

# On the Kazhdan-Lusztig basis of a spherical Hecke algebra

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## 1. Introduction

Let  $\mathcal{H}$  be an (extended) affine Hecke algebra. It contains the Hecke algebra  $\mathcal{H}_f$  of the finite Weyl group  $W_f$  as a subalgebra. The set of elements of  $\mathcal{H}$  which are “invariant” under left and right multiplication by  $\mathcal{H}_f$  is called the spherical Hecke algebra  $\mathcal{H}^{\text{sph}}$ . The Satake isomorphism identifies  $\mathcal{H}^{\text{sph}}$  with  $\mathbb{Z}[v^{\pm 1}][X^\vee]^{W_f}$  where  $X^\vee$  is the coweight lattice.

In [KL], Kazhdan and Lusztig constructed a canonical basis of  $\mathcal{H}$ . This basis is compatible with  $\mathcal{H}^{\text{sph}}$  and Lusztig has shown, [Lu1], that the Kazhdan-Lusztig elements inside  $\mathcal{H}^{\text{sph}}$  correspond, under the Satake isomorphism, to the Weyl characters of the Langlands dual group  $G^\vee$ .

The aim of this note is to give a new proof of this result and extend it to Hecke algebras with unequal parameters. This works as stated, if the parameters depend only on the root length. If the root system is of type  $C_n$  the situation is more subtle.

The main difference of our approach is that we use Demazure’s character formula while Lusztig used the formula of Weyl. Demazure’s formula is less elementary than Weyl’s but apart from that our proof appears to be simpler. Moreover, Lusztig’s proof does not work in the unequal parameter case since he uses a  $q$ -analog of Weyl’s formula, the celebrated Kato-Lusztig formula, which does not seem to generalize.

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## 2. The extended affine Weyl group

In this and most of the next section, we are setting up notation for Weyl groups and Hecke algebras and recall the properties which we are going to use. Proofs can be found, e.g., in Humphreys' book [Hum], Macdonald's book [Mac] or the nice survey [NR] of Nelsen and Ram which also presents Lusztig's approach to our main theorem.

Let  $(\Delta_f \subset X, \Delta_f^\vee \subset X^\vee)$  be a root datum. The set of affine roots is  $\Delta := \Delta_f + \mathbb{Z}\delta$  whose elements we regard as affine linear functions on  $X^\vee$  with  $\delta$  being the constant function 1. For  $\alpha = \bar{\alpha} + m\delta \in \Delta$  let

$$(2.1) \quad s_\alpha(\tau) := \tau - \alpha(\tau)\bar{\alpha}^\vee = \tau - (\bar{\alpha}(\tau) + m)\bar{\alpha}^\vee$$

be the corresponding affine reflection of  $X^\vee$ . Let  $W^a$  and  $W_f$  be the groups generated by all reflections  $s_\alpha$  with  $\alpha \in \Delta$  and  $\alpha \in \Delta_f$ , respectively. For  $\tau \in X^\vee$  let  $t_\tau$  be the translation  $t_\tau(\eta) := \eta + \tau$ . The group  $W := W_f \ltimes X^\vee$  acts on  $X^\vee$  by  $(w, \tau)(\lambda) := wt_\tau(\lambda) = w(\lambda + \tau)$ . The group  $W^a$  is a subgroup of  $W$ . More precisely,  $W^a = W_f \ltimes Q^\vee$  where  $Q^\vee \subseteq X^\vee$  is the coroot lattice, i.e., the subgroup generated by  $\Delta_f^\vee$ .

We choose a set  $\Sigma_f \subseteq \Delta_f$  of simple roots. A root  $\alpha \in \Delta_f$  is called maximal if  $(\alpha + \Sigma_f) \cap \Delta_f = \emptyset$ . Clearly, there is one maximal root for each connected component of the Dynkin diagram. The set

$$(2.2) \quad \Sigma := \Sigma_f \cup \{-\vartheta + \delta \mid \vartheta \text{ maximal}\} \subseteq \Delta.$$

is the set of simple affine roots. The groups  $W_f$  and  $W^a$  are Coxeter groups with generators  $\{s_{\bar{\alpha}} \mid \bar{\alpha} \in \Sigma_f\}$  and  $\{s_\alpha \mid \alpha \in \Sigma\}$ , respectively.

