# TOPOLOGICAL INVARIANTS OF REAL ALGEBRAIC ACTIONS

KARL HEINZ DOVERMANN AND FRIEDRICH KNOP

Max-Planck Institut für Mathematik

## 1. Introduction.

In this paper we shall construct many real algebraic group actions on non-singular real algebraic varieties (these we shall call real algebraic $G$ varieties) with non-linear topological invariants. Some basic definitions concerning these concepts are given in Section 2. There are two points of view which lead to the problems discussed in this paper. The first of them arises in algebraic transformation groups. In particular we have the

**Fixed Point Problem.** *Does an algebraic action of a reductive group action on $\mathbb{C}^n$ have a fixed point?*

Petrie and Randall [PR3] and others showed for some classes of groups that this problem has an affirmative answer (see the quoted paper for further references). On the other hand Dovermann, Masuda, and Petrie showed in [DMP]:

**Theorem 1.1.** *For any integer $k \geq 24$ there exists a fixed point free real algebraic action of the icosahedral group on a non-singular variety diffeomorphic to $\mathbb{R}^k$. (For an extension see Proposition 1.6.)*

Every linear action has a fixed point, and it is the fixed point set which is the non-linear invariant of the actions in Theorem 1.1. A fixed point free action of a reductive group on $\mathbb{C}^n$ would be a striking counter example to the Linearity Conjecture promoted by Bass and Haboush [BH]. It says that that any reductive complex algebraic action on $\mathbb{C}^n$ is congugate to a linear action. Recently it has been shown by G. Schwarz that this conjecture is false. But, these actions do not provide an answer to the Fixed Point Problem. The actions in Theorem 1.1 are also not conjugate to linear actions. On the other hand, real algebraic actions appear to be much less rigid than complex algebraic ones. Invariants which are not smooth or real algebraic may be needed in complex transformation groups.

There may not even be a big difference between smooth actions and real algebraic actions on closed manifolds (see Theorem 1.3), but the equality of these two categories is still a conjecture. In an attempt to study this difference one can study rather exotic smooth actions, and if such actions are real algebraic then one might want to believe that the same holds more generally. This is our second point of view.

There are three classical questions in smooth transformation groups which ask whether there are actions with particular non linear invariants. We formulate them in the real algebraic category, but the references refer to the smooth case. In some cases we will show that the smooth answers to the questions also hold in the real algebraic setting. As it is customary in transformation groups we suppose that all actions are effective.

**Questions.**

(1) *Which groups act real algebraically on a variety diffeomorphic to Euclidean space without fixed point? This question was first studied by Conner and Floyd in the late 50's* [CF].
(2) *Which groups act real algebraically on a homotopy sphere with exactly one fixed point? This question was posed by Montgomery and Samelson in 1946* [MS].
(3) *Which groups act real algebraically on a homotopy sphere $\Sigma$ with exactly two fixed points $x$ and $y$ such that their tangent representations $T_x\Sigma$ and $T_y\Sigma$ are not isomorphic? This question was asked by P. A. Smith in 1960* [Sm].

In the answers which we provide in this paper one particular tool is used. The construction of the exotic actions starts with a manifold which is a product of surfaces (as in 1.2) on which the action is real algebraic. The desired manifold will be equivariantly cobordant to such an action, and by Theorem 1.3 it will be algebraic as well. Our result on actions on surfaces is as follows.

**Proposition 1.2.** *Suppose a finite group $G$ acts smoothly on a closed oriented surface $S$ such that the action preserves orientations. Then $S$ is $G$ equivariantly diffeomorphic to a real algebraic $G$ variety.*

The Nash Conjecture [N] says that every closed smooth manifold is diffeomorphic to a real algebraic variety. It was proved by Tognoli [T]. This conjecture has an obvious equivariant formulation of which the following special case was verified in [DMP]. The general version does not make the bordism assumption.

**Theorem 1.3.** *Suppose $G$ is a compact Lie group and $M$ is a closed smooth $G$ manifold. Suppose $M$ is $G$ cobordant to a non-singular real algebraic $G$ variety. Then $M$ is $G$ diffeomorphic to a non-singular real algebraic $G$ variety.*

We turn to our questions from real algebraic transformation groups posed above, starting with the second one.

