

Weyl Groups of Hamiltonian Manifolds, I

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1. Introduction

Let K be a compact connected Lie group and M a compact Hamiltonian K -manifold, i.e., a symplectic K -manifold equipped with a moment map $\mu : M \rightarrow \mathfrak{k}^*$. In this paper, we determine $\text{Col}(M)$: the set of all functions on M which Poisson commute with all K -invariant functions. For this, we construct a finite reflection group W_M and show that $\text{Col}(M)$ is completely determined by $\mu(M)$ and W_M .

More precisely, from $\mu(M)$ and W_M we construct a topological space Y equipped with a differentiable structure (in fact, Y is semi-analytic) and a surjective map $\widehat{\mu} : M \rightarrow Y$ such that $\text{Col}(M)$ consists exactly of the pull-back functions via $\widehat{\mu}$. It is easy to see that, conversely, $\mathcal{C}^\infty(M)^K$ is the Poisson centralizer of $\text{Col}(M)$. Thus we obtain a symplectic dual pair

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ Y & & M/K \end{array}$$

in the sense of Weinstein.

It follows immediately from the defining property of the moment map μ that every pull-back function via μ Poisson centralizes $\mathcal{C}^\infty(M)^K$. Thus, we obtain a homomorphism of Poisson algebras $\mu^* : \mathcal{C}^\infty(\mathfrak{k}^*) \rightarrow \text{Col}(M)$. This means that the moment map μ factors through Y :

$$\begin{array}{ccc} & M & \\ \widehat{\mu} \swarrow & & \mu \downarrow \\ Y & & \mathfrak{k}^* \\ \nu \searrow & & \end{array}$$

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In fact, Guillemin and Sternberg conjectured that μ^* is surjective [GS2], or, in other words that ν is a diffeomorphism onto its image. The first counterexample was given by Lerman in [L1]. It turns out that, in his example, Y is the (half-)cone $x^2 + y^2 + z^2 = t^2$, $t \geq 0$ in \mathbb{R}^4 , $\mathfrak{k}^* = \mathbb{R}^3$, and ν is the projection to the first three coordinates. Then t is not a pull-back of a *differentiable* function via ν . But t is the pull-back of the *continuous* function $\sqrt{x^2 + y^2 + z^2}$.

This is a general phenomenon: Karshon and Lerman, [KL], proved that $\text{Col}(M) = \mu^* \mathcal{C}^0(\mathfrak{k}^*) \cap \mathcal{C}^\infty(M)$. In our language, this means that ν is a homeomorphism onto its image. Thus, as a topological space, Y is completely determined by $\mu(M)$. Therefore, for knowing $\text{Col}(M)$ we have to construct a differentiable structure on Y .

This is where the group W_M comes into the play. First, let me mention that it is a subquotient of the Weyl group of K and that it is itself a reflection group. The differentiable structure on Y is defined in terms of certain symmetry properties of the Taylor series with respect to W_M .

Instead of stating the precise result, which is quite involved, let me explain what happens in the example above. In Lerman's case we have $W_M = \{1\}$ and Y has just the differentiable structure coming from its embedding into \mathbb{R}^4 . In other cases, we might as well have $W_M = \{\pm 1\}$. Then a function f on Y should be differentiable if and only if f is a differentiable function in x, y, z, t whose Taylor series at the vertex $(0, 0, 0, 0)$ is invariant under $(x, y, z, t) \mapsto (x, y, z, -t)$. This implies that f is a differentiable function of (x, y, z) alone, i.e., Y is diffeomorphic to \mathbb{R}^3 . The general case is similar. See section 3 for a complete statement.

The problem of determining $\text{Col}(M)$ is local except that certain connectivity properties of the fibers of μ are needed. Whenever M is compact, these hold by a theorem due to Kirwan. But we need to work locally. For that we axiomatize the properties needed and introduce the notion of a *convex* Hamiltonian manifold. The point is that every Hamiltonian manifold is locally convex. Then we prove all results in this more general context. Note, that most Hamiltonian manifolds appearing in practice (e.g., compact, or complex algebraic, or cotangent bundle) are convex.

Having localized the problem this way, we apply the symplectic slice theorem. This allows us to assume that M is an open subset of a *real algebraic* Hamiltonian manifold \overline{M} . By powerful results of Tougeron, or Bierstone-Milman, the property of being pull-back can be recognized on the level of Taylor series. This is now a purely algebraic problem on \overline{M} . In particular, it can be solved by complexifying \overline{M} . We show that the complexification of \overline{M} is the cotangent bundle T_X^* of an affine G -variety X where G is the complexification of K (thus reductive). In [K1], I have already determined all regular functions on T_X^* which

Poisson commute with all G -invariants.

This paper consists roughly of three parts. The main part consists of sections 3 and 4 in which our main result is stated and proved.

The other two parts are used in the main part sections but both are also of independent interest. In one part (section 2) we introduce and study the convexity of a Hamiltonian manifold.

Finally, sections 5 through 7 provide a crucial fact (Theorem 7.5) about the geometry of T_X^* . More precisely, we establish a connectivity and reducedness property of the fibers of the moment map. Logically, these sections come first but since they are of technical nature I postponed them.

For the convenience of the reader, two appendices are added. One recalls comparison results between C^∞ -functions and power series. The other states the basic local structure theorems of Hamiltonian varieties.

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Notation: Throughout this paper, “analytic” means “real analytic”. Moreover, we use the term “smooth” always in the algebraic geometric sense: “smooth” are morphism between algebraic varieties which are surjective on tangent spaces. Smooth functions in the sense of differential geometry are called C^∞ or differentiable. Moreover, differentiable always means infinitely often differentiable.

2. Convex Hamiltonian manifolds

Let K be a connected, compact Lie group with Lie algebra \mathfrak{k}^* and M a Hamiltonian K -manifold with moment map $\mu : M \rightarrow \mathfrak{k}^*$.

Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra corresponding to a maximal torus $T \subseteq K$ and $W = W(\mathfrak{k}, \mathfrak{t})$ its Weyl group. Since \mathfrak{t} has a unique T -stable complement in \mathfrak{k} , we can regard \mathfrak{t}^* as a subspace of \mathfrak{k}^* . The map $\mathfrak{t}^*/W \rightarrow \mathfrak{k}^*/K$ is a diffeomorphism. Furthermore, its restriction to a Weyl chamber $\mathfrak{t}_+^* \subseteq \mathfrak{t}^*$ is a differentiable homeomorphism, but in general not a diffeomorphism. We use it to construct a continuous map ψ which makes the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{k}^* \\ \psi \downarrow & & \downarrow \\ \mathfrak{t}_+^* & \rightarrow & \mathfrak{t}^*/W \end{array}$$

The map ψ is also characterized by the property $K\mu(x) \cap \mathfrak{t}_+^* = \{\psi(x)\}$ for all $x \in M$.

In this section we study some topological properties of ψ . For any two (not necessarily distinct) points $u, v \in \mathfrak{t}^*$ let $[u, v]$ be the line segment joining them.

Definition: A Hamiltonian K -manifold is called *convex* if $\psi^{-1}([u, v])$ is connected for all $u, v \in \psi(M)$.

Clearly, M is convex if and only if $\psi^{-1}(B)$ is connected for every convex subset B of \mathfrak{t}^* . In practice we will use another characterization of convexity:

2.1. Theorem. *A Hamiltonian K -manifold is convex if and only if the following conditions are satisfied:*

- i) The fibers of ψ are connected.*
- ii) The image $\psi(M)$ is convex.*
- iii) The map $\psi : M \rightarrow \psi(M)$ is open.*

Proof: Assume first that *i)* through *iii)* hold. If $\psi^{-1}([u, v])$ is disconnected there are $U_1, U_2 \subseteq M$ open such that $\psi^{-1}([u, v])$ is the disjoint union of X_1 and X_2 where $X_i := \psi^{-1}([u, v]) \cap U_i \neq \emptyset$. By *i)* we have $\psi(X_1) \cap \psi(X_2) = \emptyset$ and, by *ii)*, $[u, v] = \psi(X_1) \cup \psi(X_2)$. Finally, *iii)* implies that $\psi(X_i) = [u, v] \cap \psi(U_i)$ is open in $[u, v]$. This contradicts the connectedness of $[u, v]$.

For the reverse direction, we need some preparation. Let V be a real vector space equipped with a lattice $\Gamma \subseteq V$. Our example will be $V = \mathfrak{t}^*$ with the lattice consisting of all $\frac{1}{i}d\chi$ where $T \rightarrow \mathbb{C}^*$ is a character. A *homogeneous rational cone* is a subset of the form $\sum_{i=1}^N \mathbb{R}^{\geq 0} \gamma_i$ where $\gamma_1, \dots, \gamma_N \in \Gamma$. A *rational cone* is a translate $u + C$ of a homogeneous rational cone C by a vector $u \in V$. In this case we say that u is a vertex of $u + C$. The following theorem of Sjamaar is the foundation for the results in this section.

2.2. Theorem. ([Sj] Thm. 5.5) *Let M be a Hamiltonian K -manifold. Then for every orbit $Kx \subseteq M$ there is a unique rational cone $C_x \subseteq \mathfrak{t}^*$ with vertex $\psi(x)$ such that:*

- i) There exist arbitrary small K -stable neighborhoods U of Kx such that $\psi(U)$ is a neighborhood of $\psi(x)$ in C_x .*
- ii) For $u \in \mathfrak{t}_+^*$ let x and y be in the same connected component of $\psi^{-1}(u)$. Then $C_x = C_y$.*

Sjamaar's theorem implies a general openness property of ψ :

2.3. Lemma. *Let M be a Hamiltonian K -manifold such that all fibers of $\psi : M \rightarrow \mathfrak{t}_+^*$ are connected. Let $U \subseteq M$ be open. Then also $\psi^{-1}\psi(U)$ is open.*

Proof: We may assume that U is K -stable. For every $y \in \psi^{-1}\psi(U)$ there is $x \in U$ such that $\psi(x) = \psi(y) =: u$. Since $\psi^{-1}(u)$ is connected, Theorem 2.2ii) implies $C_x = C_y =: C$. By part *i)* of that theorem there are open neighborhoods U_x, U_y of x, y , respectively,

such that $\psi(U_x)$ and $\psi(U_y)$ are neighborhoods of u in C . Hence $(\psi|_{U_y})^{-1}(\psi(U_x))$ is a neighborhood of y which is contained in $\psi^{-1}\psi(U)$. \square

2.4. Lemma. *Let M be a Hamiltonian K -manifold such that all fibers of $\psi : M \rightarrow \mathfrak{t}_+^*$ are connected. Assume moreover that every $u \in \psi(M)$ has arbitrary small neighborhoods B such that $\psi^{-1}(B)$ is connected. Then $\psi : M \rightarrow \psi(M)$ is an open map.*

Proof: Let $x \in M$ and U an open neighborhood of x . We have to show that $\psi(U)$ is a neighborhood of $u := \psi(x)$ in $\psi(M)$. For this we may assume that $\psi(U)$ is a neighborhood of u in C_x . Let B be a neighborhood of u in \mathfrak{t}_+^* such that $B \cap C_x \subseteq \psi(U)$ and such that $\psi^{-1}(B)$ is connected. Let $V_1 := \psi^{-1}\psi(U)$ which is open in M by Lemma 2.3. Clearly, also $V_2 := \psi^{-1}(\mathfrak{t}^* \setminus C_x)$ is open in M . Moreover, V_1 and V_2 are disjoint and cover $\psi^{-1}(B)$. Connectivity implies $\psi^{-1}(B) \subseteq V_1$, i.e., $B \cap \psi(M) \subseteq \psi(U)$ which proves the assertion. \square

Now we can complete the proof of Theorem 2.1. Assume that M is convex. Then *i)* and *ii)* clearly hold. Let $u \in \psi(M)$ and $B \subseteq \mathfrak{t}_+^*$ a convex neighborhood of x . Since also $\psi(M)$ is convex we have $[u, v] \subseteq B \cap \psi(M)$ for every $v \in B \cap \psi(M)$. By assumption, $\psi^{-1}([u, v])$ is connected. This implies that $\psi^{-1}(B)$ is connected. Thus *iii)* holds by Lemma 2.4. \square

Remark: I don't know of any example of a Hamiltonian K -manifold M where $\psi(M)$ is convex, the fibers of ψ are connected but where ψ is not an open map.

If M is convex then C_x depends by Theorem 2.2 only on $u = \psi(x)$. Thus we may set $C_u := C_x$. Let C be a rational cone with vertex u . For $v \in C$ let $T_v C := C + \mathbb{R}^{\geq 0}(u - v)$ be the *tangent cone* of C in v . This is the smallest rational cone containing C and having v as a vertex.

2.5. Theorem. *Let M be a convex Hamiltonian K -manifold and $u \in \psi(M)$.*

- i) The image $\psi(M)$ is contained in C_u . Moreover, it is a neighborhood of u in C_u .*
- ii) The image $\psi(M)$ is locally a polyhedral cone and, in particular, locally closed and semi-analytic.*
- iii) For every $v \in \psi(M)$ in a neighborhood of u holds $C_v = T_v C_u$.*

Proof: *i)* Let $x \in M$ with $u = \psi(x)$. Then Kx has an open neighborhood U such that $\psi(U)$ is a neighborhood of u in C_u . Let $v \in \psi(M)$. Since $\psi(U)$ is open in $\psi(M)$ and since $[u, v] \subseteq \psi(M)$ also $[u, v] \cap \psi(U)$ is open in $[u, v]$. This implies $[u, v] \subseteq C_u$, thus $\psi(M) \subseteq C_u$. Moreover, $\psi(M)$ is a neighborhood of u in C_u since already $\psi(U)$ is.

ii) Follows directly from *i)*.

iii) By *i)* there is an open neighborhood U of x such that $\psi(U)$ is open in C_u . Let $v \in \psi(U)$. Then *i)*, applied to v , implies that C_v is the cone spanned by $\psi(U)$ over the

vertex v . On the other hand, $\psi(U)$ is open in C_u . Thus $T_v C_u$ is also the cone spanned by $\psi(U)$ over v . \square

Let V be a unitary representation of K . Then every smooth K -stable complex algebraic subvariety of $\mathbf{P}(V)$ is in a canonical way a Hamiltonian K -manifold. We call a Hamiltonian K -manifold *projective* if it arises this way but possibly with the symplectic form and the moment map rescaled by some non-zero factor.

