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Stationary Nonequilibrium Solutions of Model Boltzmann Equation[1]

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We give an explicit solution of a model Boltzmann kinetic equation describing a gas between two walls maintained at different temperatures. In the model, which is essentially one-dimensional, there is a probability for collisions to reverse the velocities of particles traveling in opposite directions. Particle number and speeds (but not momentum) are collision invariants. The solution, which depends on the stochastic collision kernels at the walls, has a linear density profile and the energy flux satisfies Fourier's law.

1. INTRODUCTION

A central problem in statistical mechanics is the characterization of stationary nonequilibrium states of macroscopic systems. One way in which such states can be achieved operationally is by coupling the system to thermal baths which are maintained at constant unequal temperatures.[1,2] We expect that after some initial transient time the system will approach a stationary state in which there is a steady flux of energy going from the hot bath to the cold bath. We wish then to describe the properties of such stationary nonequilibrium systems.

We note first that in analogy with the equilibrium situation, corresponding here to the case in which all baths are at the same temperature, we can consider such systems at various levels of description: (a) the macroscopic/hydrodynamic, (b) the mesoscopic/kinetic, and (c) the

[1] This paper is dedicated to Peter Gabriel Bergmann with affection and admiration on the occasion of his 70th birthday.

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microscopic/Gibbsian. At present, there is considerable understanding on the first of these levels; i.e., a description entirely in terms of hydrodynamic variables, satisfying the Navier–Stokes equations with suitable boundary conditions, is known, at least for systems close to equilibrium.[3] There is, however, a paucity of information on the more microscopic levels.[4 9] In particular there are no explicit solutions of the Boltzmann equation or the generalized Liouville equation which describe realistic stationary non-equilibrium states. In fact even the existence and uniqueness problem of such states is far from settled.

Given this situation it seems useful to consider simple models, containing at least some elements of the real problem, for which some of these questions can be answered explicitley. This is what we do in this paper: We find the explicit stationary solutions of a Boltzmann-like kinetic equation describing a "gas" between two walls, located at $x = \pm L$, maintained at different temperatures. To accomplish this we have to assume that the system is uniform in the y and z directions. In fact we ignore these directions entirely. What is left then is a one-dimensional system of point particles moving on a line of length $2L$ (like beads on a wire) which undergo "model collisions" with each other and with the walls. The use of model collisions between the particles is essential here since particles undergoing true elastic collisions in one dimension merely exchange velocities and so leave the velocity distribution unaffected. There are many choices for the model collisions.[10,11] We assume that when two particles meet they either pass through each other unaffected (which is equivalent to saying that they exchange velocities) or have their velocities reversed, i.e., $v_1, v_2 \to -v_1, -v_2$. We simplify matters further by assuming that the first case, i.e., noncollision, occurs whenever the particles move in the same direction, that is, v_1 and v_2 have the same sign.

The resulting kinetic equation now becomes "almost" entirely tractable and we can find, more or less explicitly, the one-particle position and velocity distribution in the stationary state.

1.1. Description of Model

We consider a one-dimensional system of point particles confined to a box, $-L \leqslant x \leqslant L$, whose walls are in contact with heat baths at specified temperatures. These are modelled[12] by stochastic kernels $K_\alpha(v, v')$; $v, v' \geqslant 0$, $\alpha = \pm$. $K_\alpha(v, v')$ is the probability density for a particle with velocity $\alpha v'$ hitting the wall at αL to come off with a velocity $-\alpha v$. We simplify matters further by assuming, as is often done, that $K_\alpha(v, v') = vH_\alpha(v)$, independent of v'; i.e., the accommodation coefficient is unity.[10] Inside the box the particles can undergo binary collisions which conserve energy, i.e., if v_1' and v_2' are the velocities prior to a collision and v_1 and v_2 the velocities after a collision, then

$$v_1 = v_2', \qquad v_2 = v_1' \tag{1}$$

or

$$v_1 = -v_1', \qquad v_2 = -v_2' \tag{2}$$

(1) is what really happens in one dimension. It corresponds to an exchange of velocities between the particles and has no effect on the distribution of velocities in the system. Our modelling now consists in assuming that (2) occurs with a probability

$$p(v_1', v_2') = \bar{p}\theta(-v_1' v_2') \tag{3}$$

where $\theta(z) = 1$, $z > 0$; $\theta(z) = 0$, $z \leqslant 0$; and $\bar{p} \leqslant 1$ is a constant.