**Theorem 1.4.** *Suppose $G$ is an abelian group of odd order with at least three non-cyclic Sylow subgroups. Then $G$ acts real algebraically with exactly one fixed point (or any number of fixed points) on a variety which is a homotopy sphere.*

The question expressed in this Theorem was asked by Montgomery and Samelson [MS]. In the smooth category one fixed point actions were constructed by Stein [St] for the bi-icosahedral group, and by Petrie for several classes of groups, in particular the groups in Theorem 1.4 (see [P3] and [P4]). For a real algebraic result for the icosahedral group see our Theorem 1.1 quoted from [DMP]. Furthermore, see Morimoto's study of low dimensional one fixed point actions [Mo]. We give the proof of Theorem 1.4 in Section 3. The proof shows basically that Petrie's examples can be constructed in the following way. A careful construction produces a one fixed point action on a manifold which is a product of surfaces. This manifold is cobordant to a homotopy sphere. The theorem will then follow from Proposition 1.2 and Theorem 1.3.

Next we consider the first question. Theorem 1.5 and Propositon 1.6 provide partial results. As one may remove a fixed point from a $G$ variety and the resulting space is a $G$ manifold which is again $G$ diffeomorphic to a $G$ variety (see [M1] or [DMS]), one has as a corollary of Theorem 1.4

**Theorem 1.5.** *Suppose $G$ acts real algebraically and with exactly $k$ fixed points on a variety which is a homotopy sphere of dimension $n$ (e. g., $G$ is as in Theorem 1.4 1.6), then $G$ acts real algebraically and with exactly $k-1$ fixed points on a variety diffeomorphic to $\mathbb{R}^n$. In particular, if $k=1$, then $G$ acts with exactly one fixed point on the homotopy sphere and without fixed point on $\mathbb{R}^n$.*

The result of Theorem 1.5 may be extended by the techniques used in [DMP, Section 6]. Let $\mathcal{G}$ denote the set of all compact Lie groups which act effectively and without fixed point on a variety diffeomorphic to affine space. Then

**Proposition 1.6.** *If $H \subset G$ is a subgroup of finite index and $H \in \mathcal{G}$, then $G \in \mathcal{G}$. If $G$ surjects onto $K$ and $K \in \mathcal{G}$, then $G \in \mathcal{G}$.*

Conner and Floyd [CF] were the first to construct fixed point free actions on $\mathbb{R}^n$. These were actions of cyclic groups not of prime power order. The ultimate answer in the smooth category is based on articles by Conner and Montgomery [CM], by Hsiang and Hsiang [HH] and by Edmonds and Lee [EL]. The smoothing of their actions with the help of the Mostow-Palais embedding theorem is a standard argument.

**Theorem.** *A compact Lie group $G$ has a fixed point free action on some $\mathbb{R}^n$ if and only if $G/G_0$ is not of prime power order or $G_0$ is not abelian.*

There is another invariant which has been studied extensively in transformation groups. It is motivated by the third question from above, posed by P. A. Smith [Sm]. Consider a smooth action $\theta$ of $G$ on a homotopy sphere $\Sigma$ such that its fixed point set $\Sigma^G = \{x, y\}$ consists of exactly two points. Let $\beta(\Sigma, \theta) = T_x\Sigma - T_y\Sigma$ be the difference of the tangent representations at these fixed points. This expression is well defined up to sign. The representations $V$ and $W$ are called *Smith equivalent*, and we say that $(\Sigma, \theta)$ *realizes the Smith equivalence* between the representations $V$ and $W$ if $T_xX = V$ and $T_yX = W$. So our third question asks for which groups there are real algebraic actions $\theta$ on homotopy spheres $\Sigma$ such that $\beta(\Sigma, \theta) \neq 0$. Some smooth actions which realize non-isomorphic Smith equivalent representations have been constructed (in a way which will be explained in the proof of Theorem 1.7) from surfaces as in 1.2. It will follow that these actions are equivariantly diffeomorphic to real algebraic actions. Specifically, we have the following list of groups.

(1) $G$ is finite abelian with at least three non-cyclic Sylow subgroups.
(2) $G$ is a cyclic group of order $2^k m$ where $k \geq 0$ and $m$ is odd such that $H = \mathbb{Z}_m$ has non isomorphic representations which satisfy Assumption 1.1 in [DP2]. Values for $m$ are square free products $m = p_1 \cdots p_k$ where $k \geq 4$ such that $p_1 \equiv 5 \pmod 8$ and the Legendre symbols $\left(\frac{p_1}{p_j}\right) = 1$ for $2 \leq j \leq 4$ (see [DW]).