2.6. Proposition. *Let M be a Hamiltonian K -manifold such that for every projective Hamiltonian K -manifold X holds that $\psi_{M \times X}^{-1}(0)$ is connected. Then $\psi_M : M \rightarrow \psi(M)$ is an open map with connected fibers.*

Proof: Let \bar{X} equal X with the symplectic structure multiplied by -1 . Then we have $\mu_{M \times \bar{X}}(m, x) = \mu_M(m) - \mu_X(x)$. Thus choosing for X the coadjoint orbit Ku , $u \in \mathfrak{t}_+^*$ we obtain that $\psi_M^{-1}(u) = \psi_{M \times \bar{X}}^{-1}(0)$ is connected. Now let X_0 be projective such that $\psi_{X_0}(X_0)$ is a neighborhood of 0 , e.g., $X_0 = \mathbf{P}(V)$ where V contains stable points for the $K^{\mathbb{C}}$ -action. By rescaling, we can arrange that $\psi(X_0)$ is arbitrary small. Let $X := X_0 \times Ku$. Then $B := \psi_X(X)$ is an arbitrary small neighborhood of u in \mathfrak{t}_+^* . Moreover, projection to the first factor induces a surjective map $\psi_{M \times \bar{X}}^{-1}(0) \rightarrow \psi_M^{-1}(B)$. By assumption the first, hence the second set is connected. We conclude with Lemma 2.4, that ψ_M is open onto its image. \square

Now we can give examples of convex Hamiltonian manifolds:

2.7. Theorem. *Every Hamiltonian K -manifold M satisfying one of the conditions *i*), *ii*), *iii*) below is convex.*

- i) The moment map $\mu : M \rightarrow \mathfrak{k}^*$ is proper (e.g., if M is compact).*
- ii) The manifold M is a complex algebraic variety, the action of K is the restriction of an algebraic $K^{\mathbb{C}}$ -action, and the symplectic structure is induced by a K -invariant Kähler metric.*
- iii) The manifold M is a complex Stein space, the action of K is the restriction of a holomorphic $K^{\mathbb{C}}$ -action, and the symplectic structure is induced by a K -invariant Kähler metric.*

Proof: In all cases, it is known that ψ has connected fibers and convex image (see [HNP] or [Sj] for *i*) and [HH] for *ii*) and *iii*). Moreover, the classes *i*)-*iii*) are preserved by taking the product with a projective Hamiltonian K -manifold. Thus Proposition 2.6 implies that ψ is open. \square

Locally, every Hamiltonian manifold is convex:

2.8. Theorem. *Let M be a Hamiltonian K -manifold. Then every $x \in M$ has a convex K -stable open neighborhood U such that $\psi(U)$ is open in C_x .*

Proof: The proof is similar to that of [KL] Prop. 3.7. First, if the orbit Kx (with $\mu(x) \in \mathfrak{t}_+^*$) is not isotropic then let $L = K_{\mu(x)}$ and replace M by $M_L \cong K \times^L M(L)$ (Theorem 9.1). If there is already a neighborhood $U(L)$ of x in $M(L)$ with the properties claimed in the theorem with respect to L , then it is trivial to check that $U := K \cdot U(L)$ has the required properties for K .

Thus we may assume that Kx is isotropic. By Theorem 9.2 and the remark following it, we may assume that $M = K \times^H (\mathfrak{h}^\perp \times S)$ and $x = [1, (0, 0)]$. Here, H is a closed subgroup of K and S is a unitary representation of H .

Let D and \bar{D} be the open and the closed unit ball in S . Put $V := K \times^H (\mathfrak{h}^\perp \times D)$, $\bar{V} := K \times^H (\mathfrak{h}^\perp \times \bar{D})$. The restriction of μ to \bar{V} factors as

$$\mu|_{\bar{V}} : \bar{V} \hookrightarrow K \times^H (\mathfrak{k}^* \times \bar{D}) \cong (K \times^H \bar{D}) \times \mathfrak{k}^* \rightarrow \mathfrak{k}^*.$$

This shows that $\mu|_{\bar{V}}$ is proper.

We let $t \in \mathbb{R}^{>0}$ act on M by $t \cdot [k, (u, v)] := [k, (t^2 u, tv)]$. Then ψ becomes homogeneous, more precisely, $\psi(t \cdot y) = t^2 \psi(y)$ for all $y \in M$.

There is a K -stable open neighborhood U_0 of x in V such that $\psi(U_0)$ is a neighborhood of $0 = \psi(x)$ in C_x . From $M = \mathbb{R}^{>0} \cdot U_0$ we obtain $\psi(M) \subseteq C_x$. In particular, also $\psi(V)$ is a neighborhood of 0 in C_x . Thus there is a convex, open neighborhood B of 0 in \mathfrak{k}^* such that $B \cap C_x \subseteq \psi(V)$ and $\bar{B} \cap C_x \subseteq \psi(\bar{V})$. We put $U := V \cap \psi^{-1}(B)$.

By construction, $\psi(U) = B \cap C_x$ is convex and open in C_x . It remains to show that $\psi : U \rightarrow \psi(U)$ is open with connected fibers. The closure of U is $\bar{U} = \bar{V} \cap \psi^{-1}(\bar{B})$. Moreover, we have $U = \bigcup_{0 < t < 1} t \cdot \bar{U}$. Since increasing unions of connected sets are connected and by homogeneity it suffices to show that $\psi : \bar{U} \rightarrow \bar{B}$ has connected fibers. We show that even $\psi : \bar{V} \rightarrow C_x$ has connected fibers.

For this recall that S is a unitary vector space. In particular, $U(1)$ acts on S and thus on M . This action is Hamiltonian with moment map $\mu_0([k, (u, v)]) = \|v\|^2$. Let P be the symplectic cut of M at the level $\mu_0 = 1 - 0$: As a set we have $P := P_1 \dot{\cup} P_2$ where $P_1 := \mu_0^{-1}([0, 1))$ and $P_2 := \mu_0^{-1}(1)/U(1)$. It turns out that P is a symplectic manifold, see [L2] for details.

There is an obvious proper map $\pi : \bar{V} \twoheadrightarrow P$ where V goes isomorphically onto P_1 and the $U(1)$ -orbits in the boundary are collapsed to points in P_2 . Since the $U(1)$ -action commutes with the K -action, P is even a Hamiltonian K -manifold. Its moment map is

the unique map which makes the following diagram commutative:

$$\begin{array}{ccc} \bar{V} & & \\ \pi \downarrow & \searrow \mu & \\ P & \xrightarrow{\mu_P} & \mathfrak{t}^* \end{array}$$

In particular, μ_P is proper. This implies that μ_P , and ψ_P have connected fibers ([HNP], [Sj] 1.4). Since π is proper with connected fibers, also the fibers of $\psi|_{\bar{V}}$ are connected, which was claimed.

By Theorem 2.7*i*), the map $\psi_P : P \rightarrow \psi(P)$ is open. Since $U \subseteq P$ is open, this implies that $\psi : U \rightarrow \psi(U)$ is open, as well. \square

An application is the following well known statement.

2.9. Corollary. *Let M be any Hamiltonian K -manifold and let $\mathfrak{a}^0 \subseteq \mathfrak{t}^*$ be the affine subspace spanned by $\psi(M)$. Then \mathfrak{a}^0 is also the affine span of every local cone C_x , $x \in M$. Moreover, $\psi(M)$ is Zariski-dense in \mathfrak{a}^0 .*

Proof: With local convexity and Theorem 2.5*iii*) it follows that the affine span of C_x is locally constant, hence constant in x . \square

The last class of examples for convex Hamiltonian manifold was suggested to me by S. Tolman:

2.10. Theorem. *Let X be a K -manifold and $M := T_X^*$ with its natural Hamiltonian structure. Then M is convex.*

Proof: The map $\psi : M \rightarrow \mathfrak{t}^*$ is homogeneous for the natural scalar action on the fibers. Furthermore, X is embedded into M as the zero-section with $X \subseteq \psi^{-1}(0)$. Therefore, the cone $C = C_x$ does not depend on $x \in X$. Thus, X has a neighborhood U such that $\psi(U)$ is a neighborhood of 0 in C . Since ψ is homogeneous, we conclude $\psi(M) = C$. This shows in particular that $\psi(M)$ is convex.

Let $\pi : M \rightarrow X$ be the projection. Let $m \in M$ and $x = \pi(m)$. Then, again by homogeneity, there is a convex neighborhood V of Kx such that $m \in V$ and such that $\psi(V)$ is open in C (Theorem 2.8). Then any neighborhood of m is mapped to a neighborhood of $\psi(m)$ in $\psi(V)$. This implies that $\psi : M \rightarrow \psi(M)$ is open.

Finally, assume that the fiber over $u \in \psi(M)$ is not connected, i.e., is a disjoint union of two non-empty open pieces F_1 and F_2 . For every open K -stable subset U of X holds $\psi(\pi^{-1}(U)) = C$. Hence, $\pi(\psi^{-1}(u))$ is dense in X . Thus there is $x \in \overline{\pi(F_1)} \cap \overline{\pi(F_2)}$. Let V be a convex neighborhood of Kx in M . By homogeneity, we may assume V meets both F_1 and F_2 . But this contradicts the connectedness of fibers of $\psi|_V$. Thus, ψ has connected fibers. \square

3. Collective functions: Statement of the result

Let M be a Hamiltonian K -variety. Every element $\xi \in \mathfrak{k}$ induces a function μ_ξ in M by $\mu_\xi(x) := \langle \mu(x), \xi \rangle$. One of the defining properties of a moment map is

$$\{f, \mu_\xi\} = df(\xi_*) \quad \text{for all } f \in \mathcal{C}^\infty(M).$$

Here ξ_* is the vector field expressing the infinitesimal action of ξ on M . Thus, since K is connected, a function f is K -invariant if and only if it Poisson commutes with all functions μ_ξ . Conversely, consider the set of *collective functions*

$$\text{Col}(M) := \{g \in \mathcal{C}^\infty(M) \mid \{\mathcal{C}^\infty(M)^K, g\} = 0\}.$$

Thus every μ_ξ is collective. Slightly more generally, pull-back by μ induces a homomorphism of Poisson algebras $\mu^* : \mathcal{C}^\infty(\mathfrak{k}^*) \rightarrow \text{Col}(M)$.

The Poisson subalgebras $\mathcal{C}^\infty(M)^K$ and $\text{Col}(M)$ form a “dual pair” in $\mathcal{C}^\infty(M)$, i.e., one is the Poisson centralizer of the other. Their intersection, the set of invariant collective functions $\text{Col}^K(M)$, is the Poisson center of either algebra and is particularly important. Note that μ^* induces a homomorphism $\mathcal{C}^\infty(\mathfrak{t}^*/W) = \mathcal{C}^\infty(\mathfrak{k}^*)^K \rightarrow \text{Col}^K(M)$.

Let $U \subseteq M$ be open, K -stable. Then $\text{res}_U^M : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(U)$ has a dense image which implies that the restriction of a collective function to U is again collective.

Our aim is to describe the set of all collective functions. One step in this direction is the following criterion due to Karshon [Ka] (see [KL] Thm. 1).

3.1. Theorem. *Let M be a Hamiltonian K -manifold. Then $f \in \mathcal{C}^\infty(M)$ is collective if and only if f is locally constant on the fibers of μ .*

This theorem shows in particular that being collective is mostly a set-theoretic property. In particular, if a sequence of collective functions converges pointwise to a differentiable function f then also f is collective.

Our goal is a more precise description of $\text{Col}(M)$. For this we introduce first a basic construction. Let again $\psi : M \rightarrow \mathfrak{t}_+^*$ be the map with $K\mu(x) \cap \mathfrak{t}_+^* = \{\psi(x)\}$. Let $\mathfrak{a}^0 \subseteq \mathfrak{t}^*$ be the affine subspace spanned by $\psi(M)$ (see Corollary 2.9). We let K act on $V := \mathfrak{k}^* \times \mathfrak{a}^0$ by conjugation on the first and trivially on the second factor and consider $K\mathfrak{a}^0 \subseteq V$ where \mathfrak{a}^0 is embedded diagonally into V . This is a closed subset. In fact it equals (as a set) the fiber product $\mathfrak{k}^* \times_{\mathfrak{t}^*/W} \mathfrak{a}^0$, i.e., the set of pairs (u, v) such that u and v have the same image in $\mathfrak{k}^*/K = \mathfrak{t}^*/W$. This shows moreover that $K\mathfrak{a}^0$ is (the set of real points of) a real-algebraic closed subvariety of V . We define $\mathbb{R}[K\mathfrak{a}^0]$ to be the image of $\mathbb{R}[V] \xrightarrow{\text{res}} \mathcal{C}^\infty(K\mathfrak{a}^0)$. Note, that the kernel of $\mathbb{R}[V] \rightarrow \mathbb{R}[K\mathfrak{a}^0]$ is in general (much) larger than the ideal generated by the obvious equations expressing $K\mathfrak{a}^0$ as a fiber product. For example, the most extreme

case would be $\mathfrak{a}^0 = 0$, where the fiber product is the nilcone of \mathfrak{k}^* which happens to have just one real point but is nevertheless of positive dimension as an algebraic variety.

Definition: Let $\widetilde{K\mathfrak{a}^0}$ be (the set of real points of) the normalization of $K\mathfrak{a}^0$, i.e., $\widetilde{K\mathfrak{a}^0} = \text{AlgHom}(A, \mathbb{R})$ where A is the integral closure of $\mathbb{R}[K\mathfrak{a}^0]$ in its field of fractions. Moreover, we equip $\widetilde{K\mathfrak{a}^0}$ with the real (Hausdorff) topology.

By definition, there is a natural morphism $\nu : \widetilde{K\mathfrak{a}^0} \rightarrow K\mathfrak{a}^0$ and the K -action lifts to $\widetilde{K\mathfrak{a}^0}$.

3.2. Proposition. *The map $\nu : \widetilde{K\mathfrak{a}^0} \rightarrow K\mathfrak{a}^0$ is a homeomorphism and $\widetilde{K\mathfrak{a}^0}/K \rightarrow \mathfrak{a}^0$ is a diffeomorphism.*

Proof: Observe $(\mathfrak{k}^* \times_{\mathfrak{t}^*/W} \mathfrak{a}^0)/K = \mathfrak{k}^*/K \times_{\mathfrak{t}^*/W} \mathfrak{a}^0 = \mathfrak{a}^0$. Thus $K\mathfrak{a}^0/K = \mathfrak{a}^0$. Since \mathfrak{a}^0 is already normal we have also $\widetilde{K\mathfrak{a}^0}/K = \mathfrak{a}^0$. In particular, ν induces an isomorphism on orbit spaces. The fibers of ν are finite and all isotropy groups in $K\mathfrak{a}^0$ (being Levi subgroups) are connected which implies that ν is injective. Moreover, the dense subset of regular points of $K\mathfrak{a}^0$ is contained in the image of ν . Since ν is proper, it must be a homeomorphism. \square

Thus $\widetilde{K\mathfrak{a}^0}$ equals $K\mathfrak{a}^0$ as a topological space but it may have more differentiable functions. The image of $\mu \times \psi : M \rightarrow V$ is clearly contained in $K\mathfrak{a}^0$. So we obtain a map $\tilde{\mu} : M \rightarrow \widetilde{K\mathfrak{a}^0}$ which is continuous but in general not differentiable.