Let  $\Delta_f^+ \subseteq \Delta_f$  and  $\Delta^+ \subseteq \Delta$  be the set of positive roots, i.e., those roots which are  $\mathbb{Z}_{\geq 0}$ -linear combinations of  $\Sigma$ . With  $\Delta_f^- = -\Delta_f^+$  and  $\Delta^- = -\Delta^+$  we have  $\Delta_f = \Delta_f^+ \cup \Delta_f^-$ ,  $\Delta = \Delta^+ \cup \Delta^-$  and

$$(2.3) \quad \Delta^+ = (\Delta_f^+ + \mathbb{Z}_{\geq 0}\delta) \cup (\Delta_f^- + \mathbb{Z}_{> 0}\delta).$$

There is a natural right action of  $W$  on the space of affine linear functions on  $X^\vee$ , namely,  $(\alpha^w)(\lambda) := \alpha(w\lambda)$ . More precisely, if  $w = t_\tau \bar{w}$  and  $\alpha = \bar{\alpha} + m\delta$  then

$$(2.4) \quad \alpha^w(\lambda) = \bar{\alpha}^{\bar{w}} + \alpha(\tau)\delta = \bar{\alpha}^{\bar{w}} + (\bar{\alpha}(\tau) + m)\delta$$

Thus,  $\Delta$  is stable under  $W$  and we define the length of  $w \in W$  as

$$(2.5) \quad \ell(w) := \#\{\alpha \in \Delta^+ \mid \alpha^w \in \Delta^-\}.$$

Clearly,  $\Omega := \{w \in W \mid \ell(w) = 0\}$  is the stabilizer of  $\Delta^+$  and therefore a subgroup of  $W$  and one can show  $W = \Omega \ltimes W^a$ . We have  $\ell(w^{-1}) = \ell(w)$  and

$$(2.6) \quad \ell(w) = \min\{r \in \mathbb{N} \mid \exists \omega \in \Omega, \alpha_1, \dots, \alpha_r \in \Sigma : w = \omega s_{\alpha_1} \dots s_{\alpha_r}\}.$$

Very useful is the following explicit formula for  $\tau \in X^\vee$ ,  $w \in W_f$ :

$$(2.7) \quad \ell(wt_\tau) = \sum_{\alpha \in \Delta_f^+ \cap (\Delta_f^+)^w} |\alpha(\tau)| + \sum_{\alpha \in \Delta_f^+ \setminus (\Delta_f^+)^w} |\alpha(\tau) + 1|.$$

This implies

$$(2.8) \quad \ell(t_\tau) = \sum_{\alpha \in \Delta_f^+} |\alpha(\tau)|.$$

If  $\lambda \in X_+^\vee := \{\lambda \in X^\vee \mid \alpha(\lambda) \geq 0 \text{ for all } \alpha \in \Sigma_f\}$  this simplifies to

$$(2.9) \quad \ell(t_\lambda) = 2\rho(\lambda) \quad \text{with} \quad 2\rho := \sum_{\alpha \in \Delta_f^+} \alpha.$$

### 3. The extended affine Hecke algebra

Let  $\mathcal{L}$  be a ring. For each simple reflection  $s = s_\alpha$ ,  $\alpha \in \Sigma$  we choose an invertible element  $v^s \in \mathcal{L}$  subject to the condition that  $v^{s_1} = v^{s_2}$  if  $s_1, s_2$  are conjugate in  $W$ . In that case we may define  $v^w := v^{s_1} \dots v^{s_m}$  where  $w = \omega s_1 \dots s_m$  is any reduced expression. We also define  $v^{-w} := (v^w)^{-1}$ . We have mainly two instances of this situation in mind: first  $\mathcal{L} = \mathbb{Z}[v^{\pm s_1}, \dots, v^{\pm s_m}]$  where  $s_1, \dots, s_m \in \Sigma$  is a set of representatives of  $W$ -orbits in  $\Delta$ . Secondly,  $\mathcal{L} = \mathbb{Z}[v^{\pm 1}]$  and  $v^s = v^{n_s}$  with  $n_s \in \mathbb{Z}$ .

Let  $\mathcal{H}$  be the extended Hecke algebra associated to the root datum  $(\Delta_f \subset X, \Delta_f^\vee \subset X^\vee)$  and the weights  $v^w$ . Thus,  $\mathcal{H}$  is a free  $\mathcal{L}$ -module with basis  $\{H_w \mid w \in W\}$  and relations

$$(3.1) \quad H_w H_{w'} = H_{ww'}, \quad \text{whenever } \ell(ww') = \ell(w) + \ell(w')$$

and

$$(3.2) \quad (H_s - v^{-s})(H_s + v^s) = 0, \quad \text{for all simple reflections } s = s_\alpha, \alpha \in \Sigma.$$

The last relation implies that  $H_s$  is invertible and it can be rephrased as

$$(3.3) \quad H_s + v^s = H_s^{-1} + v^{-s}.$$

The algebra  $\mathcal{H}$  contains the finite dimensional Hecke algebra  $\mathcal{H}_f := \bigoplus_{w \in W_f} \mathcal{L} H_w$  attached to the root system  $\Delta_f$  as a subalgebra.