**Theorem 1.7.** *If $G$ is as in (1) or (2), then $G$ acts real algebraically on a homotopy sphere $\Sigma$ with exactly two fixed points $x$ and $y$ such that $T_x\Sigma \neq T_y\Sigma$.*

There is a considerable list of references containing answers to Smith's question in the smooth category. If $G$ is of odd prime power order or if $G$ acts semi-freely then Smith equivalent representations must be isomorphic, as has been shown by Atiyah and Bott [AB] and by Milnor [M2]. By a result of Sanchez [Sz] the same is true for cyclic groups of order $pq$ where $p$ and $q$ are odd primes. Bredon [B] showed that this is also true if $G$ is of order $2^k$ if the dimension of the representations is at least $\mu(k)$ for some appropriate function $\mu$. Petrie announced that there are non-isomorphic Smith equivalent representations for abelian groups of odd order with at least four non-cyclic Sylow subgroups [P1] and he proposed to find all groups which have this property. The proof of the theorem was carried out by Petrie and Randall in [PR1]. Non-isomorphic Smith equivalent representations for cyclic groups of order $4k$ with $k > 1$ were constructed by Cappell and Shaneson [CS1] and Petrie [P2]. For some classses of cyclic groups of odd order non-isomorphic Smith equivalent representations were constructed by Dovermann and Petrie [DP2].

For additional work on Smith equivalent representations see the work of Siegel [Si], Dovermann [D], Suh [Su1] and [Su2], Cho [C1] and [C2], Dovermann and Washington [DW], and Dovermann and Suh [DS], the surveys by Masuda and Petrie [MP], Cappell and Shaneson [CS2], Dovermann, Petrie, and Schultz [DPS], and a book by Petrie and Randall [PR2] on this topic.

## 2. Basic Definitions and Theorem 1.2.

Let $\Omega$ be an orthogonal representation of a compact Lie group $G$. A *real algebraic $G$ variety* is a $G$ invariant set

$$V = \{x \in \Omega \mid p_1(x) = \cdots = p_m(x) = 0\}$$

for a given set of polynomials $\{p_1, \cdots, p_m\}$. Here $G$ invariant means that $G$ maps the points of the variety again to points in the variety. We shall consider mostly affine varieties in this paper. Only in this section will we use projective $G$ varieties, which are $G$ invariant projective varieties in the projective space $P(\Omega)$ of an orthogonal or unitary representation $\Omega$ of $G$. The idea of a non singular variety is just as in the non equivariant situation.

To prove Theorem 1.2 we need the following lemma.

**Lemma 2.1.** *Let $G$ be a compact group and $\Xi$ an effective unitary representation of $G$ of dimension $N$. Let $V \subseteq \mathbf{P}(\Xi)$ be a complex projective closed $G$ subvariety. Then $V$ is equivariantly diffeomorphic to an affine real algebraic $G$ variety.*

*Proof.* By assumption we have a $G$ invariant inner product, and we denote its associated quadratic form by $q : \Xi \to \mathbb{R}$. We express the elements in $\Xi$ in terms of a unitary basis, so that $q(z_1, \ldots, z_N) = z_1\overline{z_1} + \cdots + z_N\overline{z_N}$. For $\mathbf{P}^{N-1} = \mathbf{P}(\Xi)$ we use homogeneous coordinates $[z_1 : \cdots : z_N]$. Then, with $q = q(z_1, \ldots, z_N)$,

$$\Phi : \mathbf{P}^{N-1} \to \mathbb{C}^{N^2} = \mathbb{R}^{2N^2} \text{ with } [z_1 : \cdots : z_N] \mapsto \left(\frac{z_i\overline{z_j}}{q}\right)_{i,j=1,\ldots,N}$$

identifies $\mathbf{P}^{N-1}$ with the set of hermitian matrices which have trace and rank equal to 1. Let $G$ act on the matrices in $\mathbb{C}^{N^2}$, where $u \in G \subset U_N$ acts on a matrix $H$ by mapping it to $uHu^{-1} = uH\overline{u}$. Then $\Phi$ is a $G$ equivariant embedding.