For studying the local structure of $\widetilde{K\mathfrak{a}^0}$ the following lemma is helpful. For a Levi subgroup L let $\mathfrak{t}^r \subseteq \mathfrak{t}^*$ be the set of points v with $W_v \subseteq W_L$. Put $\mathfrak{a}^r := \mathfrak{a}^0 \cap \mathfrak{t}^r$ (which might be empty).

3.3. Lemma. *For any Levi subgroup L there is a canonical map $\lambda : K \times^L \widetilde{L\mathfrak{a}^0} \rightarrow \widetilde{K\mathfrak{a}^0}$ which is a diffeomorphism over \mathfrak{a}^r .*

Proof: There is clearly a surjective map $K \times^L L\mathfrak{a}^0 \rightarrow K\mathfrak{a}^0$. Then λ is just the unique lift to the normalizations. To show that it is an isomorphism over \mathfrak{a}^r we complexify. Let $G := K_{\mathbb{C}}$. Then $(K\mathfrak{a}^0)_{\mathbb{C}}$ is the Zariski closure $\overline{G\mathfrak{a}^0_{\mathbb{C}}}$. Lemma 5.3 implies that $G \times^{L_{\mathbb{C}}} \overline{L_{\mathbb{C}}\mathfrak{a}^0_{\mathbb{C}}} \rightarrow \overline{G\mathfrak{a}^0_{\mathbb{C}}}$ is a closed embedding over $\mathfrak{a}^r_{\mathbb{C}}$. Since it is clearly surjective it is an isomorphism over $\mathfrak{a}^r_{\mathbb{C}}$. Hence the same holds for the normalizations. \square

The difference between $\widetilde{K\mathfrak{a}^0}$ and $K\mathfrak{a}^0$ is not yet completely understood. It translates into a property of certain *nilpotent* orbits of the *complexified* Lie algebra. We will state two results in this direction which might be useful. Let's call a compact group unitary if it is locally isomorphic to a product of a torus and a product of special unitary groups.

3.4. Proposition. *i) Assume \mathfrak{a}^0 contains an interior point of the Weyl chamber \mathfrak{t}^*_+ . Then ν is a diffeomorphism.*

ii) Let $v \in \mathfrak{a}^0$ and assume that the isotropy group K_v is unitary (this holds, in particular, if K itself is unitary). Then ν is a diffeomorphism in a neighborhood of v .

Proof: In case i) it suffices to show that ν is a diffeomorphism near every $v \in \mathfrak{a}^0$. Then with Lemma 3.3 we may reduce to the case $v = 0$. Let $L \subseteq K$ be the centralizer of \mathfrak{a}^0 . Then either $L = T$ or K is unitary. We are going to show that then ν is even an isomorphism of algebraic varieties.

For this we may complexify. Then $L_{\mathbb{C}} \subseteq G := K_{\mathbb{C}}$ is the Levi subgroup of a parabolic P . Let P_u be its unipotent radical. The complexification of $K\mathfrak{a}^0$ is $X := \overline{G\mathfrak{a}^0_{\mathbb{C}}}$. By the choice of L we have $\overline{P\mathfrak{a}^0} = \mathfrak{t} := \mathfrak{a}^0 \oplus \mathfrak{p}_u$. Thus the proper morphism $G \times^P \mathfrak{t} \rightarrow \mathfrak{g}$ has exactly image X . It factors through the normalization \tilde{X} of X . Thus we have $G \times^P \mathfrak{t} \xrightarrow{\pi} \tilde{X} \xrightarrow{\nu} X$. These are morphisms over \mathfrak{a}^0 .

Since L is the centralizer of \mathfrak{a}^0 , the morphism π is birational. Then it follows from [K1] 4.1 that \tilde{X} has rational singularities and is in particular Cohen-Macaulay. Let \tilde{X}_0 be the zero-fiber of $\tilde{X} \rightarrow \mathfrak{a}^0$. Since π is an isomorphism over a generic point of \tilde{X}_0 it is generically reduced. Being defined by a regular sequence it is reduced. By Nakayama's lemma, ν is an isomorphism if and only if its schematic zero-fiber $\nu^{-1}(0)$ is a (reduced) point. Since $\nu^{-1}(0) \subseteq \tilde{X}_0$ we are reduced to show that $\tilde{X}_0 \rightarrow \mathfrak{g}$ is a closed embedding. This would be true if $G\mathfrak{p}_u$ is normal and $G \times^P \mathfrak{p}_u \rightarrow G\mathfrak{p}_u$ is birational. But these conditions hold under our assumptions: if $L = T$ then $P = B$ is a Borel subgroup and we conclude by results of Kostant and Springer (Springer resolution, see e.g. [Sp]). If K is unitary then all nilpotent orbit closures are normal (Kraft-Procesi [KP]) and all stabilizers are connected modulo center. \square

The affine subspace \mathfrak{a}^0 of \mathfrak{t}^* is in general not W -stable. Thus we define

$$\begin{aligned} N_W(\mathfrak{a}^0) &:= \{w \in W \mid w\mathfrak{a}^0 = \mathfrak{a}^0\}; \\ C_W(\mathfrak{a}^0) &:= \{w \in W \mid w|_{\mathfrak{a}^0} = \text{id}\}; \\ W(\mathfrak{a}^0) &:= N_W(\mathfrak{a}^0)/C_W(\mathfrak{a}^0). \end{aligned}$$

If we let $W(\mathfrak{a}^0)$ act on the second factor of $V = \mathfrak{t}^* \times \mathfrak{a}^0$ we obtain actions on $K\mathfrak{a}^0$ and $\widetilde{K\mathfrak{a}^0}$ which commute with K . Let $W' \subseteq W(\mathfrak{a}^0)$ be any subgroup. Then also $\widetilde{K\mathfrak{a}^0}/W' \rightarrow K\mathfrak{a}^0/W'$ is a homeomorphism and $(\widetilde{K\mathfrak{a}^0}/W')/K = \mathfrak{a}^0/W'$. Now we can formulate our main result.

3.5. Theorem. *Let M be a convex Hamiltonian K -manifold and let \mathfrak{a}^0 be the affine subspace spanned by $\psi(M)$. Then there is a unique subgroup W_M of $W(\mathfrak{a}^0)$ such that:*

- i) *The map $\Phi = \tilde{\mu}/W_M : M \rightarrow \widetilde{K\mathfrak{a}^0}/W_M$ is differentiable and induces an isomorphism $\mathcal{C}^\infty(\Phi(M)) \xrightarrow{\sim} \text{Col}(M)$.*

ii) Let R be the set of all reflections in $W(\mathfrak{a}^0)$ whose reflecting hyperplane meets $\psi(M)$. Then W_M is generated by a subset of R .

The proof is given in the next section. Let me first give some conclusions.

3.6. Corollary. *The map $\varphi = \psi/W_M : M \rightarrow \mathfrak{a}^0/W_M$ is differentiable and induces an isomorphism $\mathcal{C}^\infty(\varphi(M)) \xrightarrow{\sim} \text{Col}^K(M)$. Thus $\text{Col}^K(M)$ can be identified with the algebra of differentiable functions on an r -dimensional semi-analytic subset of \mathbb{R}^r where $r = \dim \mathfrak{a}^0$.*

Proof: The first statement follows from Proposition 3.2 and Theorem 3.5i) by taking K -invariants. The second follows from the fact that the ring of invariants of a reflection group is a polynomial ring (hence $\mathfrak{a}^0/W_M \hookrightarrow \mathbb{R}^r$) and that $\psi(M)$ is locally even semi-algebraic (Theorem 2.5ii)). \square

3.7. Corollary. *i) The ring $\text{Col}(M)$ is a finitely generated $\mathcal{C}^\infty(\mu(M))$ -module.*

ii) Assume that $\psi(M)$ contains an interior point of \mathfrak{k}_+^ or that K_v is unitary for all $v \in \psi(M)$. Then $\text{Col}(M)$ is, as a $\mathcal{C}^\infty(\mu(M))$ -module, generated by finitely many invariant collective functions.*

Proof: i) The morphism of algebraic varieties $\widetilde{K\mathfrak{a}^0} \rightarrow \mathfrak{k}^*$ is finite. Hence there are f_1, \dots, f_s in $\mathbb{R}[\widetilde{K\mathfrak{a}^0}]^{W_M}$ which generate it as a $\mathbb{R}[\mathfrak{k}^*]$ -module. It follows from [Mal] Ch. V, Cor. 4.4 that for every $x \in M$ the f_i generate the stalk $\mathcal{C}_{\Phi(M), \Phi(x)}^\infty$ as a $\mathcal{C}_{\mu(M), \mu(x)}^\infty$ -module. By a partition of unity argument this shows that $\text{Col}(M) = \mathcal{C}^\infty(\Phi(M))$ is generated by the f_i as a $\mathcal{C}^\infty(\mu(M))$ -module.

ii) The assumptions imply (Proposition 3.4) that $\widetilde{K\mathfrak{a}^0} \rightarrow K\mathfrak{a}^0 \subseteq \mathfrak{k}^* \times \mathfrak{a}^0$ is a diffeomorphism in a neighborhood of $\widetilde{\mu}(M)$. Thus we can replace $\widetilde{K\mathfrak{a}^0}$ by $K\mathfrak{a}^0$ and therefore choose the f_i to be K -invariant. \square

Another characterization of W_M is:

3.8. Proposition. *A subgroup W' of $W(\mathfrak{a}^0)$ contains W_M if and only if $\psi/W' : M \rightarrow \mathfrak{a}^0/W'$ is differentiable.*

Proof: If $W_M \subseteq W'$ then $\varphi' := \psi/W'$ is clearly differentiable. Conversely, assume φ' is differentiable. Then Theorem 3.1 implies $\mathcal{C}^\infty(\varphi'(M)) \hookrightarrow \text{Col}^K(M) = \mathcal{C}^\infty(\varphi(M))$. This shows that $\varphi(M) \rightarrow \varphi'(M)$ is differentiable. Looking at the formal stalk at $u \in \psi(M)$ this implies $\mathcal{C}^\infty(\psi(M))_u^{\wedge W'_u} \subseteq \mathcal{C}^\infty(\psi(M))_u^{\wedge W_{M,u}}$ and therefore $W_{M,u} \subseteq W'_u$. Hence $W_M \subseteq W'$ since W_M is generated its isotropy groups in $\psi(M)$. \square

We conclude this section with two examples. The first one is just Lerman's example from [L1]: let $K = SU(2)$ and $M := \mathbb{C}^2$ with its symplectic structure coming from the K -

invariant Hermitian form $q(z_1, z_2) := z_1 \bar{z}_1 + z_2 \bar{z}_2$. Then a moment map $M \rightarrow \mathfrak{k}^*$ exists and is in fact given by *quadratic* polynomials. Every K -invariant on M is a composite with q . Hence, q is invariant collective. Assume there is $h \in \mathcal{C}^\infty(\mathfrak{k}^*)$ with $q = f \circ \mu$. Then h were a non-zero K -invariant function of degree one which clearly does not exist. On the other hand, one easily verifies that $\psi : M \rightarrow \mathfrak{t}^* \cong \mathbb{R}$ is given by q . Thus ψ is differentiable and $W_M = 1$.

The second example is a variant of the preceding one: Let $M := \mathbb{C}^{2n}$ and K the maximal compact subgroup of $Sp_{2n}(\mathbb{C})$. Then again a quadratic moment map $\mu : M \rightarrow \mathfrak{k}^*$ exists and K fixes a Hermitian norm q . Let $\bar{K} := U(2n)$ be the unitary group associated to q . Then there is also a quadratic moment map $\bar{\mu} : M \rightarrow \bar{\mathfrak{k}}^*$ such that μ is the composition of $\bar{\mu}$ with $\bar{\mathfrak{k}}^* \rightarrow \mathfrak{k}^*$. It happens that every K -invariant is a composite with q . Thus $\mathcal{C}^\infty(M)^K = \mathcal{C}^\infty(M)^{\bar{K}}$ and therefore, $\text{Col}_K(M) = \text{Col}_{\bar{K}}(M)$. This shows in particular that $\bar{\mathfrak{k}}^*$ embeds into $\text{Col}_K(M)$ as a space of quadratic polynomials. But the space of quadratic polynomials which pull-back from $\mu \times \psi$ has at most dimension $\dim K + 1$. This shows that for $n > 1$ the normalization of $K\mathfrak{a}^0$ is really necessary. It also shows that one needs some restrictions in Corollary 3.7.

4. Proof of Theorem 3.5

We investigate the problem of finding the collective functions first on the level of formal power series. For this, we adopt the following notation: If X is a set, $Y \subseteq X$ a subset and \mathcal{R} a ring of functions on X , then let \mathcal{R}_Y^\wedge denote the completion of \mathcal{R} with respect to the ideal of functions vanishing in Y .

Assume that Kx is an isotropic orbit of M . By Theorem 9.2 and the remark following it we may assume that $M = K \times^H (\mathfrak{h}^\perp \times S)$ and $x = [1, (0, 0)]$. The manifold M has a real algebraic structure: with $\mathbb{R}[K] \subseteq \mathcal{C}^\infty(K)$, the ring of representative (i.e., K -finite) functions, and $\mathbb{R}[\mathfrak{h}^\perp \times S]$, the ring of real polynomials, we define

$$\mathbb{R}[M] := (\mathbb{R}[K] \otimes \mathbb{R}[\mathfrak{h}^\perp \times S])^H.$$

This is a finitely generated \mathbb{R} -algebra and turns M into a real algebraic manifold. All K -orbits are Zariski closed. Also the symplectic structure and the moment map are real algebraic. Thus $\mathbb{R}[M]$ is a Poisson algebra, $\mathbb{R}[M]^K$ a Poisson subalgebra. Let $\text{Col}_{\text{alg}}(M) \subseteq \mathbb{R}[M]$ be its centralizer.