The time evolution of the macroscopic state or Gibbs ensemble of an N-particle system with the above partly deterministic partly stochastic dynamics will satisfy a generalized Liouville or master equation in the $2N$-dimensional phase space.[1,2] Equivalently, its l-particle distribution function, $l = 1,..., N$, will satisfy a BBGKY hierarchy of equations. To obtain a kinetic description in which the one-particle density satisfies an autonomous equation, analogous to the Bolzmann equation, we need to make the usual "schtoszahlansatz" or go to the Boltzmann–Grad limit.[13] The latter would correspond here to letting $N \to \infty$, $\bar{p} \to 0$ in such a way that the mean free path $(\bar{p}\, N/L)^{-1}$, remains of order unity. Under these assumptions, or limits, we obtain a Boltzmann-like equation for the (suitably normalized) one-particle distribution function, with $\bar{p} \to p$, $p > 0$. This will be given in Section 2, where we also compute the entropy production in this system. In Section 3, we obtain the stationary solution of this equation and consider the hydrodynamical limit of the solution. The solution can be written in a very explicit way in the case where the only velocities are ± 1, ± 2 and for some other simplified model like the BGK model[10], we examine such cases in Section 4.

2. KINETIC EQUATIONS

Let $f(x, v, t)\, dx\, dv$ be the density of particles in the spatial interval $(x, x + dx)$ with velocities in $(v, v + dv)$; we can write

$$f(x, v, t) = \begin{cases} f_+(x, v, t), & v > 0, \\ f_-(x, -v, t), & v > 0, \end{cases} \qquad -L \leqslant x \leqslant L$$

$f_+(x, v, t)$ and $f_-(x, v, t)$ denote, respectively, the one-particle distribution function with velocity $+v$ and $-v$. Our assumptions lead to the following Boltzmann equations for f_+ and f_- :

$$\frac{\partial f_+(x, v, t)}{\partial t} + v\frac{\partial f_+(x, v, t)}{\partial x} = p\int_0^\infty (v+v')[f_-(x, v, t)\, f_+(x, v', t)$$
$$-f_+(x, v, t)\, f_-(x, v', t)]\, dv'$$
$$\frac{\partial f_-(x, v, t)}{\partial t} - v\frac{\partial f_-(x, v, t)}{\partial x} = p\int_0^\infty (v+v')[f_+(x, v, t)\, f_-(x, v', t) \quad (4)$$
$$-f_-(x, v, t)\, f_+(x, v', t)]\, dv'$$

in $-L \leqslant x \leqslant L$, $v \geqslant 0$. The stochastic boundary conditions at $x = \alpha L$, $\alpha = \pm$, are[12]

$$vf_+(-L, v, t) = vH_-(v)\int_0^\infty v' f_-(-L, v', t)\, dv'$$
$$vf_-(L, v, t) = vH_+(v)\int_0^\infty v' f_+(L, v', t)\, dv' \quad (5)$$

Normalization (particle conservation) at the boundaries requires

$$\int_0^\infty vH_\alpha(v)\, dv = 1, \qquad \alpha = \pm \quad (6)$$

When the walls represent thermal reservoirs at temperatures $T_\alpha = (K\beta_\alpha)^{-1}$ $\alpha = \pm 1$, then[10]

$$H_\alpha(v) = \beta_\alpha m \exp(-\beta_\alpha m v^2/2), \qquad \alpha = \pm \quad (7)$$

We may also consider, however, different kinds of $H_\alpha(v) \geqslant 0$ as long as (6) is satisfied and there is no "strong singularity" as $v \to 0$.

We note here that the right side of (4) vanishes whenever

$$f_+(x, v) = g(x)\, f_-(x, v) \quad (8)$$

This reflects the fact that the density of particles with a given speed is a collision invariant. In particular, any x-independent velocity distribution satisfying (8) would be a stationary solution of our kinetic equation for an isolated system in a periodic box or in infinite space. In a box with reflecting walls stationarity requires $g = 1$, i.e., $f(v)$ is even in v. We shall not pursue this further here. Instead we shall consider now the entropy production in the system with stochastic boundary conditions (5).