Let $p_\nu \in \mathbb{C}[z_1, \ldots, z_N]$ be homogeneous polynomials which define $V$. Then $\Phi(V)$ is defined by the equations

$$p_\nu\left(\frac{z_1\overline{z_\mu}}{q}, \ldots, \frac{z_N\overline{z_\mu}}{q}\right) \text{ for all } \nu \text{ and } \mu = 1, \ldots, N$$

together with the equations defining $\Phi(\mathbf{P}^{N-1})$. These can obviously be rewritten as real polynomial equations in $\mathbb{C}^{N^2} = \mathbb{R}^{2N^2}$. □

*Proof of Proposition 1.2.* Let $s$ be a point in the surface $S$ with isotropy group $G_s$ and tangent representation $T_sS$. By assumption, $G$ acts on $S$ preserving the orientation. Therefore $G_s$ is cyclic and acts by rotations on $T_sS$. We can identify $T_sS$ with $\mathbb{C}$ such that $G_s$ acts by multiplication with roots of unity. The map $z \mapsto z^l$ identifies $T_sS/G_s$ with $\mathbb{C}$. Here $l$ denotes the order of $G_s$. Therefore the quotient $S' = S/G$ is again a smooth oriented closed surface.

We identify $S'$ with some Riemann surface. By a classical result [G, §10a] there is a unique lifting of the complex structure on $S'$ to one on $S$, and the uniqueness implies that the action of $G$ on $S$ is analytic.

Any Riemann surface is isomorphic to a complex algebraic projective subvariety of a projective space $\mathbf{P}^{N-1}$ [GH, Chapter 2.1]. Let $G = \{g_1, \ldots, g_h\}$ and define

$$\phi : S \hookrightarrow \mathbf{P}^{hN-1} = \mathbf{P}((\mathbb{C}^N)^{\otimes h}) \text{ by } s \mapsto (g_1 s) \otimes \cdots \otimes (g_h s).$$

The $G$ action on $(\mathbb{C}^N)^{\otimes h}$ permutes the factors, and the action on $\mathbf{P}((\mathbb{C}^N)^{\otimes h})$ is the induced one. This is a $G$ equivariant complex algebraic embedding. The claim of Proposition 1.2 is now an immediate consequence of the previous lemma. □

## 3. Proof of Theorem 1.4 and Theorem 1.7.

Our proof of Theorem 1.4 will be based on a proof of the following result of Petrie [P1].

**Theorem 3.1.** *Suppose $G$ is an abelian group of odd order with at least three non-cyclic Sylow subgroups. Then $G$ acts smoothly on a homotopy sphere with exactly one fixed point.*

Our Theorem 1.4 states that the action in Theorem 3.1 can be chosen to be equivariantly diffeomorphic to a real algebraic action on a variety. Petrie started out with a carefully chosen one fixed point action of $G$ on a manifold $X$, and he showed that this manifold $X$ is $G$ cobordant to a homotopy sphere $\Sigma$, relative to the fixed point set. We follow the same approach, but we show that $X$ can be chosen as a product of surfaces with smooth action of $G$, such that each factor satisfies the assumptions in 1.2. This will imply that $\Sigma$ is $G$ diffeomorphic to a real algebraic $G$ variety. We continue with the details of the approach.

Let $V$ be a representation of $G$ (as in Theorem 1.4 or 3.1) which has

**Properties 3.2.**

(1) *$K \subset G$ is an isotropy group of $V$ if $G/K$ is not of prime power order.*
(2) *$\dim V^K = 0$ if $G/K$ is of prime power order.*

Then we construct a manifold $X(V)$ with smooth $G$ action which has the following

**Properties 3.3.**

(1) *$X(V)^G$ has exactly one fixed point at which the tangent representation is $V$.*
(2) *$\mathrm{Res}_K X(V)$ is a $K$ equivariant boundary whenever $G/K$ is not of prime power order.*
(3) *The equivariant tangent bundle $TX(V)$ is stably $G$ isomorphic to a product bundle, and restricted to $K$ as in (2) the stable isomorphism extends $K$ equivariantly over the zero-cobordism.*
(4) *Equivariantly, $X(V)$ is a product of surfaces $X(\psi)$, where $\psi$ varies over the irreducible summands of $V$. Let $G(\psi)$ be the kernel of the $G$ action on $\psi$ and $L(\psi) = G/G(\psi)$. Then $L(\psi)$ acts effectively on $X(\psi)$ and the singular orbits are isolated. The action of $G$ on $X(\psi)$ is induced via the projection $G \to L(\psi)$. (So $X(\psi)$ satisfies the assumptions in Proposition 1.2.)*
(5) *The smooth action of $G$ on $X(V)$ is equivariantly diffeomorphic to a real algebraic action of $G$ on a variety. (This is an immediate consequence of (4) and Theorem 1.2)*

This construction of $X(\psi)$ as well as the $K$ equivariant zero cobordism in (2) are given in [DP2, Section 3], but the extension of the bundle isomorphism over the zero cobordism is not mentioned. For this we refer the reader to [DS, 3.1 Addendum].