4.1. Lemma. *Let M and x be as above. Then there is a subgroup $W(x)$ of $W(\mathfrak{a}^0)$ which is generated by reflections such that $\Phi = \tilde{\mu}/W(x) : M \rightarrow \widetilde{K\mathfrak{a}^0}/W(x)$ is algebraic and induces an isomorphism $\mathbb{R}[\widetilde{K\mathfrak{a}^0}/W(x)]_0^\wedge \xrightarrow{\sim} \text{Col}_{\text{alg}}(M)_{Kx}^\wedge$.*

Proof: We are going to determine the complexification of the real-algebraic variety M . Since H is compact, the symplectic structure of S comes from an H -stable Hermitian form. In particular, S has also the structure of a complex vector space. Hence $S_{\mathbb{C}} = S \otimes_{\mathbb{R}} \mathbb{C} = S \oplus \bar{S} = S \oplus S^* = T_S^*$ where T^* denotes the complex algebraic cotangent bundle. With $G := K_{\mathbb{C}}$ consider the complex algebraic G -variety $X := G \times^{H_{\mathbb{C}}} S$. Then $T_X^* = G \times^{H_{\mathbb{C}}} Z$ where Z is the restriction of T_X^* to the fiber $S \subseteq X$. Let $\mathfrak{q} \subseteq \mathfrak{g}$ be a $H_{\mathbb{C}}$ -stable complement. Then for every point $s \in S$ we have a splitting $T_{X,s} = \mathfrak{q}s \oplus T_{S,s}$. Since $\mathfrak{q}^* \cong \mathfrak{h}_{\mathbb{C}}^{\perp}$ we obtain $Z = \mathfrak{q}^* \times T_S^* \cong (\mathfrak{h}^{\perp} \times S)_{\mathbb{C}}$. Therefore, we have constructed an isomorphism $T_X^* \xrightarrow{\sim} M_{\mathbb{C}}$. It is easy to see that the moment map on M induces just the natural cotangent bundle moment map on T_X^* . Thus we have a commutative diagram

$$\begin{array}{ccc} M & \hookrightarrow & T_X^* \\ \downarrow \mu & & \downarrow \mu_{\mathbb{C}} \\ \mathfrak{t}^* & \hookrightarrow & \mathfrak{g}^* \end{array}$$

In [K3] 3.3, a subspace $\mathfrak{a}'_{\mathbb{C}} \subseteq \mathfrak{t}_{\mathbb{C}}^*$ was constructed such that the closure of the image of $\mu_{\mathbb{C}}$ equals $\overline{G\mathfrak{a}'_{\mathbb{C}}}$. The complexified space $\mathfrak{a}_{\mathbb{C}}^0$ has the same property (see Corollary 2.9). Moreover, both have the same image in \mathfrak{t}^*/W thus they are conjugate by an element w of W . The construction of $\mathfrak{a}'_{\mathbb{C}}$ in [K3] depended on the choice of a Borel subgroup B . Thus replacing B by wBw^{-1} we may assume $\mathfrak{a}'_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}}^0$.

Now consider the fiber product $\tilde{T} = T_X^* \times_{\mathfrak{t}^*/W} \mathfrak{a}_{\mathbb{C}}^0$. Then the group $W(\mathfrak{a}^0)$ acts transitively on the irreducible components of \tilde{T} (or at least on those which map dominantly to the image of T_X^* in $\mathfrak{t}_{\mathbb{C}}^*/W$). There is one such component which is distinguished. Consider the fiber product $\tilde{M} := M \times_{\mathfrak{t}^*/W} \mathfrak{t}_+^*$ which is an irreducible semi-algebraic space. Thus the Zariski closure of the image of $\tilde{M} \rightarrow \tilde{T}$ is an irreducible component \hat{T} . Observe that \hat{T} , being the closure of a set of real points, is defined over \mathbb{R} . The component \hat{T}_X in [K3] depended again on the choice of B . So changing it we may assume that $\hat{T} = \hat{T}_X$.

Now we let $W(x)$ be the stabilizer of \hat{T} in $W(\mathfrak{a}^0)$. It is generated by reflections by [K1] 6.6. The morphism $\hat{T}/W(x) \rightarrow T_X^*$ is finite and birational, hence an isomorphism. Since \hat{T} maps to $\mathfrak{a}_{\mathbb{C}}^0$ we obtain morphisms $T_X^* \rightarrow \mathfrak{a}_{\mathbb{C}}^0/W(x)$ and $\Phi_{\mathbb{C}} : T_X^* \rightarrow \widetilde{G\mathfrak{a}_{\mathbb{C}}^0}/W(x)$ where $\widetilde{G\mathfrak{a}_{\mathbb{C}}^0}$ is the normalization of the closure of $G\mathfrak{a}_{\mathbb{C}}^0$ in $\mathfrak{g}^* \times \mathfrak{a}_{\mathbb{C}}^0$.

By definition, the variety M_X of [K1] is normal and finite over $\widetilde{G\mathfrak{a}_{\mathbb{C}}^0}/W(x)$. Moreover $M_X \rightarrow \mathfrak{a}_{\mathbb{C}}^0/W(x)$ is the categorical quotient. The generic orbits are closed and the generic stabilizers of $\widetilde{G\mathfrak{a}_{\mathbb{C}}^0}/W(x)$ are connected. We conclude that $M_X = \widetilde{G\mathfrak{a}_{\mathbb{C}}^0}/W(x)$.

By [K4] 9.4, $\Phi_{\mathbb{C}}$ induces an isomorphism

$$\mathbb{C}[\widetilde{G\mathfrak{a}_{\mathbb{C}}^0}/W(x)] \xrightarrow{\sim} \text{Col } T_X^*.$$

Since all maps and varieties are defined over \mathbb{R} we get an algebraic map between real points $\Phi : M \rightarrow \widetilde{K\mathfrak{a}^0}/W(x)$ which induces an isomorphism $\mathbb{R}[\widetilde{K\mathfrak{a}^0}/W(x)] \xrightarrow{\sim} \text{Col}_{\text{alg}}(M)$.

Finally, to get an isomorphism between completions, we use that that $\mathbb{R}^{>0}$ acts on $M = G \times^H (\mathfrak{h}^\perp \times S)$ by $t \cdot [g, (\xi, s)] := [g, (t^2\xi, ts)]$ which equips all algebras with a grading. Moreover, Φ is homogeneous. Since Gx is exactly the fixed point set of this action, we see that completion at Gx is the same as replacing the direct sum of homogeneous components by their direct product. This shows the claim. \square

We proceed with the following comparison result:

4.2. Lemma. *Let M and x be as above and let U be some K -stable open neighborhood of x . Then there is an isomorphism $\text{Col}_{\text{alg}}(M)_{Kx}^\wedge \xrightarrow{\sim} \text{Col}(U)_{Kx}^\wedge$.*

Proof: The image of $\mathbb{R}[M]^K \hookrightarrow \mathcal{C}^\infty(U)^K$ is dense (Stone-Weierstraß), thus we obtain $\text{Col}_{\text{alg}}(M) \hookrightarrow \text{Col}(U)$ and $\text{Col}_{\text{alg}}(M)_{Kx}^\wedge \hookrightarrow \text{Col}(U)_{Kx}^\wedge$.

To show surjectivity we again use the $\mathbb{R}^{>0}$ -action on M . For any subspace S of $\mathcal{C}^\infty(M)$ let S_d be its subset of elements which are homogeneous of degree d with respect to this $\mathbb{R}^{>0}$ action. Let $I \subseteq \mathcal{C}^\infty(U)$ be the ideal of functions vanishing in K/H . Then one sees easily $\mathcal{C}^\infty(U)/I^{d+1} = \bigoplus_{i=0}^d \mathcal{C}^\infty(M)_d$. Thus $\mathcal{C}^\infty(U)_{Kx}^\wedge = \prod_{d=0}^\infty \mathcal{C}^\infty(M)_d$. The Poisson bracket on M is homogeneous of degree -2 , i.e., $\{\mathcal{C}^\infty(M)_a, \mathcal{C}^\infty(M)_b\} \subseteq \mathcal{C}^\infty(M)_{a+b-2}$. This implies that an element of $\mathcal{C}^\infty(U)_{Kx}^\wedge$ is collective if and only if all its homogeneous components are. An analogous result holds for $\mathbb{R}[M]_{Kx}^\wedge$. Thus, it suffices to show that $\text{Col}_{\text{alg}}(M)_d \rightarrow \text{Col}(M)_d$ is surjective for every $d \in \mathbb{N}$.

For an irreducible representation η of K and a K -module V let V^η denote its η -isotypic component. Let $\mathcal{E} \rightarrow G/H$ be a K -equivariant vector bundle of finite rank. Then $\Gamma(G/H, \mathcal{E})^\eta$ is finite-dimensional where Γ denotes differentiable sections. This implies in particular $\dim \mathcal{C}^\infty(M)_d^\eta < \infty$. From Stone-Weierstraß we obtain $\mathbb{R}[M]_d^\eta \xrightarrow{\sim} \mathcal{C}^\infty(M)_d^\eta$. Thus also $\text{Col}_{\text{alg}}(M)_d^\eta \xrightarrow{\sim} \text{Col}(M)_d^\eta$.

Since $\mathbb{R}[\widetilde{K\mathfrak{a}^0}/W(x)]$ is a finite $\mathbb{R}[\mathfrak{k}^*]$ -module we have $\dim \mathbb{R}[\widetilde{K\mathfrak{a}^0}/W(x)]_d < \infty$. Thus, by Lemma 4.1, also $\text{Col}_{\text{alg}}(M)_d$ is finite dimensional. We conclude that $\text{Col}_{\text{alg}}(M)_d \xrightarrow{\sim} \bigoplus_\eta \text{Col}(M)_d^\eta =: \text{Col}(M)_d^{\text{fin}}$ is finite dimensional. But $\text{Col}(M)_d^{\text{fin}}$ is dense in $\text{Col}(M)_d$ (Peter-Weyl). Thus equality holds and we obtain $\text{Col}_{\text{alg}}(M)_d \xrightarrow{\sim} \text{Col}(M)_d$. \square

From this we deduce an analogue of Theorem 3.5 on the level of formal power series.

4.3. Lemma. *Let M be any Hamiltonian K -manifold. For every $x \in M$ there is a (unique) subgroup $W(x) \subseteq W(\mathfrak{a}^0)_{\psi(x)}$ which is generated by reflections and an open K -stable neighborhood U of x such that:*

- i) The map $\Phi = \tilde{\mu}/W(x) : M \rightarrow \widetilde{K\mathfrak{a}^0}/W(x)$ is differentiable on U and induces an isomorphism $\mathcal{C}^\infty(\widetilde{K\mathfrak{a}^0}/W(x))_{K\Phi(x)}^\wedge \xrightarrow{\sim} \text{Col}(U)_{Kx}^\wedge$.
- ii) U carries an analytic structure such that $\Phi|_U$ is analytic.

Proof: We may change x in its orbit such that $\mu(x) \in \mathfrak{t}_+^*$. Let $L := K_{\mu(x)} \subseteq K$. Then Lx is isotropic in $M(L)$ (see Theorem 9.1). Thus Lemma 4.1 provides us with a group $W(x) \subseteq W_L$ such that $\Phi_L : M(L) \rightarrow Y := \widetilde{L\mathfrak{a}^0}/W(x)$ is differentiable and even analytic in a neighborhood of Lx . We obtain a commutative diagram

$$\begin{array}{ccc} K \times^L M(L) & \rightarrow & M \\ \downarrow \Phi' & & \downarrow \Phi \\ K \times^L Y & \rightarrow & \widetilde{K\mathfrak{a}^0}/W(x) \end{array}$$

where the horizontal arrows are diffeomorphisms (Theorem 9.1, Lemma 3.3) and $\Phi' = K \times^L \Phi_L$ is differentiable in a neighborhood of Kx or its image. Thus also Φ is differentiable and analytic in a neighborhood of Kx .

By Lemma 4.1 and Lemma 4.2 we have an isomorphism $\mathcal{C}^\infty(Y)_{\Phi'(x)}^\wedge \xrightarrow{\sim} \text{Col}(M(L))_{Lx}^\wedge$. We need a parameter dependent version of it. Let $\text{Col}_K(M(L))$ be the set of differentiable functions $h_k(x) = h(k, x)$ on $K \times M(L)$ which are collective for every fixed k . Choose functions $f_i \in \mathcal{C}^\infty(Y)$ whose Taylor series form a topological basis of $\mathcal{C}^\infty(Y)_{\Phi'(x)}^\wedge$. Then the image of h_k in $\mathcal{C}^\infty(M(L))_{Lx}^\wedge$ can be written as $\sum_i a_i(k) f_i(\Phi(x))$. The coefficients a_i are unique and can in fact be expressed in terms of finitely many derivatives of h_k . Thus they are differentiable functions in k . This shows

$$\mathcal{C}^\infty(K \times Y)_{K \times \Phi'(x)}^\wedge \xrightarrow{\sim} \text{Col}_K(M(L))_{K \times Lx}^\wedge$$

and therefore

$$\mathcal{C}^\infty(K \times^L Y)_{K\Phi'(x)}^\wedge \xrightarrow{\sim} \text{Col}_K(M(L))_{K \times Lx}^\wedge.$$

A differentiable function h on $K \times^L M(L)$ is collective if and only if it is locally constant on fibers of Φ' (Theorem 3.1). Therefore, for every $k \in K$ the restriction h_k of h to $[k, M(L)] \cong M(L)$ is collective. Thus we get an inclusion

$$\text{Col}(K \times^L M(L))_{Kx}^\wedge \hookrightarrow \text{Col}_K(M(L))_{K \times Lx}^\wedge.$$

This implies

$$\mathcal{C}^\infty(K \times^L Y)_{K\Phi'(x)}^\wedge \xrightarrow{\sim} \text{Col}(K \times^L M(L))_{Kx}^\wedge$$

which finishes the proof. \square

Now we come to a key lemma which compares the little Weyl groups $W(x)$ of nearby points. The proof rests on an algebraic statement which is proved in section 7.

4.4. Lemma. *Let M be a Hamiltonian K -manifold. Then every $x \in M$ has an open K -stable neighborhood U such that $W(y) = W(x)_{\psi(y)}$ for all $y \in U$.*

Proof: With the help of the the Cross Section Theorem 9.1 and the Slice Theorem 9.2 we can assume we are in the situation $M = K \times^H (\mathfrak{h}^\perp \times S)$ and $x = [1, (0, 0)]$. Then we have an algebraic morphism $\varphi : M \rightarrow \mathfrak{a}^0/W(x)$ such that the formal power series ring at 0 in $\mathfrak{a}^0/W(x)$ identifies with the formal invariant collective functions along Kx .

For $y \in M$ and $u := \psi(y)$, put $W_y := W(x)_u$. Let u' be the image of u in \mathfrak{a}^0/W_y . Then $\mathfrak{a}^0/W_y \rightarrow \mathfrak{a}^0/W(x)$ is invertible in a neighborhood of u' , i.e., $M \rightarrow \mathfrak{a}^0/W_y$ is differentiable in Ky . As in Proposition 3.8 this implies $W(y) \subseteq W_y$.