2.1. Entropy Production

The total entropy production, that is, the sum of the entropy production in the system and in the reservoirs, is given by[1]

$$\sigma = \dot{S} + \sigma_R \tag{9}$$

where S is the Boltzmann entropy of the system

$$S = -K \iint dx\, dv\, f(x, v, t) \ln f(x, v, t) \tag{10}$$

and $\sigma_R = \sum J_\alpha \beta_\alpha$ is the entropy production in the reservoirs, J_α being the energy flux to the αth wall.
When $H_\alpha(v)$ is given by (7) we have, after some manipulations,[12]

$$\begin{aligned}\sigma = 1/2 \iint dv\, dv'(v+v')\, f_+(x, v', t)\, f_-(x, v, t) \left[1 - \frac{f_+(x, v, t)\, f_-(x, v', t)}{f_-(x, v, t)\, f_+(x, v', t)}\right] \\ \times \ln \frac{f_+(x, v, t)\, f_-(x, v', t)}{f_-(x, v, t)\, f_+(x, v', t)} \\ + \iint dv\, dv'\, R_-(v, v')\, \nu_-(-L, v, t)[\ln \nu_-(-L, v, t) - \ln \nu_+(-L, v', t)] \\ + \iint dv\, dv'\, R_+(v, v')\, \nu_+(L, v, t)[\ln \nu_+(L, v, t) - \ln \nu_-(L, v', t)] \geqslant 0\end{aligned} \tag{11}$$

where

$$\nu_\pm(\alpha L, v, t) = f_\pm(\alpha L, v, t) \exp(\beta_\alpha m v^2/2)$$

and

$$R_+(v, v') = (\beta_\alpha m)\, v v' \exp(-\beta_\alpha m(v^2 + v'^2)/2), \qquad \alpha = \pm$$

The inequality in equation (8) follows since

$$y(\ln y - \ln x) - y + x \geqslant 0$$

We note here that when $\beta_+ = \beta_-$, σ is just the time derivative of (minus) the free energy.[1,2] Also for $\beta_+ \neq \beta_-$ but $p = 0$, no collisions, σ can be written as a time derivative of a Lyapunov functional,[12] but this fails when $p \neq 0$. In all cases in the stationary state $J_+ = -J_- = \hat{J}$, $\dot{S} = 0$, and

$$\sigma(f_s) = \sigma_R(f_s) = \hat{J}(\beta_+ - \beta_-) \geqslant 0$$

3. STATIONARY SOLUTIONS

We are interested primarily in the stationary distributions $f_{\pm}(x, v)$, i.e., solutions of (4) and (5) with the time derivatives set equal to zero. We expect further that these will be approached, as $t \to \infty$, from arbitrary initial states, $f_{\pm}(x, v, 0)$. (In order to avoid trivial complications arising from having an accumulation of particles with zero velocities we shall assume that neither $f_{\pm}(x, v, 0)$ nor $H_{\alpha}(v)$ have any singularities as $v \searrow 0$.) Uniqueness but not approach will be shown for certain classes of $H_{\alpha}(v)$. In the general case, however, we shall be satisfied with explicit construction of stationary solutions without proving uniqueness. Before considering the general case, however, we remark that when $H_{+}(v) = H_{-}(v) = H(v)$ then there is always a stationary solution of the form

$$f_{\pm}(x, v) = \rho H(v) \Big/ \int_{-\infty}^{\infty} H(v)\, dv \tag{12}$$

where ρ is the average particle density which is specified by the initial state and is conserved by (1) and (2). In particular, for the choice (7) with $\beta_{+} = \beta_{-} = \beta$, i.e., the two walls are at the same temperature, the solution (14) is just the Maxwellian velocity distribution

$$\rho h_{\beta}(v) = \rho(\beta m/2\pi)^{1/2} \exp\left(-\beta \frac{m}{2} v^2\right)$$