We impose a few more technical assumptions on the representation $V$, namely that

**Continuation of Properties 3.2.**

(3) *$V$ is stable, satiesfies the gap hypothesis, and* $\dim V^K$ *is zero or at least* 6 *for all* $K \subset G$.

We explain the terms in 3.2 (3).

**Definiton 3.4.** *A representation $W$ is said to satisfy the gap hypothesis if whenever $W^K \subset W^L$ and $W^K \neq W^L$, then $2 \dim W^K + 1 < \dim W^L$, for all $K$, $L \subset G$. A smooth $G$ manifold $M$ satisfies the gap hypothesis if for every $x \in M$ the $G_x$ tangent representation $T_x M$ satisfies the gap hypothesis.*

**Definition 3.5.** *A $G$ manifold $M$ is defined to be stable if for each $x \in M$ and $K = G_x$, the multiplicity $m_\chi(T_x M)$ of the irreducible representation $\chi$ in $T_x M$ is either zero or $d_\chi m_\chi(V) \geq \dim_{\mathbb{R}} V^K$. Here $d_\chi = \dim_{\mathbb{R}} D_\chi$, and $D_\chi$ is the algebra of real $K$ endomorphisms of $\chi$.*

*Proof of Theorem 1.4.* Above we outlined the general strategy of proof (see the paragraph after Theorem 3.1). Let $V$ be a representation of $G$ which satisfies 3.2 (1)–(3). An example of such a representation is $V = 2(\mathbb{C}[G] - V_0)$, where $\mathbb{C}[G]$ is the complex regular representation of $G$, and $V_0$ is the sum of all irreducible representations $\psi$ of $G$ for which $G/G(\psi)$ is of prime power order. As in 3.3 (4), $G(\psi)$ is the kernel of $\psi$. It is elementary to check that $V$ satisfies 3.2 (1)–(3).

Let $X(V)$ be a $G$ manifold which has all of the properties stated in 3.3. As we said, such manifolds exist. Set $X = kX(V)$, so $X$ is a disjoint union of $k$ copies of $X(V)$. The manifold has exactly $k$ fixed points.

We will show that $X$ is $G$ cobordant relative to the fixed point set to a homotopy sphere on which the action will then have exactly $k$ fixed point. The cobordism will be obtained from the Induction Theorem of [DP1, Theorem 2.8], applied to $X$. It has been reformulated in [P3, Theorem 2.2] to tailor it to our situation.

The induction theorem is usually applied to $G$ normal maps. We obtain such a normal map by collapsing an equivariant neighbourhood of a fixed point $x \in X$ to a point. So we get $f : X \to S(T_x X \oplus \mathbb{R})$, and this map is equivariant and of degree one (with correctly chosen orientation). The stable trivialization of $TX$ provides bundle data $b$ and $(X, f, b)$ will be an equivariant normal map. A few technical assumptions concerning dimensions and Euler characteristics need to be satisfied such that the data $(X, f, b)$ satisfies the assumptions in the definition of a normal map as in [DP1]. The dimension assumptions follow from those for $V$ made in 3.2 (3) and they are derived from those listed for $X(V)$ in 3.3. The assumption on Euler characteristics follows as in the proof of Lemma 2.11 in [P1] from our Assumption 3.2 (2).

The approximate statement of the induction theorem is as follows (for the complete formulation see the references quoted above):

*If $G$ is an odd order abelian group and $\mathrm{Res}_H(X, f, b)$ is $H$ equivariantly a boundary for all hyperelementary subgroups $H$ of $G$, then $(X, f, b)$ is $G$ equivariantly cobordant to a normal map whose underlying function is equivariant and a homotopy equivalence.* (In the literature such maps are called pseudo equivalences.)