For brevity, we write  $H_{t_\tau} = H_\tau$  for  $\tau \in X^\vee$ . Then equations (3.1) and (2.9) imply

$$(3.4) \quad H_\lambda H_\mu = H_{\lambda+\mu} = H_\mu H_\lambda \quad \text{for all } \lambda, \mu \in X_+^\vee.$$

Since every  $\tau \in X^\vee$  is of the form  $\lambda - \mu$  with  $\lambda, \mu \in X_+^\vee$  we can define commuting elements

$$(3.5) \quad Y_\tau := H_\lambda H_\mu^{-1}.$$

This way, we get a homomorphism

$$(3.6) \quad \Phi : \mathcal{L}[X^\vee] \rightarrow \mathcal{H} : e^\tau \mapsto Y_\tau.$$

Moreover, the map

$$(3.7) \quad \mathcal{L}[X^\vee] \otimes_{\mathcal{L}} \mathcal{H}_f \xrightarrow{\sim} \mathcal{H} : \xi \otimes u \mapsto \Phi(\xi)u$$

is a linear isomorphism. To see the ring structure on the left-hand side we have to know the commutation relation between the  $H_s$  and  $Y_\tau$ . To explain them we need to set up some notation.

**Definition:** A simple root  $\alpha \in \Sigma_f$  is called *special* if

- i)  $\alpha$  is the long simple root in a component of  $\Delta_f$  of type  $C_n$  ( $n \geq 1$ , with  $C_1 = A_1$ ) and
- ii)  $v^{s\alpha_0} \neq v^{s\alpha}$  where  $\alpha_0 := -\vartheta + \delta \in \Sigma$  and  $\vartheta \in W_f\alpha$  is the highest root.

In that case we put  $v_0^s := v^{s\alpha_0}$ .

With this notation we have according to [Lu2]:

**3.1. Theorem.** *Let  $\alpha \in \Sigma_f$ ,  $s := s_\alpha$ , and  $\xi \in \mathcal{L}[X^\vee]$ . Then*

$$(3.8) \quad H_s \Phi(\xi) - \Phi(s\xi)H_s = (v^{-s} - v^s) \Phi\left(\frac{\xi - s\xi}{1 - e^{-\alpha^\vee}}\right)$$

*if  $\alpha$  is not special and*

$$(3.9) \quad H_s \Phi(\xi) - \Phi(s\xi)H_s = (v^{-s} - v^s) \Phi\left(\frac{\xi - s\xi}{1 - e^{-2\alpha^\vee}}\right) + (v_0^{-s} - v_0^s) \Phi\left(e^{-\alpha^\vee} \frac{\xi - s\xi}{1 - e^{-2\alpha^\vee}}\right)$$

*if it is.*

**Remarks:** 1. For non-special  $\alpha$  we define  $v_0^s := v^s$ . Then (3.8) is a special case of (3.9).

2. If  $\alpha \in \Sigma_f$  is special then the simple affine root  $\alpha_0 = -\vartheta + \delta$  is not  $W$ -conjugate to any element of  $\Sigma_f$ . Therefore, the parameter  $v^{s\alpha_0}$  is possibly different from every parameter  $v^{s\beta}$ ,  $\beta \in \Sigma_f$ . But it has to occur somewhere in any presentation of  $\mathcal{H}$ . This explains why not all commutation relations can be of the form (3.8).

These formulas imply in particular:

**3.2. Corollary.** *The image of  $\Phi : \mathcal{L}[X^\vee]^{W_f} \longrightarrow \mathcal{H}$  is in the center of  $\mathcal{H}$ .*

**Remark:** If the parameters  $v^s$  are sufficiently general then one can show that the image is the entire center, see [Lu2], but we won't need that in the sequel.

In the definition of  $\Phi$  there is nothing special about the dominant Weyl chamber. For a fixed  $w \in W_f$  we can define

$$(3.10) \quad Y_\tau^{(w)} := H_\lambda H_\mu^{-1}$$

where  $\lambda, \mu \in w(X_+^\vee)$  with  $\tau = \lambda - \mu$ . Again, we get a homomorphism

$$(3.11) \quad \Phi_w : \mathbb{Z}[X^\vee] \rightarrow \mathcal{H} : e^\tau \mapsto Y_\tau^{(w)}.$$

This homomorphism can be expressed in terms of  $\Phi$ :