Turning now to the general case, we first note that if $f(x, v)$ is a stationary solution of the Boltzmann equation (4), then necessarily the difference $f_{-}(x, v) - f_{+}(x, v)$ is independent of x and thus

$$f_{-}(x, v) = f_{+}(x, v) + c(v) \tag{13}$$

The stationary equation for f_{+} then becomes (we set $s = x/L$, $-1 \leqslant s \leqslant 1$)

$$v \frac{\partial f_{+}}{\partial s}(s, v) = q \int_{0}^{\infty} (v + v')(c(v) f_{+}(s, v') - c(v') f_{+}(s, v)\, dv' \tag{14}$$

with $q = pL$: q^{-1} corresponds to the mean free path on the macroscopic length scale, usually denoted by ε in the Boltzmann equation.[4] Moreover, it is easy to see that there is no net transport of particles in the stationary state:

$$\int_{0}^{\infty} v c(v)\, dv = \int_{0}^{\infty} v(f_{+}(s, v) - f_{-}(s, v))\, dv = 0 \tag{15}$$

and that $\int_0^\infty vf_\pm(s, v)\,dv = d$ is independent of s. Using this information we rewrite Eq. (5) and (6) as

$$v\frac{\partial f_+}{\partial s} = qvc(v)\,n_+(s) + qdc(v) - qvaf_+(s, v) \tag{16}$$

$$vf_+(-1, v) = vdH_-(v)$$
$$vf_+(1, v) = vdH_+(v) - vc(v) \tag{17}$$

Here $a = \int_0^\infty c(v)\,dv$ and $n_+(s) = \int_0^\infty f_+(s, v)\,dv$ is the partial density of particles with positive velocities. $n_+(s)$ satisfies the equation

$$\frac{\partial}{\partial s} n_+(s) = qdb, \qquad b = \int_0^\infty \frac{c(v)}{v}\,dv \tag{18}$$

It follows that $n_+(s)$ has the form

$$n_+(s) = \frac{n_+(1) + n_+(-1)}{2} + \frac{n_+(1) - n_+(-1)}{2}s$$

$$n_+(-1) = d\int_0^\infty H_-(v)\,dv = dh_-$$

$$n_+(1) = dh_+ - a, \qquad h_\pm = \int_0^\infty H_\pm(v)\,dv$$

Combining this with Eq. (15) gives the particle density $n(s)$:

$$n(s) = d(h_+ + h_-) + d(h_+ - h_-) - a]s \tag{19}$$

with $d = \rho/(h_+ + h_-)$. The density profile in thus always linear for our model system.

We can now find the general solution of Eq. (18) by writing $f_+(s, v) = e^{-\gamma s}\phi_+(s, v)$, $\gamma = qa$; $\phi_+(s, v)$ satisfies the equation

$$\frac{\partial\phi_+(s, v)}{\partial s} = qc(v)\,e^{\gamma s}\,n_+(s) + qd\frac{c(v)}{v}\,e^{\gamma s} \tag{20}$$

After integration this equation yields

$$f_+(s, v) = \hat{c}(v)\,n_+(s) + e^{-\gamma(s+1)}\,z_1(v) + e^{-\gamma(s-1)}\,z_2(v) + \chi(v) \tag{21}$$

$$\hat{c}(v) = c(v)/a$$

$$z_1(v) = d/2\,H_-(v) + \hat{c}(v)\left\{-\frac{dh_-}{2} + \frac{dh_+ - dh_- - \gamma/q}{4\gamma} - \frac{d}{2v}\right\}$$

$$\chi(v) = \hat{c}(v)\left\{\frac{d}{v} - \frac{dh_+ - dh_- - \gamma/q}{2\gamma}\right\}$$

and $z_2(v)$ is obtained from $z_1(v)$ by interchanging the + and − subcripts. Now, from (17), we can get $\hat{c}(v)$

$$\hat{c}(v)=\frac{\{H_+(v)(1+e^{2\gamma})-H_-(v)(1+e^{-2\gamma})\}\,2\gamma v}{v\left\{2\gamma[h_+(1+e^{2\gamma})-h_-(1+e^{-2\gamma})] + \left(h_+-h_- -\frac{\gamma}{dq}\right)(e^{-2\gamma}-e^{2\gamma})\right\}+2\gamma(e^{2\gamma}-e^{-2\gamma})} \tag{22}$$

or more concisely

$$\hat{c}(v)=A\left[\frac{vH_+(v)(1+e^{2\gamma})}{1+Bv}-\frac{vH_-(v)(1+e^{-2\gamma})}{1+Bv}\right] \tag{23}$$

where $A=1/(e^{2\gamma}-e^{-2\gamma})$ and

$$B=\frac{h_+(1+e^{2\gamma})-h_-(1+e^{-2\gamma})}{e^{2\gamma}-e^{-2\gamma}}-\frac{1}{2\gamma}\left(h_+-h_- -\frac{\gamma}{dq}\right)$$