The assumption in this theorem that $\mathrm{Res}_H(X, f, b)$ is a boundary follows immediately from 3.3 (2) (compare [P1]). The induction theorem implies that $X$ is equivariantly cobordant to a homotopy sphere $\Sigma$ relative to the fixed point set, because we chose $Y$ as a homotopy sphere. The action on $\Sigma$ has exactly the same number of fixed points as the action on $X$, namely $k$ of them. Because $X$ is equivariantly diffeomorphic to a real algebraic $G$ variety (see 3.3 (5)) it follows from Theorem 1.3 that $\Sigma$ is equivariantly diffeomorphic to a real algebraic $G$ variety. □

*Proof of Theorem 1.7.* There are two cases, $G$ as in (1) or (2). The corresponding result in the smooth case and for $G$ as in (1) was shown in [DS]. This was a generalization of the theorem announced in [P1] and proved in [PR1]. In case $G$ is as in (2), $k = 0$, and supposing that the action is smooth, Theorem 1.7 has been proved in [DP2]. It has been generalized for $k > 0$, also in the smooth category, in [DS]. In either of these cases one starts out with a representation $U$ of $G$ and a sufficiantly large collection $\mathcal{S}$ of representations of $G$ which satisfy a cartain list of conditions. For $V$ and $W$ in $\mathcal{S}$ one constructs manifolds $X(V, W)$ which are of the form $X(V) \sqcup X(W) \sqcup Z$. Here $X(V)$ and $X(W)$ are products of surfaces as in 3.3. The manifold $X(V, W)$ is equivariantly diffeomorphic to a real algebraic $G$ variety, and $Z$ is an equivariant boundary. Next one shows that among the representations in $\mathcal{S}$ one can find non-isomorphic representations such that $X(V, W)$ is equivariantly cobordant to a homotopy sphere $\Sigma$, relative to the fixed point set. Then $\Sigma^G$ consists of exactly two points $x$ and $y$ with tangent representations $T_x\Sigma = V$ and $T_y\Sigma = W$. It follows from 3.3 (5) and Theorem 1.3 that $\Sigma$ is equivariantly diffeomorphic to a real algebraic $G$ variety. As $V$ and $W$ were chosen non-isomorphic the claim of our theorem follows. □

## References


[**AB**] M. Atiyah and R. Bott, *The Lefschetz Fixed Point Theorem for Elliptic Complexes II*, Annals of Math. **88** (1968), 451–491.

[**BH**] H. Bass and W. Haboush, *Linearizing Certain Reductive Group Actions*, Transctions of the Amer. Math. Soc. **292** (1984), 463–482.

[**B**] G. Bredon, *Representations at Fixed Points of Smooth Actions of Compact Groups*, Annals of Math. **89** (1969), 515–532.

[**CS1**] S. Cappell and J. Shaneson, *Fixed Points of Periodic Differentiable Maps*, Inventiones Math. **68** (1982), 1–19.

[**CS2**] S. Cappell and J. Shaneson, *Representations at Fixed Points*, in: Group Actions on Manifolds, Contemp. Math. **36** (1985), 151–158.

[**C1**] E. C. Cho, *Smith Equivalent Representations of Generalized Quaternion Groups*, Thesis, Rutgers University (1984).

[**C2**] E. C. Cho, *s-Smith Equivalent Representations of Dihedral Groups*, Pac. J. of Math. **135** (1988), 17–28.

[**CF**] P. E. Conner and E. E. Floyd, *On the Construction of Periodic Maps without Fixed Points*, Proc. Amer. Math. Soc. **10** (1959), 354–360.

[**CM**] P. E. Conner and D. Montgomery, *An Example for SO*(3), Proc. Nat. Acad. Sci. U. S. A. **48** (1962), 1918–1922.

[**D**] K. H. Dovermann, *Even Dimensional Smith Equivalent Representations*, in: Algebraic Topology, Åarhus 1982, Springer Lecture Notes in Mathematics **1051** (1983), 587–602.

[**DMP**] K. H. Dovermann, M. Masuda, and T. Petrie, *Fixed Point Free Algebraic Actions on Varieties Diffeomorphic to* $\mathbb{R}^n$, preprint 1988.

[**DP1**] K. H. Dovermann and T. Petrie, *An Induction Therem for Equivariant Surgery, G-Surgery III*, Amer. J. Math. **105** (1983), 1369–1403.