**3.3. Lemma.** *For  $w \in W_f$ ,  $\xi \in \mathcal{L}[X^\vee]$  holds*

$$(3.12) \quad \Phi_w(\xi) = H_w \Phi(w^{-1}\xi) H_w^{-1}.$$

*Proof:* It suffices to prove the formula for  $\xi = e^\tau$  where  $\tau$  is in the interior of  $wX_+^\vee$ . Let  $\tau_+ = w^{-1}(\tau) \in X_+^\vee$ . Then formula (2.7) implies  $\ell(wt_{\tau_+}) = \ell(t_{\tau_+}) + \ell(w)$ . Hence  $H_{wt_{\tau_+}} = H_w H_{t_{\tau_+}} = H_w \Phi(e^{w^{-1}\tau})$ . On the other hand, formula (2.8) implies  $\ell(t_\tau) = \ell(t_{\tau_+})$ , and therefore  $\ell(t_\tau w) = \ell(wt_{\tau_+}) = \ell(t_{\tau_+}) + \ell(w)$ . Thus we get

$$(3.13) \quad H_w \Phi(e^{w^{-1}\tau}) = H_{wt_{\tau_+}} = H_{t_\tau w} = H_{t_\tau} H_w = \Phi_w(e^\tau) H_w. \quad \square$$

**3.4. Corollary.** *The homomorphisms  $\Phi$  and  $\Phi_w$  coincide on  $\mathcal{L}[X^\vee]^{W_f}$ .*

Now assume there is an automorphism  $x \mapsto \bar{x}$  of  $\mathcal{L}$  with  $\overline{v^w} = v^{-w}$  for all  $w \in W$ . Then the automorphism extends uniquely to a duality map  $d : \mathcal{H} \rightarrow \mathcal{H}$  by putting

$$(3.14) \quad d(H_w) = H_{w^{-1}}^{-1} \text{ for all } w \in W.$$

**3.5. Lemma.** *Let  $w_0$  be the longest element of  $W_f$  and  $\xi \in \mathbb{Z}[X^\vee]$ . Then*

$$(3.15) \quad d(\Phi(\xi)) = \Phi_{w_0}(\xi) = H_{w_0} \Phi(w_0 \xi) H_{w_0}^{-1}.$$

*Proof:* We may assume  $\xi = e^\tau$  with  $\tau \in X_+^\vee$ . Then

$$(3.16) \quad d(\Phi(e^\tau)) = d(Y_\tau) = d(H_{t_\tau}) = H_{t_{-\tau}}^{-1} = (Y_{-\tau}^{(w_0)})^{-1} = Y_\tau^{(w_0)} = \Phi_{w_0}(e^\tau) \quad \square$$

**3.6. Corollary.** *Let  $\xi \in \mathbb{Z}[X^\vee]^{W_f}$  and  $h = \Phi(\xi)$ . Then  $d(h) = h$ .*

#### 4. The right spherical submodule

Every right coset in  $W/W_f$  is of the form  $t_\tau W_f$  with a unique  $\tau \in X^\vee$ . It contains a unique element  $m_\tau := t_\tau w_\tau$  of minimal length. Explicitly,  $w_\tau \in W_f$  is minimal with  $w_\tau^{-1}(\tau) \in -X_+^\vee$ .

The Bruhat order on  $W$  induces an order relation on  $X^\vee$ :

$$(4.1) \quad \lambda \leq \mu \stackrel{\text{def}}{\iff} m_\lambda \leq m_\mu.$$

This order relation satisfies (see [Hum] Prop. 5.7):

$$(4.2) \quad s_\alpha(\tau) \stackrel{\geq}{\iff} \tau \iff \alpha(\tau) \stackrel{\geq}{\iff} 0 \quad \text{for all } \alpha \in \Delta^+, \tau \in X^\vee$$

and is, in fact, the coarsest order relation with this property.

**4.1. Lemma.** For  $\tau \in X^\vee$  and  $\alpha \in \Delta_f^+$  with  $N := \alpha(\tau) \geq 0$  let

$$(4.3) \quad \tau_0 = \tau, \tau_1 = \tau - \alpha^\vee, \dots, \tau_N = \tau - N\alpha^\vee = s_\alpha(\tau)$$

be the  $\alpha^\vee$ -string through  $\tau$ . Then

$$(4.4) \quad \tau_N > \tau_0 > \tau_{N-1} > \tau_1 > \tau_{N-2} > \tau_2 > \dots > \tau_{\lfloor N/2 \rfloor}.$$

*Proof:* If  $N = 0$  there is nothing to show. If  $N > 0$  we get  $\tau_N > \tau_0$  by (4.2). If  $N = 1$  we are done, so assume  $N > 1$  and consider the affine root  $\beta = -\alpha + \delta$ . Then  $\beta(\tau_{N-1}) = N - 1 > 0$  and  $s_\beta(\tau_{N-1}) = \tau_0 > \tau_{N-1}$ , again by (4.2). The remaining inequalities follow by replacing  $\tau$  by  $\tau - \alpha^\vee$ .  $\square$