The maps $M \rightarrow \mathfrak{a}^0/W(y) \rightarrow \mathfrak{a}^0/W_y \rightarrow \mathfrak{a}^0/W(x)$ are all differentiable in y . So they induce

$$\mathbb{R}[\mathfrak{a}^0]_u^{\wedge W(x)} \xrightarrow{\simeq} \mathbb{R}[\mathfrak{a}^0]_u^{\wedge W_y} \hookrightarrow \mathbb{R}[\mathfrak{a}^0]_u^{\wedge W(y)} \hookrightarrow \mathbb{R}[M]_{Ky}^{\wedge}.$$

It suffices to show that the second homomorphism is an isomorphism. For this, we complexify. Then Gy is a closed orbit in T_X^* and we get

$$\mathbb{C}[\mathfrak{a}^0]_u^{\wedge W(x)} \xrightarrow{\simeq} \mathbb{C}[\mathfrak{a}^0]_u^{\wedge W_y} \rightarrow \mathbb{C}[\mathfrak{a}^0]_u^{\wedge W(y)} \hookrightarrow \mathbb{C}[T_X^*]_{Gy}^{\wedge}.$$

Recall, that both W_y and $W(y)$ are generated by reflections. Thus, if $W(y) \neq W_y$ then the map $\mathfrak{a}^0/W(y) \rightarrow \mathfrak{a}^0/W_y$ is ramified over u' . That means that \mathfrak{a}^0/W_y contains a divisor whose preimage in $\mathfrak{a}^0/W(y)$ is not reduced. Thus the same happens for the spectra of completions. Hence also $\mathbb{C}[\mathfrak{a}^0]_u^{\wedge W(x)} \rightarrow \mathbb{C}[T_X^*]_{Gy}^{\wedge}$ has non-reduced fibers in codimension one. Since the completion of a reduced algebra of finite type is reduced we obtain a contradiction to Theorem 7.5. Thus $W_y = W(y)$. \square

4.5. Lemma. *Assume M to be convex. Let $W_M \subseteq W(\mathfrak{a}^0)$ be the group generated by all $W(x)$, $x \in M$. Then $W(x) = (W_M)_{\psi(x)}$ for all $x \in M$.*

Proof: Let $S := \psi(M)$. Since M is convex, ψ has connected fibers. Thus Lemma 4.4 implies that $W(u) := W(x)$ (with $u = \psi(x)$) is well defined. Since W_M is generated by reflections and since $S \subseteq \mathfrak{t}_+^*$ there is a unique Weyl chamber \mathfrak{a}_+^0 with respect to W_M which contains S . Let Δ be a root system attached to W_M (we consider roots as affine functions on \mathfrak{a}^0).

Also every $W(u)$ is a reflection group giving rise to a root subsystem $\Delta(u) \subseteq \Delta$. The chamber \mathfrak{a}_+^0 determines a chamber for $W(u)$ and therefore a set $\Sigma(u) \subseteq \Delta(u)$ of

simple roots. Let $w \in S$ and $w' \in S$ any nearby point. Since w' is in the fundamental chamber for W_M it is also in the fundamental chamber for $W(w)$. Thus, by Lemma 4.4, $\Sigma(w') = \{\gamma \in \Sigma(w) \mid \gamma(w') = 0\}$.

Let Σ be the union of all $\Sigma(u)$, $u \in S$. It suffices to show that Σ is the set of simple roots for W_M because then the isotropy groups in \mathfrak{a}_+^0 are generated by simple reflections. Since Δ is certainly generated by the reflection corresponding to Σ it suffices to show that Σ is a set of simple roots for some root system. This is equivalent to $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta \in \Sigma$ where (\cdot, \cdot) is some W -invariant scalar product on Δ . So assume $\alpha \in \Sigma(u)$, $\beta \in \Sigma(v)$. Then we have $\alpha(u) = \beta(v) = 0$. Moreover, $u, v \in \mathfrak{a}_+^0$ implies $\alpha(v) \geq 0$, $\beta(u) \geq 0$.

Assume first $\alpha(v) = 0$. Since M is convex, the line segment $[u, v]$ is entirely contained in S . Moreover, α vanishes on it. This implies that $\alpha \in \Sigma(w)$ for all $w \in [u, v]$. In particular, $\alpha, \beta \in \Sigma(v)$ which shows $(\alpha, \beta) \leq 0$.

The case $\beta(u) = 0$ is handled the same way. Thus we are left with the case where both $\alpha(v)$ and $\beta(u)$ are positive. Let $\gamma := \alpha - \beta$. Then $\gamma(u) < 0$ and $\gamma(v) > 0$. Hence γ is not a root which implies $(\alpha, \beta) \leq 0$. \square

Proof of Theorem 3.5: The group W_M of Lemma 4.5 certainly satisfies *ii*). We have $W(x) \subseteq W_M$ for all $x \in M$. Thus $\Phi = \tilde{\mu}/W_M$ is differentiable by Lemma 4.3*i*). It remains to show that every $h \in \text{Col}(M)$ is pull-back from $\Phi(M)$. For this, we verify the conditions of Theorem 8.3. Put $Y := \Phi(M)$. Since $Y = \nu^{-1}(K\psi(M))/W_M$ it is locally semi-algebraic (Theorem 2.5*ii*). Thus condition *i*) is satisfied. Condition *ii*) follows from the convexity of M and $\tilde{\mu}(M)/K = \psi(M)$ (Proposition 3.2). Condition *iii*) is Lemma 4.3*ii*). Finally, condition *iv*) follows again from $\tilde{\mu}(M)/K = \psi(M)$ and from the convexity of M .

Thus it remains to show that h pushes down formally. For every $x \in M$ we have $W(x) = (W_M)_{\psi(x)} = (W_M)_{\tilde{\mu}(x)}$ (Lemma 4.5). Thus

$$\mathcal{C}^\infty(\widetilde{K\mathfrak{a}^0}/W_M)_{K\Phi(x)}^\wedge \xrightarrow{\sim} \mathcal{C}^\infty(\widetilde{K\mathfrak{a}^0}/W(x))_{K\Phi(x)}^\wedge.$$

The assertion follows from Lemma 4.3*i*). \square

5. Hamiltonian actions in the algebraic category

In the proof of Lemma 4.4 we crucially used a result on complex algebraic cotangent bundles. The purpose of the next sections is to provide a proof of this result. All varieties, groups and maps will be complex algebraic. The group G will always be connected and reductive. First we develop algebraic versions of the two structure theorems for Hamiltonian group actions in the algebraic setting. First, we consider the equivariant Darboux-Weinstein Theorem:

5.1. Theorem. For $i \in \{1, 2\}$ let Z_i be a smooth, affine, symplectic G -variety and $Y_i \subseteq Z_i$ a closed G -orbit. Let $\varphi : Y_1 \rightarrow Y_2$ be a G -isomorphism and suppose that the induced isomorphism $d\varphi : T_{Y_1} \xrightarrow{\sim} \varphi^*T_{Y_2}$ between tangent bundles extends to a G -isomorphism of symplectic vector bundles $T(\varphi) : T_{Z_1}|_{Y_1} \rightarrow \varphi^*T_{Z_2}|_{Y_2}$. Let \widehat{Z}_i be the spectrum of the completion of $\mathbb{C}[Z_i]$ along Y_i . Then:

- i) There exists an isomorphism $\widehat{\varphi} : \widehat{Z}_1 \rightarrow \widehat{Z}_2$ of symplectic G -schemes which induces φ and $T(\varphi)$.
- ii) Let $\mu_i : Z_i \rightarrow \mathfrak{g}^*$ be moment maps with $\mu_1|_{Y_1} = \varphi^*\mu_2|_{Y_2}$. Let $\widehat{\mu}_i$ be the induced moment map on \widehat{Z}_i . Then $\widehat{\mu}_1 = \widehat{\varphi}^*\widehat{\mu}_2$.

Proof: Let $x_1 \in Y_1$ and $x_2 = \varphi(x_1)$. Then the isotropy group $H = G_{x_1} = G_{x_2}$ is reductive. Hence there are H -stable smooth subvarieties S_i of Z_i which meet Y_i transversally in x_i . Moreover, we can choose the slices S_i in such a way that $T(\varphi)$ maps $V_1 := T_{x_1}(S_1)$ isomorphically to $V_2 := T_{x_2}(S_2)$. Thus if we put $Z := G \times^H V_1$, $Y = G/H \subseteq Z$, and \widehat{Z} the completion of Z along Y we obtain G -isomorphisms $\widehat{Z} \rightarrow \widehat{Z}_i$ such that φ and $T(\varphi)$ correspond to the identity on Y and $T_{\widehat{Z}}|_Y$, respectively. Moreover, we get two symplectic forms ω_i on \widehat{Z} . Our assumptions imply $\omega_1|_Y = \omega_2|_Y$. Under these conditions we have to show that there is a G -automorphism of \widehat{Z} which maps ω_1 to ω_2 and which is the identity on Y and $T_{\widehat{Z}}|_Y$.

The proof of this is completely analogous to the proof of the classical Darboux-Weinstein theorem given in [GS1], we just have to make sure that all constructions work in the algebraic category.

Since Z is a vector bundle it carries a scalar multiplication $\varphi : \mathbf{A}^1 \times Z \rightarrow Z : (t, x) \mapsto \varphi_t(x)$. Since $\varphi(\mathbf{A}^1 \times Y) = Y$ this induces a morphism $\widehat{\varphi} : \widehat{Z}_t := (\mathbf{A}^1 \times Z)_{\mathbf{A}^1 \times Y} \rightarrow \widehat{Z}$. The regular functions on \widehat{Z}_t can be described as follows. Let y_1, \dots, y_s be generators of the sheaf of sections of $G \times^H V_1^*$. Then

(*) every function on \widehat{Z}_t is a formal power series in the y_i with coefficients in $\mathbb{C}[Y][t]$.

The scalar multiplication also induces an Euler vector field ξ on Z , \widehat{Z} , and \widehat{Z}_t . Let $\sigma := \omega_1 - \omega_0$ which is a closed 2-form on \widehat{Z} . Then we can construct the 1-form

$$\beta := \int_0^1 \iota_{\xi}(\varphi^*\sigma) dt.$$

By (*), the integration is over polynomials in t , hence well-defined. Moreover, since both ξ and σ vanish in Y , the form β vanishes quadratically.

Next we form $\omega_t := \omega_0 + t\sigma = (1-t)\omega_0 + t\omega_1$. Since ω_0 is non-degenerate and $\sigma|_Y = 0$, this is a non-degenerate closed 2-form on \widehat{Z}_t (relative to \mathbf{A}^1). Thus we can define a vector field η_t on \widehat{Z}_t by $\eta_t \lrcorner \omega_t = -\beta$.

I claim that one can integrate the time dependent vector field η_t on \widehat{Z} . For this, it suffices to show that the derivation $\frac{\partial}{\partial t} + \eta_t$ acts pointwise topologically nilpotently on $\mathbb{C}[\widehat{Z}_t]$. But η_t also vanishes quadratically in Y . Thus it strictly increases the order of vanishing along Y . On the other hand, a sufficiently high power of $\frac{\partial}{\partial t}$ also increases the order of vanishing by (*).

Thus, we obtain a one-parameter family f_t of morphisms of \widehat{Z} into itself such that $f_0 = \text{id}$. Moreover, f_t is the identity on Y and $T_{\widehat{Z}}|_Y$ since η_t vanishes quadratically along Y . This implies that f_t is an automorphism for every t , in particular, for $t = 1$. Furthermore, everything being G -equivariant, also f_1 commutes with the G -action.

Finally, the point of all this is of course the relation $\omega_0 = f_1^* \omega_1$. But this is a formal consequence of the identity (22.1) in [GS1] and several other identities which hold also in the algebraic setting. One way to see this is: they hold for holomorphic forms and those are dense in the forms with coefficients in formal power series.

This shows *i*). Part *ii*) follows from the fact that a moment map is unique up to a translation in \mathfrak{g}^* . \square

As consequence we obtain the algebraic analogue of Theorem 9.2.

5.2. Corollary. *Let $Gx \subseteq Z$ be a closed isotropic orbit. Put $S_x := (\mathfrak{g}x)^\perp/\mathfrak{g}x$ which is a symplectic representation of G_x . Let $u_x := \mu(x) \in (\mathfrak{g}^*)^G$. Then the triple (G_x, S_x, u_x) determines a formal neighborhood of Gx uniquely up to Hamiltonian isomorphism. Conversely, any such triple occurs.*

We temporarily identify \mathfrak{g}^* with \mathfrak{g} by means of a G -invariant scalar product. Then every element $v \in \mathfrak{g}$ has a Jordan decomposition $v = v_s + v_u$. Let $L \subseteq G$ be a Levi subgroup, i.e., the centralizer of a semisimple element in \mathfrak{g} . Let W_L be the Weyl group of L . Consider $\mathfrak{t}^r := \{v \in \mathfrak{t}^* \mid W_v \subseteq W_L\}$. In other words \mathfrak{t}^r is obtained from \mathfrak{t} by removing all reflecting hyperplanes which don't belong to W_L . Hence it is an open subset of \mathfrak{t}^* . It has the property that $G_v \subseteq L$ whenever $v \in L\mathfrak{t}^r$.

5.3. Lemma. *The morphism*

$$\lambda: G \times_{\mathfrak{t}/W_L}^L (\mathfrak{l} \times_{\mathfrak{t}} \mathfrak{t}) \rightarrow \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}: [g, (v, w)] \mapsto (gv, w)$$

is an isomorphism over \mathfrak{t}^r . Moreover, if $\mathfrak{l}^r \subseteq \mathfrak{l}$ is the preimage of $\mathfrak{t}^r/W_L \subseteq \mathfrak{l}/W$ then we have

$$G \times \mathfrak{l}^r \xrightarrow{\sim} \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}^r/W_L$$

Proof: Since \mathfrak{t}/W is a polynomial ring, the variety $\mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}$ is a complete intersection, hence Cohen-Macaulay. Moreover, the quotient morphism $\mathfrak{g} \rightarrow \mathfrak{t}/W$ is smooth in regular

points. This implies that $\mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}$ is smooth in codimension 1, hence normal, by Serre's criterion. It follows that also $G \times^L (\mathfrak{l} \times_{\mathfrak{t}/W_L} \mathfrak{t})$ is a normal variety. So, by Zariski's main theorem (or just the Richardson lemma), it suffices to show that λ is bijective over \mathfrak{t}^r .

Assume $\lambda([g, (v, w)]) = \lambda([g', (v', w)])$. Let $v = v_s + v_n$, $v' = v'_s + v'_n$ be the Jordan decompositions. Since v and v' have the same image as w in $\mathfrak{t}/W = \mathfrak{l}/L$ there is $l \in L$ such that $v_s = lv'_s \in Lw$. Furthermore, $gv = g'v'$ implies $gv_s = g'v'_s$. Thus $lg'^{-1}g \in G_{v_s}$. Since $v_s \in L\mathfrak{t}^r$ we have $g'^{-1}g \in L$. Thus $[g, (v, w)] = [g', (g'^{-1}gv, w)] = [g', (v', w)]$.

For surjectivity, let $v \in \mathfrak{g}$ and $w \in \mathfrak{t}^r$ have the same image in \mathfrak{t}/W . Then there is $g \in G$ with $v_s = gw$. Hence $u := g^{-1}v \in \mathfrak{g}_w \subseteq \mathfrak{l}$ and we obtain $\lambda([g, (u, w)]) = (v, w)$.

The second isomorphism follows from the first by dividing out W_L on both sides. \square

Next, we consider the cross section theorem.

