightarrow \bar{\mathfrak{t}}^*/\overline{W} = \mathfrak{t}^*_G/W_G \times \mathfrak{t}^*_H/W_H$$

are irreducible. From $\mathfrak{a}^*_X = \bar{\mathfrak{t}}^*$ it follows that the little Weyl group W_X is generated by reflections of $\overline{W}$. Hence, it decomposes uniquely as $W_X = W_1 \times W_2$ with $W_1 \subseteq W_G$ and $W_2 \subseteq W_H$. Thus, $L_X = \bar{\mathfrak{t}}^*/W_X = \mathfrak{t}^*_G/W_1 \times \mathfrak{t}^*_H/W_2$ decomposes accordingly. That means, that Ψ has irreducible general fibers if and only if the components

$$T^*_X \longrightarrow \mathfrak{t}^*_G/W_G \quad \text{and} \quad T^*_X \longrightarrow \mathfrak{t}^*_H/W_H$$

have this property. Because $T^*_X \cong G \times \mathfrak{g}^*$ it suffices to check the maps

$$\mathfrak{g}^* \longrightarrow \mathfrak{t}^*_G/W_G \quad \text{and} \quad \mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^* \longrightarrow \mathfrak{t}^*_H/W_H$$

where it is well known. This proves $W_X = \overline{W}$.

From that we get $M_X = \bar{\mathfrak{g}}^*$ and thus,

$$\mathfrak{U}(X) = \mathfrak{U}(\mathfrak{g}) \underset{k}{\otimes} \mathfrak{U}(\mathfrak{h}) \subseteq \mathfrak{U}(\mathfrak{g}) \underset{\mathfrak{Z}(\mathfrak{g})}{\otimes} \mathfrak{U}(\mathfrak{g}) \subseteq \mathcal{D}(X)$$

which is by Theorem 9.5 the full commutant inside $\mathcal{D}(X)$ of

$$\mathcal{D}(X)^{\overline{G}} = \mathfrak{U}(\mathfrak{g})^{\mathfrak{h}} \overset{1 \otimes \mathrm{id}}{\hookrightarrow} \mathfrak{U}(\mathfrak{g}) \underset{\mathfrak{Z}(\mathfrak{g})}{\otimes} \mathfrak{U}(\mathfrak{g}) \subseteq \mathcal{D}(X).$$

Restricting to right invariant operators, proves the theorem. □

Remarks: 1. For the proof, the Vanishing Theorem 4.1 is not needed. In fact, Theorem 10.1 claims that in certain cases all central invariant differential operators come from the center of the universal enveloping algebra. The whole purpose of Theorem 4.1 is to construct operators that are not induced by $\mathfrak{U}$.

2. Joseph [Jos] proved Theorem 10.1 for many cases. He has even results for non-reductive groups.

3. Kostant (unpublished) classified all cases, where $\mathfrak{U}(\mathfrak{g})^{\mathfrak{h}}$ is commutative: A non-trivial, indecomposable pair $\mathfrak{h} \subset \mathfrak{g}$ has this property if and only if it is isomorphic to $\mathfrak{gl}_{n-1} \subset \mathfrak{sl}_n$ ($n \geq 2$) or $\mathfrak{so}_{n-1} \subset \mathfrak{so}_n$ ($n \geq 2$). Then Cooper [Coo] proved $\mathfrak{U}(\mathfrak{g})^{\mathfrak{h}} = \mathfrak{Z}(\mathfrak{g}) \otimes_k \mathfrak{Z}(\mathfrak{h})$ in these cases. His proof was later simplified by Howe [How]. All this is still unpublished, and I was not aware of it, when I wrote [Kn2], which contains the classification and the equality. With Theorem 10.1 this last part becomes obvious: Because in these cases X is spherical, $\mathfrak{U}(\mathfrak{g})^{\mathfrak{h}} = \mathcal{D}(X)^G$ is commutative.

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