Consider the following left submodule of  $\mathcal{H}$ :

$$(4.5) \quad \mathcal{M} := \{h \in \mathcal{H} \mid hH_w = v^{-w}h \quad \text{for all } w \in W_f\}.$$

It is easy to see that  $\mathcal{M} \cap \mathcal{H}_f = \mathcal{L}\theta$  with

$$(4.6) \quad \theta := \sum_{w \in W_f} v^{ww_0} H_w \in \mathcal{H}_f.$$

Then (3.7) implies  $\mathcal{M} \cong \mathcal{H} \otimes_{\mathcal{H}_f} \mathcal{L}\theta$ . Since the elements  $m_\tau, \tau \in X^\vee$ , represent the cosets  $W/W_f$  we conclude that the elements

$$(4.7) \quad M_\tau := H_{m_\tau} \theta = v^{m_\lambda w_0} \sum_{w \in t_\lambda W_f} v^{-w} H_w \quad \text{with } \tau \in X^\vee$$

form an  $\mathcal{L}$ -basis of  $\mathcal{M}$ . On the other hand, the map

$$(4.8) \quad \Psi : \mathcal{L}[X^\vee] \rightarrow \mathcal{M} : p \mapsto \Phi(p)\theta$$

is an isomorphism of  $\mathcal{L}$ -modules. In particular, we obtain a second basis of  $\mathcal{M}$  namely the elements  $\Psi(e^\tau) = Y_\tau \theta$ ,  $\tau \in X^\vee$ .

By transport of structure, the Hecke algebra  $\mathcal{H}$  acts also on  $\mathcal{L}[X^\vee]$ . Explicitly, we have

$$(4.9) \quad Y_\eta(e^\tau) = e^{\tau+\eta} \quad \text{for } \eta \in X^\vee$$

and

$$(4.10) \quad H_s(e^\tau) = v^{-s} e^{s(\tau)} + (v^{-s} - v^s + (v_0^{-s} - v_0^s) e^{-\alpha^\vee}) \frac{e^\tau - e^{s(\tau)}}{1 - e^{-2\alpha^\vee}} \quad \text{for } s = s_\alpha, \alpha \in \Sigma_f.$$

The basis  $M_\tau$  of  $\mathcal{M}$  gives rise to a basis  $p_\tau := \Psi^{-1}(M_\tau)$  of  $\mathcal{L}[X^\vee]$ .

**4.2. Lemma.** *For  $\tau \in X^\vee$  choose any  $w \in W_f$  with  $\tau_+ := w^{-1}(\tau) \in X_+^\vee$ . Then*

$$(4.11) \quad p_\tau = v^{w_\tau} v^w H_w(e^{\tau_+}).$$

*Proof:* Equation (3.12) implies

$$(4.12) \quad H_{m_\tau} H_{w_\tau^{-1}} = H_{t_\tau} = \Phi_w(e^\tau) = H_w \Phi(e^{\tau_+}) H_w^{-1}.$$

Hence

$$(4.13) \quad M_\tau = H_w \Phi(e^{\tau_+}) H_w^{-1} H_{w_\tau^{-1}}^{-1} \theta = v^{w_\tau} v^w H_w \Psi(e^{\tau_+}). \quad \square$$

**4.3. Lemma.** *For every  $\tau \in X^\vee$  holds  $p_\tau \in \sum_{\eta \leq \tau} \mathcal{L} e^\eta$ .*

*Proof:* Let  $w \in W_f$  be the shortest element with  $w^{-1}(\tau) \in X_+^\vee$  and let  $w = s_1 \dots s_m$  be a reduced expression. If  $m = 0$  then  $p_\tau = v^{w_\tau} e^\tau$  and we are done. For  $m \geq 1$  put  $\tau' := s_1(\tau)$  and  $w' = s_1 w$ . Then we have  $p_\tau = H_{s_1}(p_{\tau'})$ . By induction we may assume that every monomial  $e^\eta$  occurring in  $p_{\tau'}$  satisfies  $\eta \leq \tau'$ . The monomials  $e^{\eta'}$  occurring in  $H_{s_1}(e^\eta)$  are all in the  $\alpha_1$ -string with endpoint  $\eta$ . If  $\alpha(\eta) \leq 0$  then Lemma 4.1 implies  $\eta' \leq \eta \leq \tau' < \tau$  and we are done. If  $\alpha(\eta) > 0$  the same holds except for  $\eta' = s_1(\eta) > \eta$ . But then  $\tau = s_1(\tau') > \tau' \geq \eta$  implies  $\tau \geq \eta'$ .  $\square$