The constant γ can now be found as a solution of the nonlinear equation obtained by first multiplying both sides of (23) by v and then integrating over v and using (15). The solution can be shown to exist for any $H_-(v)$ and $H_+(v)$ satisfying Eq. (6). Moreover, γ goes to zero when $H_+(v)$ goes to $H_-(v)$, and the stationary solution is then given by (12). In all cases the stationary solution is positive and satisfies the symmetry condition

$$f_+(s,v)=\tilde{f}_-(-s,v), \qquad f_-(s,v)=\tilde{f}_+(-s,v)$$

where $\tilde{f}_\pm$ corresponds to $f_\pm$ when H_+ and H_- are interchanged, $\tilde{H}_+=H_-$, $\tilde{H}_-=H_+$.

3.1. Uniqueness

The question of uniqueness is reduced in our case to showing that there is a unique value of γ which satisfies Eq. (22). We are able to prove this generally only when $|H_+(v)-H_-(v)|$ is small (Appendix 1), or when $h_\alpha=\int H_\alpha(v)\,dv=h$ independent of α (Appendix 2). In the case which we are really interested in, when $H_\alpha(v)$ is given by (7), uniqueness can be shown explicitly by numerical analysis (Appendix 3).

3.2. Hydrodynamical Limit

We consider now the asymptotic behavior of the stationary solution as $q=pL\to\infty$. We note that if $\gamma(q)$ is any solution of Eq. (22) then $\gamma(q)$

remains finite as $q \to \infty$. If we call $\bar{\gamma} = \lim_{n \to \infty} (q_n a)$, where $\{q_n\}$ is such that $q_n \to \infty$, $\bar{n}(s) = \lim n(s)$, etc., then we have

$$\bar{n}(s) = \rho\left(1 + \frac{h_+ - h_-}{h_+ + h_-} s\right) \tag{24}$$

$$\bar{f}_+(s, v) = A_+(s, v)\, H_+(v) + A_-(s, v)\, H\ \ (v) = \bar{f}_-(s, v) \tag{25}$$

where

$$A_+(s, v) = \frac{\bar{A}v}{1 + \bar{B}v}(1 + 2^{2\bar{\gamma}})\{\bar{n}_+(s) + e^{-\bar{\gamma}(s+1)} z_+(v) + z(v)\}$$

$$A_-(s, v) = \frac{\bar{A}v}{1 + \bar{B}v}(1 + e^{-2\bar{\gamma}})\{\bar{n}_+(s) + e^{-\gamma(s-1)} z_-(v) + z(v)\}$$

$$z_+(v) = -dh_- + \frac{d(h_+ - h\ \ )}{2\bar{\gamma}} - \frac{d}{v},\ z(v) = \frac{d}{v} - \frac{d(h_+ - h_-)}{2\bar{\gamma}}$$

$$z_-(v) = -dh_+ + \frac{d(h_+ - h_-)}{2\bar{\gamma}} - \frac{d}{v}$$

As already noted in Section 2, any even distribution $f_+(v) = f_-(v)$ is left stationary by the collisions. The particular form of (23) and (25) is produced by the boundaries which, due to the fact that $f_e(v)$ is a collision invariant, can propagate indefinitely into the bulk. (This will not be the case for the real Boltzmann equation.) The odd part of the distribution $c(v)$ is proportional to the "temperature gradient" $L^{-1}[H_+ - H_-]$ and so goes to zero in this limit.

In particular, despite the pathologies of the model, Fourier's law is satisfied for this system (when $p \neq 0$).

3.3. Simplified Models

3.3.1. Discrete Volocity Model

The discrete velocity model is like the previous one, but now we allow only particles with velocities ± 1, ± 2. This is accomplished by setting

$$H_-(v) = \gamma_1 \delta(v-1) + \gamma_2 \delta(v-2)$$

$$H_+(v) = \bar{\gamma}_1 \delta(v-1) + \bar{\gamma}_2 \delta(v-2)$$

with the normalization conditions $\gamma_1 + 2\gamma_2 = 1$, $\bar{\gamma}_1 + 2\bar{\gamma}_2 = 1$. The number of particles in dx with velocity ± 1 (± 2) is denoted by $f_{\pm 1}(x, t)\, dx$ $(f_{\pm 2}(x, t)\, dx)$.