5.4. Theorem. *Let Z be an algebraic Hamiltonian G -manifold with moment map μ . For a Levi subgroup $L \subseteq G$ put $Z(L) := \mu^{-1}(\mathfrak{t}^r)$ and $Z_L := G \times^L Z(L)$. Assume, $Z(L)$ is not empty. Then*

- i) The set $Z(L)$ is a Hamiltonian L -variety (possibly disconnected): its symplectic form is the restriction of that of Z ; its moment map is $Z(L) \rightarrow \mathfrak{t}^r \hookrightarrow \mathfrak{t}^*$.*
- ii) The canonical morphism $Z_L \rightarrow Z$ is an étale Poisson morphism.*
- iii) If Z is affine then $Z(L)$ and Z_L are affine as well. Moreover, let $Y \subseteq Z_L$ be a closed orbit. Then its image Y' in Z is also closed and $Y \rightarrow Y'$ is an isomorphism.*

Proof: From Lemma 5.3 we obtain an isomorphism

$$Z_L = G \times^L Z(L) \xrightarrow{\sim} Z \times_{\mathfrak{t}/W} \mathfrak{t}^r/W_L.$$

Then $Z_L \rightarrow Z$ is étale since $\mathfrak{t}^r/W_L \rightarrow \mathfrak{t}^*/W$ is. The other assertions are either trivial or can be deduced exactly as in the classical case (Theorem 9.1). \square

Remark: The Slice Corollary 5.2 works best for closed orbits Gx with $\mu(x) = 0$. On the other hand, Theorem 5.4 allows reduction to the case where $\mu(x)$ is nilpotent (take L to be the centralizer of $\mu(x)_s$). Thus, unlike in the classical case, there is a gap between the ranges of applicability of these theorems.

6. The moment map of complex cotangent bundles: connectedness of fibers

Again, G is a connected reductive group, $B \subseteq G$ a Borel subgroup, $U \subseteq B$ its unipotent radical, and $T \subseteq B$ a maximal torus. Let X be a smooth G -variety. Then the cotangent bundle T_X^* is equipped with a canonical symplectic structure and a moment map $\mu : T_X^* \rightarrow \mathfrak{g}^*$. Then, in [K1], we have constructed a subspace \mathfrak{a}^* of \mathfrak{t}^* , a subquotient W_X of the Weyl group W and a morphism $\varphi : T_X^* \rightarrow \mathfrak{a}^*/W_X$ which factors $T_X^* \rightarrow \mathfrak{g}^*/G = \mathfrak{t}^*/W$. It is known to be flat. Almost by definition its generic fibers are connected. We prove a refinement.

6.1. Theorem. *Let X be a smooth G -variety. Then all fibers of $\varphi : T_X^* \rightarrow \mathfrak{a}^*/W_X$ are connected.*

Proof: Let us assume first that $X = G/H$ is homogeneous. Then there is a nice way to compactify the moment map μ . The Lie algebra \mathfrak{h} of H corresponds to an H -fixed point in the Grassmannian $\text{Gr}(\mathfrak{g})$ of \mathfrak{g} . Thus we get a G -morphism $X \rightarrow \text{Gr}(\mathfrak{g})$. Choose any G -equivariant compactification $X \hookrightarrow \tilde{X}$ of X . Then G/H maps diagonally into $\text{Gr}(\mathfrak{g}) \times \tilde{X}$. Let \bar{X} be the normalization of closure of the image. Thus, by construction, \bar{X} is a compactification of X such that the canonical map $X \rightarrow \text{Gr}(\mathfrak{g})$ extends to $\bar{X} \rightarrow \text{Gr}(\mathfrak{g})$.

Let $V \rightarrow \text{Gr}(\mathfrak{g})$ be the vector bundle whose fiber over a point $\mathfrak{m} \subseteq \mathfrak{g}$ is $\mathfrak{m}^\perp \subseteq \mathfrak{g}^*$. Let $\bar{T} := \bar{X} \times_{\text{Gr}(\mathfrak{g})} V$. Then, by construction, the restriction of \bar{T} to X is T_X^* . Since $V \subseteq \text{Gr}(\mathfrak{g}) \times \mathfrak{g}^*$ we obtain a morphism $\bar{\mu} : \bar{T} \rightarrow \mathfrak{g}^*$ which extends the moment map. Moreover, by construction, $\bar{\mu}$ is proper.

Let $\bar{T} \rightarrow M \rightarrow \mathfrak{g}^*$ be the Stein factorization of $\bar{\mu}$. Then M is an affine G -variety with $M//G = \mathfrak{a}^*/W_X$. In particular, the fibers of $M_X \rightarrow \mathfrak{a}^*/W_X$ are connected. By Zariski's connectedness theorem, all fibers of $\bar{T} \rightarrow M$ are connected. This implies that also all fibers of $\bar{\varphi} : \bar{T} \rightarrow \mathfrak{a}^*/W_X$ are connected.

The fiber \bar{T}_x of \bar{T} over $x \in \bar{X}$ is $\mathfrak{m}(x)^\perp \subseteq \mathfrak{g}^*$ where $\mathfrak{m}(x) \subseteq \mathfrak{g}$ is a limit of conjugates of \mathfrak{h} . It follows that \mathfrak{m} is also an algebraic Lie algebra with the same rank and complexity as \mathfrak{h} (see [K1] 2.5). This implies that the map $\mathfrak{m}(x) \rightarrow \mathfrak{a}^*/W_X$ is equidimensional ([K1] 6.6). Thus also $\bar{\varphi} : \bar{T} \rightarrow \mathfrak{a}^*/W_X$ is equidimensional, hence flat, since both domain and range are smooth varieties.

Consider now the fiber $F := \bar{\varphi}^{-1}(u)$ over a point $u \in \mathfrak{a}^*/W_X$. Since all maps $\mathfrak{m}(x)^\perp \rightarrow \mathfrak{a}^*/W_X$ are equidimensional, no irreducible component of F is contained in $\partial\bar{T} := \bar{T} \setminus T_X^*$. Let C_1, C_2 be irreducible components of $\varphi^{-1}(u)$. Then their closures \bar{C}_i are irreducible components of F . Since F is locally a complete intersection it is connected in codimension one ([Ha]). Thus, to show that $\varphi^{-1}(u)$ is connected it suffices to prove the following claim: assume $\bar{C}_1 \cap \bar{C}_2$ is non-empty, of codimension one in $\bar{\varphi}^{-1}(u)$, and contained in $\partial\bar{T}$. Then

$C_1 \cap C_2 \neq \emptyset$.

Consider the subset $\bar{X}_0 \subseteq \bar{X}$ of $x \in X$ where $\dim(\mathfrak{m}(x) + \mathfrak{u})$ and $\dim(\mathfrak{m}(x) + \mathfrak{b})$ are maximal. Since $\mathfrak{m}(x)$ is an algebraic Lie algebra of the same complexity and rank as \mathfrak{h} , the set X_0 meets all G -orbits of \bar{X} . Now consider the subbundles $\mathcal{B} \subseteq \mathcal{U} \subseteq \bar{T}|_{\bar{X}_0}$ whose fibers at x are $\mathcal{B}_x = (\mathfrak{m}(x) + \mathfrak{b})^\perp$ and $\mathcal{U}_x := (\mathfrak{m}(x) + \mathfrak{u})^\perp$. The quotient \mathcal{U}/\mathcal{B} can be identified with the trivial bundle $\mathfrak{a}^* \times \bar{X}_0$. Moreover, the restriction of $\bar{\varphi}$ to \mathcal{U} is just the projection onto \mathfrak{a}^* followed by the quotient map $\mathfrak{a}^* \rightarrow \mathfrak{a}^*/W_X$ (see [K1] 6.2). Thus, the intersection $\bar{C}_i \cap \mathcal{U}$ is contained in the preimage of a finite subset $S \subseteq \mathfrak{a}^*$. On the other hand, its codimension is less or equal $\dim \mathfrak{a}^*$. We conclude that $\bar{C}_i \cap \mathcal{U}$ equals the preimage of some S_i . Therefore, we are done if we can show that $\bar{C}_1 \cap \bar{C}_2 \cap \mathcal{U}$ is not empty since then $S_1 \cap S_2 \neq \emptyset$.

Since $I := \bar{C}_1 \cap \bar{C}_2 \subseteq \partial \bar{T}$ and since it is of codimension one in $\bar{\varphi}^{-1}(u)$, it must be the union of irreducible components of $\partial \bar{T} \cap \bar{\varphi}^{-1}(u)$. Thus there is $x \in \bar{X}_0$ such that $I \cap \mathfrak{m}(x)^\perp$ is a union of components of $\varphi_x^{-1}(u)$ where φ_x is the restriction of $\bar{\varphi}$ to $\mathfrak{m}(x)^\perp = \bar{T}_x$. Write $\mathcal{U}_x = \mathcal{B}_x \oplus \mathfrak{a}^*(x)$. Then we are done with the following lemma:

6.2. Lemma. *Every irreducible component C of $\varphi_x^{-1}(u)$ meets $\mathfrak{a}^*(x)$.*

Proof: We use the \mathbb{C}^* -action by scalar multiplication. Then $\varphi_x^{-1}(\mathbb{C}^*u)$ is closed in $\mathfrak{m}(x)^\perp \setminus \varphi_x^{-1}(0)$. Thus $\overline{\mathbb{C}^*C} \subseteq \mathbb{C}^*C \cup \varphi_x^{-1}(0)$. Since $\overline{\mathbb{C}^*C} \cap \mathfrak{a}^*(x) \neq \emptyset$ and $\text{codim } \mathbb{C}^*C = \dim \mathfrak{a}^* - 1$ we have $\dim \overline{\mathbb{C}^*C} \cap \mathfrak{a}^*(x) \geq 1$. Moreover, this intersection is not contained in $\varphi_x^{-1}(0)$. Thus $\mathbb{C}^*C \cap \mathfrak{a}^*(x) \neq \emptyset$ which implies the claim. \square

This settles the case where X is homogeneous. In the general case let $X_0 \subseteq X$ be non-empty, open, G -stable such that the orbit space $X_1 := X_0/G$ exists. Then the morphism $T_{X_0}^* \rightarrow \mathfrak{a}^*/W_X$ factors through the relative cotangent bundle T_{X_0/X_1}^* . The discussion of the homogeneous case implies that the fibers of $T_{X_0/X_1}^* \rightarrow Y \times \mathfrak{a}^*/W_X$ are connected. Thus the assertion of the theorem holds for X_0 .

Let $Y \subseteq X$ be an orbit and let $\pi : T_X^* \rightarrow X$ be the projection. The restriction of φ to $\pi^{-1}(Y)$ factors through T_Y^* . Let C be any component of $\pi^{-1}(Y) \cap \varphi^{-1}(u)$. Then, by Lemma 6.2, there is $x \in Y$ and a subspace $\mathfrak{a}_Y^*(x) \subseteq T_{X,x}^*$ such that $\mathfrak{a}_Y^*(x) \cap C \neq \emptyset$. Now we are applying the local structure theorem to Y ([K2] 2.10): there is a parabolic subgroup P with Levi part L and an affine subvariety Z such that $P_u \times Z \rightarrow X : (u, z) \mapsto uz$ is an open embedding, with $Z \cap Y \neq \emptyset$ and (L, L) acts trivially on $Z \cap Y$. Moreover $(\text{Lie } L/L_x)^* = \mathfrak{a}_Y^*$. For $x \in Z \cap Y$ we can find a L_x -stable slice S in Z . Then for every $z \in S$ choose a complement $\mathfrak{a}_Y^*(z)$ of $(T_{Z,z})^\perp$ in $(T_{S,z})^\perp \subseteq T_{X,z}^*$ which can be canonically identified with \mathfrak{a}_Y^* . Moreover, the restriction of φ to $\mathfrak{a}_Y^*(z)$ is just the composition $\mathfrak{a}_Y^*(z) \hookrightarrow \mathfrak{a}^* \rightarrow \mathfrak{a}^*/W_X$. This shows that the intersection $C \cap \mathfrak{a}_Y^*(x)$ can be moved continuously into X_0 . Thus, $\varphi^{-1}(u)$ is connected. \square

7. The moment map of complex cotangent bundles: reducedness of fibers

Next we investigate the local structure of the fibers of $\varphi : T_X^* \rightarrow S := \mathfrak{a}^*/W_X$. In [K5] I have shown how to interpret it as a moment map: there is a flat abelian group scheme \mathcal{A}_X/S which acts on T_X^*/S . Moreover, there is an isomorphism $T_S^* \rightarrow \text{Lie } \mathcal{A}_X$ (in particular, the fibers of \mathcal{A}_X are $\dim \mathfrak{a}^*$ dimensional) such that the following diagram commutes (where $Z := T_X^*$):

$$\begin{array}{ccc} T_S^* \times_S Z & \xrightarrow{\varphi^*} & T_Z^* \\ \sim \downarrow & & \sim \downarrow \alpha \\ (\text{Lie } \mathcal{A}_X) \times_S Z & \xrightarrow{\beta} & T_Z \end{array}$$

Here, α is induced by the symplectic structure on Z and β by the action of \mathcal{A}_X on Z . If \mathcal{A}_X were a constant group scheme then the diagram above would be equivalent to the defining property of the moment map.

The commutative diagram above shows in particular, that φ is smooth in a point x if and only if its \mathcal{A}_X -isotropy group is finite.

The group scheme \mathcal{A}_X/S is intimately connected with the quotient map $\mathfrak{a}^* \rightarrow S = \mathfrak{a}^*/W_X$. As any commutative linear group, the fiber $\mathcal{A}_{X,u}$ over a point $u \in S$ is the direct product of its semisimple part $\mathcal{A}_{X,u}^s$ and its unipotent part $\mathcal{A}_{X,u}^u$. Choose $x \in \mathfrak{a}^*$ over u and let $W_{X,x}$ be its isotropy group. Then $\mathcal{A}_{X,u}^s$ is an open subgroup of $A_X^{W_{X,x}}$. This determines also the dimension of $\mathcal{A}_{X,u}^u$. In particular, $\mathcal{A}_{X,u}$ is a torus if and only if u is not in the branching divisor Δ of $\mathfrak{a}^* \rightarrow S$. Furthermore, $\dim \mathcal{A}_{X,u}^u = 1$ if and only if $|W_{X,x}| = 2$ if and only if u is a smooth point of Δ .

7.1. Example. Take $X = G$ with the action by left translations. The cotangent bundle T_G^* is trivial and the G -action on it becomes $g(h, \lambda) = (gh, \lambda)$. The moment map is $\mu(g, \lambda) = g\lambda$. The action of \mathcal{A}_X on T_X^* commutes with the G -action and preserves G -invariants. This implies that for every $a \in \mathcal{A}_G$ and $\lambda \in \mathfrak{g}^*$ which have the same image in \mathfrak{t}^*/W , there is $c(a, \lambda) \in G$ such that $a \cdot (g, \lambda) = (gc(a, \lambda)^{-1}, \lambda)$. Since the \mathcal{A}_X -action also preserves μ , we have $c(a, \lambda) \in G_\lambda$. Thus, the \mathcal{A}_G -action on T_G^* is described by a homomorphism $\mathcal{A}_G \times_{\mathfrak{t}^*/W} \mathfrak{g}^* \rightarrow G \times \mathfrak{g}^*$ such that $\mathcal{A}_{G, \pi(\lambda)} \rightarrow G_\lambda$. One can show that this is an isomorphism whenever λ is regular. Moreover, we obtain homomorphisms between Lie algebras

$$T_{\mathfrak{t}^*/W}^* \times_{\mathfrak{t}^*/W} \mathfrak{g}^* \cong \text{Lie } \mathcal{A}_G \times_{\mathfrak{t}^*/W} \mathfrak{g}^* \rightarrow \mathfrak{g} \times \mathfrak{g}^* = T_{\mathfrak{g}^*}^*$$

which is just the derivative of $\pi : \mathfrak{g}^* \rightarrow \mathfrak{t}^*/W$.