To cover special simple reflections, we introduce the root system  $\tilde{\Delta}_f \subseteq X$  which is generated by  $\tilde{\Sigma}_f := \{\varepsilon(\alpha)\alpha \mid \alpha \in \Sigma_f\}$  with

$$(4.14) \quad \varepsilon(\alpha) := \begin{cases} \frac{1}{2} & \text{if } \alpha \text{ is special;} \\ 1 & \text{otherwise.} \end{cases}$$

Correspondingly,  $\tilde{\Delta}_f^\vee$  is generated by  $\tilde{\Sigma}_f^\vee := \{\varepsilon(\alpha)^{-1} \alpha^\vee \mid \alpha^\vee \in \Sigma_f^\vee\}$ . In other words,  $\tilde{\Delta}_f^\vee = \Delta_f^\vee$  if none of the simple roots are special while  $\tilde{\Delta}_f^\vee = C_n$  if  $\Delta_f = C_n$  and the long root is special.

We recall the Demazure operators (see, e.g., [De]). For each simple reflection  $s = s_\alpha$ ,  $\alpha \in \tilde{\Sigma}_f$  we define

$$(4.15) \quad \Delta_s := s + (1 - e^{-\alpha^\vee})^{-1}(1 - s)$$

which acts on  $\mathbb{Z}[X^\vee]$ . If  $w = s_1 \dots s_m \in W_f$  is a reduced expression then  $\Delta_w = \Delta_{s_1} \dots \Delta_{s_m}$  depends only on  $w$ . For  $w \in W_f$  and  $\lambda \in X_+^\vee$  the element  $\Delta_w(e^\lambda)$  is called a Demazure character. We parameterize it as follows: for  $\tau \in X^\vee$  let  $w \in W_f$  be such that  $\tau_+ = w^{-1}(\tau) \in X_+^\vee$ . Then put  $\delta_\tau := \Delta_w(e^{\tau_+})$ . This does not depend on the choice of  $w$ .

Now we can be more specific about the coefficients in Lemma 4.3.

**4.4. Lemma.** *Let  $\mathcal{L}_{++} \subseteq \mathcal{L}$  be a non-unital subring which contains all  $v^s$  where  $s := s_\alpha$ ,  $\alpha \in \Sigma_f$  and, moreover,  $v^s v_0^{\pm s}$  in case  $\alpha$  is special. Let  $\tau \in X^\vee$ . Then*

$$(4.16) \quad p_\tau \in v^{w_\tau} (\delta_\tau + \mathcal{L}_{++}[X^\vee]).$$

*Proof:* Let again  $w \in W_f$  with  $\tau_+ = w^{-1}(\tau) \in X_+^\vee$ . Choose a reduced expression  $w = s_1 \dots s_m$ . Then, by Lemma 4.2,

$$(4.17) \quad p_\tau = v^{w_\tau} (v^{s_1} H_{s_1}) \dots (v^{s_m} H_{s_m}) (e^{\tau_+}).$$

By (4.10) the operator  $v^s H_s$ ,  $s = s_\alpha$ ,  $\alpha \in \Sigma_f$  can be expressed as

$$(4.18) \quad v^s H_s = \Delta_s - (v^s)^2 (1 - e^{-\alpha^\vee})^{-1} (1 - s)$$

if  $s$  is not special and

$$(4.19) \quad v^s H_s = \Delta_s - [(v^s)^2 - (v^s v_0^{-s} - v^s v_0^s) e^{-\alpha^\vee}] (1 - e^{-2\alpha^\vee})^{-1} (1 - s)$$

if  $s$  is special. This implies the assertion.  $\square$

## 5. The spherical Hecke algebra

Every double coset in  $W_f \backslash W / W_f$  is of the form  $W_f t_\lambda W_f$  with a unique  $\lambda \in X_+^\vee$ . It contains a unique longest element namely  $n_\lambda := w_0 t_\lambda$ . Thus we have  $n_\lambda = m_{w_0 \lambda} w_0$ .

We put

$$(5.1) \quad \begin{aligned} \mathcal{H}^{\text{sph}} &:= \{h \in \mathcal{M} \mid H_w h = v^{-w} h \text{ for all } w \in W_f\} = \\ &= \{h \in \mathcal{H} \mid H_w h = h H_w = v^{-w} h \text{ for all } w \in W_f\}. \end{aligned}$$

Since  $\mathcal{M} = \mathcal{H}\theta$ , we obtain  $\mathcal{H}^{\text{sph}} = \theta\mathcal{H} \cap \mathcal{H}\theta$ . The bijection  $\Psi : \mathcal{L}[X^\vee] \xrightarrow{\sim} \mathcal{M}$  induces a bijection

$$(5.2) \quad \Psi : \mathcal{L}[X^\vee]^{W_f} \xrightarrow{\sim} \mathcal{H}^{\text{sph}} : p \mapsto \Phi(p)\theta = \theta\Phi(p),$$

called the *Satake isomorphism*.