[**DP2**] K. H. Dovermann and T. Petrie, *Smith Equivalence of Representations for Odd Order Cyclic Groups*, Topology **24** (1985), 283–305.

[**DPS**] K. H. Dovermann, T. Petrie and R. Schultz, *Transformation Groups and Fixed Point Data*, in: Group Actions on Manifolds, Contemp. Math. **36** (1985), 159–189.

[**DS**] K. H. Dovermann and D. Y. Suh, *Smith Equivalence for Finite Abelian Groups*, preprint 1989.

[**DW**] K. H. Dovermann and L. D. Washington, *Relations between Cyclotomic Units and Smith Equivalence of Representations*, to appear, Topology (1989).

[**EL**] A. L. Edmonds and R. Lee, *Compact Lie Group which Act on Euclidean Space without Fixed Points*, Proc. Amer. Math. Soc. **55** (1976), 416–418.

[**GH**] P. Griffith and J. Harris, *Principals of Algebraic Geometry*, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1978.

[**G**] R. C. Gunning, *Lectures on Riemann Surfaces*, Mathematical Notes, Princeton University Press, Princeton N. J., 1966.

[**HH**] W. C. Hsiang and W. Y. Hsiang, *Differentiable Actions of connected classical groups, I*, Amer. J. of Math. **89** (1967), 705–786.

[**MP**] M. Masuda and T. Petrie, *Lectures on Transformation Groups and Smith Equivalence*, in: Group Actions on Manifolds, Contemp. Math. **36** (1985), 191–242.

[**M1**] J. Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Math. Studies, Princeton University Press, Princeton N. J., 1968.

[**M2**] J. Milnor, *Whitehead torsion*, Bull. of the Amer. Math. Soc. **72** (1966), 358–426.

[**MS**] D. Montgomery and H. Samelson, *Fiberings with Singularities*, Duke Math. J. **13** (1946), 51–56.

[**Mo**] M. Morimoto, *On One Fixed Point Actions on Spheres*, Proc. of the Japan Academy **63** (1987), 95–97.

[**N**] J. Nash, *Real Algebraic Manifolds*, Annals of Math. **56** (1952), 405–421.

[**P1**] T. Petrie, *G Surgery I–A Survey*, in: Algebraic and Geometric Topology, Proceedings, Santa Barbara 1977, Springer Lecture Notes in Mathematics **664** (1978), 197–223.

[**P2**] T. Petrie, *Smith Equivalence of Representations*, Math. Proc. Camb. Phil. Soc. **94** (1983), 61–99.

[**P3**] T. Petrie, *One Fixed Point Actions on Spheres, I*, Advances in Math. **46** (1982), 3–14.

[**P4**] T. Petrie, *One Fixed Point Actions on Spheres, II*, Advances in Math. **46** (1982), 15–70.

[**PR1**] T. Petrie and J. Randall, *Spherical Isotropy Representations*, Publ. I. H. E. S. **62** (1985), 5–40.

[**PR2**] T. Petrie and J. Randall, *Transformation Groups on Manifolds*, Dekker Series in Pure and Applied Mathematics Vol. 82, Marcel Dekker, New York, 1984.

[**PR3**] T. Petrie and J. Randall, *Finite-order Algebraic Automorphisms of Affine Varieties*, Comment. Math. Heletici **61** (1986), 203–221.

[**Sz**] C. Sanchez, *Actions of Groups of Odd Order on Compact Orientatable Manifolds*, Proc. Amer. Math. Soc. **54** (1976), 445–448.

[**Si**] A. Siegel, *Fixed Point Representations of Group Actions on Spheres*, Thesis, Rutgers University (1982).

[**Sm**] P. A. Smith, *New Results and Old Problems in Finite Transformation Groups*, Bull. Amer. Math. Soc. **66** (1960), 401–415.

[**Su1**] D. Y. Suh, *s-Smith Equivalence of Representations*, Thesis, Rutgers University 1983.

[**Su2**] D. Y. Suh, *Isotropy Representations of cyclic Group Actions on Homotopy Spheres*, to appear, Bull. Korean Math. Soc..

[**St**] E. Stein, *Surgery on Products with Finite Fundamental Group*, Topolgy **16** (1977), 473–493.

[**T**] A. Tognoli, *Su una Congettura di Nash*, Annali della Scuola Normale Superiore di Pisa **27** (1973), 167–185.