7.2. Theorem. *Let $u \in S \setminus \Delta$. Then the fiber of $\varphi : T_X^* \rightarrow S$ over u is reduced.*

Proof: Let $\mathfrak{a}^r \subseteq \mathfrak{a}^*$ be the preimage of $S \setminus \Delta$. Since $\mathfrak{a}^r \rightarrow S$ is étale, $Z := T_X^* \times_S \mathfrak{a}^r$ is a smooth symplectic variety. The pull-back $\mathcal{A}_X \times_S \mathfrak{a}^r$ is just the trivial group scheme with fiber A_X . Thus A_X acts on Z . The moment map of this action is the projection $Z \rightarrow \mathfrak{a}^r \subseteq \mathfrak{a}^*$. Since the assertion of the lemma is stable under étale base-change, we have reduced it to a statement about moment maps of tori on affine varieties.

Let $z \in \varphi^{-1}(u)$. Since $A := A_X$ is a torus, there is an open affine A -stable neighborhood of z in which Az is closed. Moreover, Az is automatically isotropic. Thus, by Theorem 5.1, a formal neighborhood of Az is isomorphic to $Z' := A \times^H (\mathfrak{h}^\perp \oplus S)$ where $H := A_z$. The fiber $\varphi^{-1}(u)$ is reduced in Az if and only if its completion is (see [ZS] VIII Thm. 31). Thus we may replace Z by Z' . But then it suffices to look at the moment map for the H^0 -action on the vector space S .

Since S is symplectic there are linear coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ of S and characters χ_1, \dots, χ_n of H^0 such that H^0 acts on S by $t \cdot (q_i, p_i) = (\chi_i(t)q_i, \chi_i(t)^{-1}p_i)$. Let $d\chi := \text{Lie } \chi_i \in \mathfrak{h}^* := (\text{Lie } H^0)^*$. Then the moment map is given by $\mu(q_i, p_i) = \sum_i q_i p_i d\chi_i$. Thus, μ is the composition of $\pi : S \rightarrow \mathbb{C}^n : (q_i, p_i) \mapsto (q_i p_i)$ and the linear map $\mathbb{C}^n \rightarrow \mathfrak{h}^* : (t_i) \mapsto \sum_i t_i d\chi_i$. Any fiber of the latter is an affine space V , hence reduced. Moreover, since all fibers of π are reduced the generic fiber of $\pi^{-1}(V) \rightarrow V$ is reduced. Because π is flat, this implies that $\pi^{-1}(V)$ itself is reduced. \square

It remains to study the fibers of φ over Δ . There, our results are rather incomplete. A first subcase is:

7.3. Lemma. *Assume X to be affine. Let $Gz \subseteq T_X^*$ be a closed orbit such that $G\mu(z) \subseteq \mathfrak{g}^*$ is not closed. Assume moreover, that $u := \varphi(z)$ is a smooth point of Δ . Then $\varphi^{-1}(\Delta)$ is smooth in z .*

Proof: Since Δ is smooth in u , the unipotent part of $\mathcal{A}_{X,u}$ is one-dimensional. The Lie algebra of it corresponds via the identification $\text{Lie } \mathcal{A}_{X,u} = T_{S,u}^*$ to $(T_u \Delta)^\perp$. We conclude that C is smooth in z if and only if z is not a fixed point for $\mathcal{A}_{X,u}^u$. This is what we are going to show.

Let $\mu(z) =: \lambda = \lambda_s + \lambda_u$ be the Jordan decomposition (observe $\mathfrak{g}^* \cong \mathfrak{g}$) and put $L := G_{\lambda_s}$. We may assume $\lambda_s \in \mathfrak{a}^* \subseteq \mathfrak{t}^*$ and thus $\mathfrak{t}^* \subseteq \mathfrak{l}^*$. Let $u' \in \mathfrak{t}^*/W_L$ and $u'' \in \mathfrak{t}^*/W_G$ be the images of λ . Let \mathcal{A}_G and \mathcal{A}_L be the group schemes associated to G and L (see

Example 7.1). Then we obtain the following commutative diagram (with $Z := T_X^*$):

$$\begin{array}{ccccccc}
\mathfrak{l} & \cong & T_\lambda^* \mathfrak{l}^* & \leftarrow & T_{u'}^*(\mathfrak{t}^*/W_L) & \cong & \text{Lie } \mathcal{A}_{L,u'} \\
\uparrow \alpha & & \uparrow \alpha' & & \uparrow \sim & & \uparrow \sim \\
\mathfrak{g} & \cong & T_\lambda^* \mathfrak{g}^* & \xleftarrow{\beta} & T_{u''}^*(\mathfrak{t}^*/W_G) & \cong & \text{Lie } \mathcal{A}_{G,u''} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T_z Z & \cong & T_z^* Z & \leftarrow & T_u^*(\mathfrak{a}^*/W_X) & \cong & \text{Lie } \mathcal{A}_{X,u}
\end{array}$$

The inclusion $\mathfrak{c}_{\mathfrak{g}}(\lambda) \subseteq \mathfrak{l}$ implies $\mathfrak{l}^\perp \subseteq \mathfrak{c}_{\mathfrak{g}}(\lambda)^\perp = \mathfrak{g}\lambda$. This shows that the image of β is contained in $(\mathfrak{g}\lambda)^\perp = \mathfrak{l}$. This implies that, if we replace α by the natural inclusion $\mathfrak{l} \hookrightarrow \mathfrak{g}$ (and remove α') then the diagram still commutes.

Let q_0 be any L -invariant non-degenerate quadratic form on \mathfrak{l}^* and $q(\tau) := q_0(\tau - \lambda_s)$ for $\tau \in \mathfrak{l}^*$. Then the 1-form dq_0 can be viewed as a map $\mathfrak{l}^* \rightarrow \mathfrak{l}$. It is affine linear and maps λ_s to 0 and λ to a non-zero nilpotent element $\xi \in \mathfrak{l}$. On the other hand, q_0 , being L -invariant, is a pull-back from \mathfrak{t}^*/W_L . This show that ξ is the image of some element of $\eta \in \text{Lie } \mathcal{A}_{L,u'}$. The map $\text{Lie } \mathcal{A}_{L,u'} \rightarrow \mathfrak{l}$ is induced by a homomorphism between algebraic groups (see Example 7.1). Thus we can choose η to be nilpotent. Hence there is also an element of $\text{Lie } \mathcal{A}_{X,u}^u$ which is mapped to $\xi z \in T_z Z$.

The orbit Gz is closed by assumption. Hence its isotropy group G_z is reductive. Moreover, λ and ξ are G_z -fixed. Since \mathfrak{g}_z does not contain nilpotent elements in its center we conclude that $\xi \notin \mathfrak{g}_z$, i.e., $\xi z \neq 0$. \square

For the next result note that $\varphi^{-1}(\Delta)$ is a divisor in T_X^* , i.e., purely of codimension one and each component has a multiplicity attached to it.

7.4. Lemma. *Assume X to be affine. Let $K \subseteq T_X^*$ be a closed, G -stable subvariety of codimension two. Assume $\varphi(K)$ is an irreducible component of Δ . Then the multiplicity of all components of $\varphi^{-1}(\Delta)$ which pass through K is the same. Moreover, it is either one or two.*

Proof: Since we want to apply the Cross Section and the Slice Theorem, we have to check that the property to be proved can be detected in formal neighborhoods. Let $f = 0$ be a local equation of $\varphi^{-1}(\Delta)$ in z . Since the local ring $\mathcal{O}_{Z,z}$ is regular, it is a UFD. Therefore f has a prime factorization $f = ag_1^{m_1} \dots g_s^{m_s}$. Here, a is a unit, the g_i correspond to the different components of $\varphi^{-1}(\Delta)$ passing through z and the m_i are their multiplicities. We want to show $m_1 = \dots = m_s = 1$ or $m_1 = \dots = m_s = 2$. The completion $\widehat{\mathcal{O}}_{Z,z}$ is also regular, thus a UFD. The g_i are in general not prime in $\widehat{\mathcal{O}}_{Z,z}$ but one can say two things: each g_i is square-free in $\widehat{\mathcal{O}}_{Z,z}$ (since $\widehat{\mathcal{O}}_{Z,z}/(g_i)$ has no nilpotents, see [ZS] VIII Thm. 31), and: if $i \neq j$ then g_i and g_j are coprime in $\widehat{\mathcal{O}}_{Z,z}$ (since $\dim \widehat{\mathcal{O}}_{Z,z}/(g_i, g_j) = \dim Z - 2$). This implies easily that our assertion can be checked in the completion alone.

Let $Gz \subseteq K$ be a generic closed orbit. If $G\mu(z) \subseteq \mathfrak{g}^*$ is not closed then $\varphi^{-1}(\Delta)$ is even smooth in z (Lemma 7.3). Thus we have yet to consider the case where $\xi := \mu(z)$ is semisimple. First, we are applying the Cross Section Theorem 5.4 to $Z := T_X^*$ and $L = G_\xi$. Then it suffices to prove the assertion for a formal neighborhood of Lz in $Z(L)$ and the restricted morphism $Z(L) \rightarrow S$.

Next we apply Corollary 5.2. Thus, a formal neighborhood of Lx in $Z(L)$ is isomorphic to a formal neighborhood of the zero-section of $Z' := L \times^{L_x} (\mathfrak{l}_x^\perp \oplus S)$ where S is a symplectic representation of L_x . From $\mathfrak{g} = \mathfrak{p}_u^+ \oplus \mathfrak{l} \oplus \mathfrak{p}_u^-$ we obtain isomorphisms of symplectic L_x -representations:

$$T_z Z \cong \mathfrak{p}_u^+ z \oplus \mathfrak{p}_u^- z \oplus T_x Z(L) \cong [\mathfrak{p}_u^+ z \oplus \mathfrak{p}_u^- z] \oplus [\mathfrak{l}z \oplus (\mathfrak{l}z)^*] \oplus S$$

It is easy to see that every symplectic representation of a reductive group is uniquely of the form $V \oplus V^* \oplus \bigoplus_i M_i$ where V and V^* are isotropic submodules and the M_i are irreducible and pairwise non-isomorphic. Since $T_z Z$ has a Lagrangian submodule we conclude that $S \cong V \oplus V^*$ for some Lagrangian submodules V and V^* . This means that we can identify Z' with the cotangent bundle $T_{X'}^*$, where $X' := L \times^{L_x} V$.

The morphism $Z' \rightarrow S$ factors through $\varphi' : Z' = T_{X'}^* \rightarrow S' := \mathfrak{a}^*/W_{X'}$. Let Δ' be the branch divisor of S' . If $\varphi'(K) \not\subseteq \Delta'$ then we are in the situation of Theorem 7.2, i.e., all fibers of φ' outside Δ' are reduced. Moreover all components of the preimage of Δ in $S' \setminus \Delta'$ have multiplicity two. Thus, the multiplicities of all components of $\varphi^{-1}(\Delta)$ passing through K are 2. (Remark: it will turn out (Theorem 7.5) that this case actually never occurs).

We are left with the case $\varphi'(K) \subseteq \Delta'$. The group $W_{X'}$ is contained in the isotropy group of W_X over $\varphi(u)$. Since that point is generic in the codimension one subvariety $\varphi(K)$ we must have $|W_{X'}| = 2$. Therefore, the morphism $S' \rightarrow S$ is unramified in the generic point of Δ' . Thus we may replace the map $Z' \rightarrow S$ by $Z' \rightarrow S'$. The moment map on Z' coming from $Z(L)$ differs from the canonical moment map of Z' as a cotangent bundle by a shift by a L -fixed element. This means that we can also replace the original moment map by the cotangent bundle moment map. The conclusion of the discussion is: we can replace G, X by L, X' and thus assume that $|W_X| = 2$.

Consider the projection $\pi : Z = T_X^* \rightarrow X$. I show first that every component of $\varphi^{-1}(\Delta)$ maps dominantly to X . Otherwise, there is a G -stable divisor Y of X such that $\pi^{-1}(Y) \subseteq \varphi^{-1}(\Delta)$. The divisor Y induces a G -stable valuation v_Y of the function field $\mathbb{C}(X)$. Since the restriction of the moment map to $\pi^{-1}(Y)$ factors through the moment map on T_Y^* we have $\dim \mathfrak{a}_Y^* = \dim \Delta = \dim \mathfrak{a}^* - 1$. This implies that v_Y is a so-called central valuation (see [K2] 7.3). These form a cone in the group $\mathcal{H} := X_*(A_X)$ of one-

parameter subgroups of A_X which is a W_X -module. The image of $\mathfrak{a}_Y^* \subseteq \mathfrak{a}^*$ in S is Δ . This implies ([K2] 7.4) that $\sigma \cdot v_Y = -v_Y \in \mathcal{H}$ where σ is the non-trivial element of W_X .

Let $\lambda : \mathbb{C}^* \rightarrow A_X$ be the one-parameter subgroup attached to v_Y . In [K3] 7.3, I constructed a class of maps $\psi : A_X \hookrightarrow X$ with the property $\lim_{t \rightarrow 0} \psi(\lambda(t)) \in Y$. With ψ also $\psi \circ \sigma$ is of this class. Thus $\sigma(\lambda) = \lambda^{-1}$ implies

$$\lim_{t \rightarrow \infty} \psi(\lambda(t)) = \lim_{t \rightarrow 0} (\psi \circ \sigma)(\lambda(t)) \in Y$$

This means that $\psi \circ \lambda : \mathbb{C}^* \rightarrow X$ extends to a non-constant morphism from \mathbf{P}^1 to X which is absurd since X is affine.

This shows that we can “recognize” all components of $\varphi^{-1}(\Delta)$ by looking at a generic (possibly non-affine) orbit Gy of X . The restriction of the moment map to $\pi^{-1}(Gy)$ factors through $T_{Gy}^* = G \times^{G_y} \mathfrak{g}_y^\perp$. Now observe that $\Delta \subseteq S$ is given by the vanishing of a function of degree two. Since the map $\mathfrak{g}_y^\perp \rightarrow S$ is homogeneous, also the intersection of $\varphi^{-1}(\Delta)$ with \mathfrak{g}_y^\perp is given by the vanishing of a *quadratic* polynomial. We conclude, that $\varphi^{-1}(\Delta)$ is either irreducible with multiplicity at most two, or it consists of two components of multiplicity one. That concludes the proof. \square

The following is a technical result which was already used in section 4.