**Remark:** Let  $P := v^{-w_0} \sum_{w \in W_f} (v^w)^2 \in \mathcal{L}$ . Then  $\theta^2 = P\theta$  which implies  $\Psi(\xi_1 \xi_2) = P\Psi(\xi_1)\Psi(\xi_2)$ . Hence, if  $P$  is not a zero divisor, we can define a new multiplication  $h_1 * h_2 := \frac{1}{P} h_1 h_2$  on  $\mathcal{H}^{\text{sph}}$  for which  $\mathcal{H}^{\text{sph}}$  becomes a commutative ring with identity element  $\theta$  and (5.2) is an isomorphism of rings.

For every  $\lambda \in X_+^\vee$  put

$$(5.3) \quad N_\lambda := \sum_{\tau \in W_f \lambda} v^{w_\tau} M_\tau = v^{n_\tau} \sum_{w \in W_f t_\lambda W_f} v^{-w} H_w.$$

Then the  $N_\lambda$  form an  $\mathcal{L}$ -basis of  $\mathcal{H}^{\text{sph}}$ . Via  $\Psi$ , they give rise to a basis  $P_\lambda := \Psi^{-1}(N_\lambda)$  of  $\mathcal{L}[X^\vee]^{W_f}$ . For the root system  $A_{n-1}$  they are basically the Hall-Littlewood polynomials.

For  $\lambda \in X_+^\vee$  we denote the Demazure character  $\delta_{w_0(\lambda)} = \Delta_{w_0}(e^\lambda)$  by  $s_\lambda$ . It is well known that the  $s_\lambda$  form a  $\mathbb{Z}$ -basis of  $\mathbb{Z}[X^\vee]^{W_f}$ . For the root system  $A_{n-1}$  they are basically the Schur polynomials.

**5.1. Theorem.** *Let  $\mathcal{L}_{++} \subseteq \mathcal{L}$  be as in Lemma 4.4 and let  $\lambda \in X_+^\vee$ . Then*

$$(5.4) \quad P_\lambda \in s_\lambda + \sum_{\substack{\mu \in X_+^\vee \\ \mu < \lambda}} \mathcal{L}_{++} s_\mu.$$

*Proof:* The definition (5.3) and Lemma 4.4 imply  $P_\lambda \in s_\lambda + r_\lambda$  with  $r_\lambda \in \mathcal{L}_{++}[X^\vee]$ . Since  $s_\lambda$  is  $W_f$ -invariant we also have  $r_\lambda \in \mathcal{L}_{++}[X^\vee]^{W_f}$ . Thus  $r_\lambda$  is a  $\mathcal{L}_{++}$ -linear combination of  $s_\mu$ 's. Finally, Lemma 4.3 implies that every  $s_\mu$  occurring in  $r_\lambda$  has  $\mu < \lambda$ .  $\square$

## 6. Kazhdan-Lusztig elements

In [KL], Kazhdan and Lusztig constructed their celebrated basis of  $\mathcal{H}$ . Recall, that  $\mathcal{L}$  was supposed to be equipped with an involution  $x \rightarrow \bar{x}$  such that  $\overline{v^w} = v^{-w}$  for all  $w \in W$ . Moreover, fix an additive subgroup  $\mathcal{L}_{++} \subseteq \mathcal{L}$  and put  $\mathcal{H}_{++} := \sum_{w \in W} \mathcal{L}_{++} H_w \subseteq \mathcal{H}$ .

**Definition:** A *KL-element* for  $w \in W$  is an element  $\underline{H}_w \in \mathcal{H}$  with

- i)  $d(\underline{H}_w) = \underline{H}_w$  and
- ii)  $\underline{H}_w \in H_w + \mathcal{H}_{++}$ .

As for existence and uniqueness, we have the following theorem. Its proof is quite easy and can be found in [Lu3]. See also a revised version of [Soe] on the Soergel's homepage.

**6.1. Theorem.** *For  $\mathcal{L}^\pm := \{x \in \mathcal{L} \mid \bar{x} = \pm x\}$  consider the homomorphism  $\varphi : \mathcal{L}_{++} \rightarrow \mathcal{L}^- : x \mapsto x - \bar{x}$ .*

- i) Assume that  $\varphi$  is injective, i.e.,  $\mathcal{L}_{++} \cap \mathcal{L}^+ = 0$ . Then for every  $w \in W$  there is at most one KL-element  $\underline{H}_w$ .
- ii) Assume that  $\varphi$  is surjective. Then for every  $w \in W$  there is a KL-element  $\underline{H}_w$  which is even triangular, i.e.,  $\underline{H}_w \in \sum_{v \leq w} \mathcal{L}H_v$ .

We come to the main theorem of our paper where we explicitly compute the KL-elements for  $n_\lambda$ . This generalizes a result of Lusztig [Lu1] who proved the same in case of equal parameters. As mentioned in the introduction, his proof is quite different from ours.

**6.2. Theorem.** *Let  $\mathcal{L}_{++} \subseteq \mathcal{L}$  be a non-unital subring which contains all  $v^s$  where  $s := s_\alpha$ ,  $\alpha \in \Sigma_f$  and moreover  $v^s v_0^{\pm s}$  in case  $\alpha$  is special. Let  $\lambda \in X_+^\vee$ . Then  $\Psi(s_\lambda)$  is a KL-element for  $n_\lambda$ .*

*Proof:* We verify that  $\Psi(s_\lambda)$  satisfies the defining properties of  $\underline{H}_{n_\lambda}$ .

First, we have  $\mathcal{M} \cap \mathcal{H}_f = \mathcal{L}\theta$ . Hence  $d(\theta) \in \mathcal{L}\theta$  and therefore  $d(\theta) = \theta$ . Together with Corollary 3.6 this implies that all elements of  $\Psi(\mathbb{Z}[X^\vee]^{W_f})$ , in particular  $\Psi(s_\lambda)$ , are selfdual.

The spherical algebra  $\mathcal{H}^{\text{sph}}$  has two bases, namely  $N_\lambda$  and  $\Psi(s_\lambda)$  with  $\lambda \in X_+^\vee$ . By Theorem 5.1, the transition matrix from the former to the latter is unitriangular with nondiagonal coefficients in  $\mathcal{L}_{++}$ . Thus the same holds for its inverse. This shows

$$(6.1) \quad \Psi(s_\lambda) \in N_\lambda + \sum_{\mu} \mathcal{L}_{++} N_\mu \subseteq H_{n_\lambda} + \mathcal{H}_{++}. \quad \square$$

**Remark:** The most important case is the one considered by Lusztig: here  $\mathcal{L} = \mathbb{Z}[v, v^{-1}]$  with  $\bar{v} = v^{-1}$  and  $\mathcal{L}_{++} = v\mathbb{Z}[v]$ . Then  $\varphi : \mathcal{L}_{++} \rightarrow \mathcal{L}^-$  is bijective which implies that all KL-elements exist and are unique. Moreover, the parameters are of the form  $v^s = v^{n_s}$  with  $n_s \in \mathbb{Z}$ . Then the conditions of the Theorem boil down to  $n_s > 0$  for  $s = s_\alpha$ ,  $\alpha \in \Sigma_f$  and  $|n_{s_0}| < n_s$  if  $\alpha$  is special and  $s_0$  is the associated affine reflection. In particular,  $n_{s_0}$  may be negative.

**6.3. Corollary.** *Assume additionally, that  $\varphi : \mathcal{L}_{++} \rightarrow \mathcal{L}^-$  is injective. For  $\lambda \in X_+^\vee$  let  $L_\lambda$  be the irreducible  $\tilde{G}^\vee$ -module with highest weight  $\lambda$ . Let  $m_{\lambda\mu}^\nu$  be the multiplicity of  $L_\nu$  in  $L_\lambda \otimes L_\mu$ . Then*

$$(6.2) \quad \underline{H}_{n_\lambda} * \underline{H}_{n_\mu} = \sum_{\nu \in X_+^\vee} m_{\lambda\mu}^\nu \underline{H}_{n_\nu}.$$

*Proof:* This expresses the fact that  $s_\lambda$  is the character of  $L_\lambda$  (Demazure's character formula). □

Assume we have a (evaluation) homomorphism  $\varepsilon : \mathcal{L} \rightarrow \mathbb{Z}$  with  $\varepsilon(v^w) = 1$  for all  $w \in W$ .  
Then

**6.4. Corollary.** *Let  $\underline{H}_w = \sum_{u \in W} p_{uw} H_u$ . For  $\lambda \in X, \mu \in X_+^\vee$  let  $L_\lambda(\mu)$  be the  $\mu$ -weight space in the irreducible representation of  $\tilde{G}^\vee$  with highest weight  $\lambda$ . Then  $\dim L_\lambda(\mu) = \varepsilon(p_{n_\mu n_\lambda})$ .*

*Proof:* This uses the fact that for  $v^s = 1$ , the Hecke algebra degenerates to the group algebra of  $W = X^\vee \rtimes W_f$  and  $\Psi$  becomes the “obvious” map  $e^\tau \mapsto \sum_{w \in W_f} t_\tau w$ .  $\square$

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