7.5. Theorem. *Let X be a smooth affine G -variety and let $D \subseteq \mathfrak{a}^*/W_X$ be a prime divisor. Then $\varphi^{-1}(D)$ is reduced.*

Proof: If $D \not\subseteq \Delta$ then $\varphi^{-1}(D)$ is reduced by Theorem 7.2. Thus let D be a component of Δ . Let C_1, \dots, C_s be the irreducible components of $\varphi^{-1}(D)$. Define a graph Γ whose vertices are the C_i . There is an edge between C_i and C_j if $K_{ij} = C_i \cap C_j$ is of codimension two in T_X^* and $\varphi(K_{ij}) = D$. I claim, that Γ is connected. Indeed, let $D_0 \subseteq D$ be the complement of the union of all $\varphi(K_{ij}) \neq D$. Then $\varphi^{-1}(D_0)$ has the same number of components as $\varphi^{-1}(D)$ since φ is flat. Moreover, $\varphi^{-1}(D_0)$ is connected by Theorem 6.1. Being locally a complete intersection one can remove from $\varphi^{-1}(D_0)$ all K_{ij} which are not of codimension two in T_X^* and the result is still connected ([Ha]). But this means that Γ is connected.

It follows now from Lemma 7.4 that if $\Phi^{-1}(D)$ is not reduced then all of its components have multiplicity two. The divisor D is defined by an equation $f_0 = 0$. Thus $\varphi^{-1}(D)$ is the zero scheme of $f := f_0 \circ \varphi = 0$. Thus, if $\varphi^{-1}(D)$ is not reduced then f is locally a square times a unit. Therefore, for every $x \in X$, the restriction of f to $T_{X,x}^*$ is either zero or a square h_x^2 where h_x is unique up to a sign. This implies that there is a ramified cover $\tilde{X} \rightarrow X$ of degree at most two such that the pull back of f to $T_{\tilde{X}}^*$ is a square h^2 . By [K1] 6.5, we have $W_{\tilde{X}} = W_X$. Thus $\mathbb{C}[\mathfrak{a}^*/W_X] = \mathbb{C}[\mathfrak{a}^*/W_{\tilde{X}}]$ is integrally closed in $\mathbb{C}[T_{\tilde{X}}^*]$. This implies that f_0 itself is a square which is absurd. \square

7.6. Corollary. *Let X be affine. Then $\varphi : T_X^* \rightarrow \mathfrak{a}^*/W_X$ is smooth in codimension one, i.e., the set of points where φ is not smooth is of codimension at least two.*

Proof: Let $D \subseteq T_X^*$ be a prime divisor. Since φ is flat, $D' := \varphi(D)$ is a divisor in S . Since $D \rightarrow D'$ is generically smooth and D is a reduced component of $\varphi^{-1}(D')$ we conclude that φ is smooth in a generic point of D . \square

Remark: Some restriction on X is necessary for the above Corollary to hold: there is a spherical homogeneous variety X of rank two for which $\varphi : T_X^* \rightarrow \mathfrak{a}^*/W_X$ is not smooth in codimension one. That example is not even quasi-affine, though.

8. Appendix A: Lifting and pushing down differentiable maps

Let X be any smooth manifold and $Y \subseteq X$ a subset. A function f on Y is differentiable if every $y \in Y$ has an open neighborhood U in X and a differentiable function \bar{f} on U such that $f = \bar{f}|_{Y \cap U}$. The set of differentiable functions is denoted by $\mathcal{C}^\infty(Y)$. We can also form the sheaf \mathcal{C}_Y^∞ of differentiable functions on Y . In this paper we consider only locally closed subsets, i.e., $Y = Z \cap U$ where $Z \subseteq X$ is closed and $U \subseteq X$ is open. In this case, U is a manifold and an easy argument involving partitions of unity shows that $\mathcal{C}^\infty(Y)$ are just the restrictions of differentiable functions on U to Y .

If K is a compact Lie group acting on X let $\pi : X \rightarrow X/K$ be its quotient. Then a function f on X/K is called differentiable if $f \circ \pi \in \mathcal{C}^\infty(X)$. Very often this case can be reduced to the first one (Schwarz [Sch]):

8.1. Theorem. *Let V be finite dimensional representation (over \mathbb{R}) of K and $X \subseteq V$ locally closed, analytic and K -stable. Let f_1, \dots, f_s be generators of the ring of K -invariant polynomials on V and $\pi : V \rightarrow \mathbb{R}^s$ the mapping with components f_i . Then $\pi(X)$ is a locally closed subset of \mathbb{R}^s and $X/K \rightarrow \pi(X)$ is an homeomorphism and induces an isomorphism $\mathcal{C}^\infty(\pi(Y)) \xrightarrow{\sim} \mathcal{C}^\infty(X/K)$.*

Note, that Koszul's slice theorem implies that every K -manifold meets the assumptions of the theorem at least locally.

For any point x of a manifold M let $\widehat{\mathcal{C}}_x(M)$ be the completion of $\mathcal{C}^\infty(M)$ with respect to the \mathfrak{m}_x -adic topology. It is a formal power series ring and the image of $f \in \mathcal{C}(M)$ in $\widehat{\mathcal{C}}_x(M)$ is its Taylor series \widehat{f}_x . We are now stating two basic theorems about the relationship of f with its Taylor series. They are easy consequences of deep theorems by Hörmander [Hör], Malgrange [Mal], Tougeron [Tou], and Bierstone-Milman [BM].

Consider the following diagram:

$$\begin{array}{ccc} & & X \\ & \beta \nearrow & \downarrow \pi \\ U & \xrightarrow{\alpha} & Y \end{array}$$

Given α and π , we say α *lifts to* X if there is β with $\alpha = \pi \circ \beta$. For every $u \in U$, $x \in X$ with $y := \alpha(u) = \pi(y)$ we obtain homomorphisms $\widehat{\alpha}$ and $\widehat{\pi}$ between completions:

$$\begin{array}{ccc} & & \widehat{\mathcal{C}}_x(X) \\ & \widehat{\beta} \nearrow & \uparrow \widehat{\pi} \\ \widehat{\mathcal{C}}_u(U) & \xrightarrow{\widehat{\alpha}} & \widehat{\mathcal{C}}_y(Y) \end{array}$$

We say, α *lifts formally to* X if for every $u \in U$ there is $x \in X$ as above and a homomorphism $\widehat{\beta}$ with $\widehat{\alpha} = \widehat{\beta} \circ \widehat{\pi}$.

Clearly, if α lifts then it lifts formally. A converse is given by the following theorem:

8.2. Theorem. *We assume*

- i) X and Y are real-algebraic manifolds and π is a morphism.*
- ii) There is $h \in \mathbb{R}[Y]$ such that π is a closed embedding over $Y_0 := \{y \in Y \mid h(y) \neq 0\}$.*
- iii) The function $\bar{h} := h \circ \alpha \in \mathcal{C}^\infty(U)$ is non-zero and locally analytic.*

Then α lifts if and only if it lifts formally. Moreover, the lift β is unique.

Proof: The map β is unique since by *iii)* the zero-set of \bar{h} is nowhere dense and π is injective over Y_0 .

As for existence, since X and Y are algebraic we have to show that $\mathbb{R}[Y] \xrightarrow{\alpha^*} \mathcal{C}^\infty(U)$ extends to $\mathbb{R}[X] \xrightarrow{\beta^*} \mathcal{C}^\infty(U)$. Because α lifts formally, the image of α is contained in the image of π . Thus, if I is the kernel of $\mathbb{R}[Y] \xrightarrow{\pi^*} \mathbb{R}[X]$ then $\alpha^*(I) = 0$. Therefore, we obtain a homomorphism $\alpha^* : R := \mathbb{R}[Y]/I \rightarrow \mathcal{C}^\infty(U)$ and we can think of R as a subring of $\mathbb{R}[X]$. Because of *ii)*, every $f \in \mathbb{R}[X]$ is of the form g/h^N where $g \in R$ and $N \in \mathbb{N}$. Therefore, $\beta^*(f)$ exists if and only if $\alpha^*(g)$ is divisible by \bar{h}^N . Because \bar{h} is locally analytic, this can be checked by looking at Taylor series ([Mal] Thm. 1.1). But for them divisibility holds since α lifts formally. \square

We consider now the dual situation:

$$\begin{array}{ccc} X & & \\ \pi \downarrow & \searrow \alpha & \\ Y & \xrightarrow{\beta} & U \end{array}$$

Given α and π we say α *pushes down to* Y if there is a β with $\alpha = \beta \circ \pi$. Analogously, α *pushes down formally to* Y if for every $x \in X$, $y = \pi(x)$, $u = \alpha(x)$ there is a homomorphism

$\widehat{\beta}$ completing the following diagram:

$$\begin{array}{ccc} \widehat{\mathcal{C}}_x(X) & & \\ \widehat{\pi} \uparrow & \swarrow \widehat{\alpha} & \\ \widehat{\mathcal{C}}_y(Y) & \xleftarrow{\widehat{\beta}} & \widehat{\mathcal{C}}_u(U) \end{array}$$

8.3. Theorem. *We assume*

- i) X is a manifold and Y is semi-analytic.*
- ii) π is equivariant with respect to an action of a compact Lie group K and the map on orbit spaces $\pi/K : X/K \rightarrow Y/K$ is open.*
- iii) Every orbit $Kx \subseteq X$ has a neighborhood V with an analytic structure such that $\pi|_V$ is analytic.*
- iv) All fibers of π are non-empty and connected.*

Then α pushes down if and only if it pushes down formally. Moreover, the push-down β is unique.

Proof: The map π is surjective by *iv)*. So β is unique if it exists. Moreover, β exists if and only if $f \circ \alpha$ can be pushed down for every differentiable function f on U . Thus we may assume $U = \mathbb{R}$. Since π is locally analytic, the set of smooth points in any fiber of π is dense. Since α can be pushed down formally, the derivatives of its restriction to a fiber of π in a smooth point is zero. Because of *iv)*, we conclude that α is constant on the fibers of π . In particular, β exists set-theoretically.

Let $V \subseteq X$ as in *iii)* which we may choose to be K -stable. Then $\pi(V)$ is open in Y hence again semi-analytic. Let $Z \subseteq Y_0$ be compact. For every $y \in Z$ choose $x \in V$ with $z = \pi(x)$ and a compact K -stable neighborhood V_y of Kx in V . Because of *ii)*, the image $\pi(V_y)$ is a neighborhood of z . Thus there are finitely many sets $V_i := V_{y_i}$ such that Z is covered by $\pi(V_i)$. Put $Z' := \pi^{-1}(Z) \cap (\cup_i V_i)$. Then Z' is a compact subset of V with $\pi(Z') = Z$. This means by definition that $\pi : V \rightarrow \pi(V)$ is semi-proper.

This means that $\alpha|_V$ satisfies all conditions of [BM]. Therefore, α can be pushed down to a differentiable function β_V on $\pi(V)$. By uniqueness we have $\beta_V = \beta_{\pi(V)}$. Because Y is covered by open sets of the form $\pi(V)$ we have proved that β is differentiable. \square

9. Appendix B: The local structure of Hamiltonian manifolds

For any $u \in \mathfrak{k}_+^*$ let $L = L(u) := K_u$. A subgroup of this type is called a Levi subgroup of K . Since L contains T , its Lie algebra \mathfrak{l} has a unique complement \mathfrak{q} in \mathfrak{k} . Thus we can regard $\mathfrak{l}^* \subseteq \mathfrak{k}^*$. Let $\mathfrak{l}^r = \{v \in \mathfrak{l}^* \mid K_v \subseteq L\}$ and let \mathfrak{l}^0 be its connected component which contains u . Both sets are open in \mathfrak{l}^* . Moreover, $K \times^L \mathfrak{l}^0 \rightarrow \mathfrak{k}^*$ is a diffeomorphism onto an open subset.

Let Y be a Hamiltonian L -manifold such that $\mu_L(Y) \subseteq \mathfrak{l}^0$. Then there is a Hamiltonian structure on $X = K \times^L Y$ as follows: let $x = [1, y] \in X$. Then $T_x X = T_y Y \oplus \mathfrak{q}x$ is an orthogonal decomposition and the symplectic form on $\mathfrak{q}x$ is given by $\omega(\xi x, \eta x) = \langle \mu(y), [\xi, \eta] \rangle$ for all $\xi, \eta \in \mathfrak{q}$. The moment map on X is $\mu_X([k, y]) = k\mu_Y(y)$.

Now we can state the local cross-section theorem of Guillemin-Sternberg:

9.1. Theorem. *Let M be a Hamiltonian K -manifold. For a Levi subgroup $L \subseteq K$ put $M(L) := \mu^{-1}(\mathfrak{l}^0)$ and $M_L := K \cdot M(L)$. Assume, $M(L)$ is not empty. Then*

- i) The set $M(L)$ is a Hamiltonian L -manifold: its symplectic form is the restriction of that of M ; its moment map is $M(L) \rightarrow \mathfrak{l}^0 \hookrightarrow \mathfrak{l}^*$.*
- ii) The set M_L is open, dense, connected, and the map $\bar{M} := K \times^L M(L) \rightarrow M_L$ is an Hamiltonian isomorphism.*
- iii) Let $x \in M$ with $\mu(x) \in \mathfrak{k}_+^*$. Put $L = K_{\mu(x)}$. Then Lx is an isotropic orbit in $M(L)$.*

The next theorem describes the neighborhood of an isotropic orbit:

9.2. Theorem. *Let $Kx \subseteq M$ be an isotropic orbit. Put $S_x := (\mathfrak{k}x)^\perp / \mathfrak{k}x$ which is a symplectic representation of K_x . Let $u_x := \mu(x) \in (\mathfrak{k}^*)^{K_x}$. Then the triple (K_x, S_x, u_x) determines a neighborhood of Kx uniquely up to Hamiltonian isomorphism.*

Conversely, every such triple occurs: let $H \subseteq K$ be a closed subgroup, S a symplectic representation of H and $u_0 \in (\mathfrak{k}^*)^{K_x}$. We choose a H -invariant unitary structure on S . This induces a moment map $\mu_S : S \rightarrow \mathfrak{h}^*$ by $\mu_S(v)(\xi) := \frac{i}{2} \langle \xi v, v \rangle$. Furthermore, choose an H -stable complement \mathfrak{r} of the Lie algebra \mathfrak{h} in \mathfrak{k} such that $\mathfrak{h}^* \hookrightarrow \mathfrak{k}^*$. Then $M := K \times^H (\mathfrak{h}^\perp \oplus S)$ carries a Hamiltonian structure with moment map $\mu([k, (u, v)]) = u_0 + k \cdot (u + \mu_S(v))$. One verifies that the K -orbit of $x := [1, (0, 0)]$ is isotropic with $(K_x, S_x, u_x) = (H, S, u_0)$.

10. References

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