From (5) and (14), we obtain for the stationary solution

$$f_{-1}(x)=f_{+1}(x)+c(1), \qquad f_{-2}(x)=f_{+2}(x)+c(2)$$
$$f_{+1}(x)+2f_{+2}(x)=d, \qquad f_{-1}(x)+2f_{-2}(x)=d, \qquad c(1)+2c(2)=0$$

This gives

$$f_{+1}(s)=[(\bar{\gamma}_1-\gamma_1)\,d-c(1)]\frac{s}{2}+(\bar{\gamma}_1+\gamma_1)\frac{d}{2}-\frac{c(1)}{2}, \qquad s=\frac{x}{L} \tag{26}$$

which is now linear in s. As before, we can compute $c(1)$ from the boundary conditions and d from the normalization chosen for f:

$$c(1)=\frac{(\gamma_1-\gamma_1)\,d}{3qd+1}, \qquad d=\frac{2\rho}{\bar{\gamma}_1+\gamma_1+2}, \qquad q=pL$$

When $q\to\infty$ the solution becomes

$$\lim_{q\to\infty} f_{+1}(s)=\frac{\rho\gamma_1}{\bar{\gamma}_1+\gamma_1+2}(1-s)+\frac{\rho\bar{\gamma}_1}{\bar{\gamma}_1+\gamma_1+2}(1+s)=\bar{f}_{\ 1}(s)$$

3.3.2. The BGK Model

The Bhatnagar, Gross, and Krook[10] (BGK) equation is a simplified version of the Boltzmann equation which has the same collision invariants.

For our system the BGK analog of (4) is

$$\begin{aligned}\frac{\partial f_+}{\partial t}+v\frac{\partial f_+}{\partial x}&=\frac{1}{\tau}[f_-(x,v)-f_+(x,v)]\\ \frac{\partial f_-}{\partial t}-v\frac{\partial f_-}{\partial x}&=\frac{1}{\tau}[f_+(x,v)-f_-(x,v)]\end{aligned} \tag{27}$$

with boundary conditions given by (5). Here τ, "the collision frequency," can be a function of the local state of the system. Equations (27) now become

$$v\frac{\partial f_+}{\partial x}=\frac{c(v)}{\tau}, \qquad v\frac{\partial f_-}{\partial x}=\frac{c(v)}{\tau}, \qquad c(v)=f_-(x,v)-f_+(x,v) \tag{28}$$

with boundary conditions

$$\begin{aligned}vf_+(-L,v)&=vH_-(v)\,d\\ vf_-(+L,v)&=vH_+(v)\,d_+\end{aligned} \tag{29}$$

where d_- and d_+ are the values at $x=-L$ and $x=L$, respectively, of the function

$$d(x)=\int_{-\infty}^{\infty} vf_-(x, v)\,dv=\int_{-\infty}^{\infty} vf_+(x, v)\,dv$$

The stationary solution has the form

$$f_+(x, v)=\frac{H_+(v)\,d_+ + H_-(v)\,d_-}{2}+c(v)\left(\frac{x}{v\tau}-\frac{1}{2}\right) \tag{30}$$

From (30), we can get $c(v)$

$$c(v)=\frac{H_+(v)\,d_+ - H_-(v)\,d_-}{\left(1+\dfrac{2L}{v\tau}\right)} \tag{31}$$

If we put $\tau=l/v$, where l is a constant playing the role of the mean free path, then $d=d_-=d_+=\rho/(h_+ + h_-)$,

$$c(v)=\frac{ld(H_+(v)-H_-(v))}{2L+l} \tag{32}$$

$$f_\alpha(x, v)=dH_\alpha(v)\left(\frac{L+\alpha x}{2L+l}\right)+dH_{-\alpha}\left(\frac{L+l-\alpha x}{2L+l}\right), \qquad \alpha=\pm \tag{33}$$

When $H_\alpha(v)$ is given by (7), we can compute the heat flux as

$$\begin{aligned} Q&=\frac{1}{2}\int_0^\infty v^3[f_+(x, v)-f_-(x, v)]\,dv=-\frac{1}{2}\int_0^\infty v^3c(v)\,dv \\ &=\frac{l}{2L+l}d(T_- - T_+)=\rho\sqrt{2/\pi}\,\frac{l}{(1/\sqrt{T_+})+(1/\sqrt{T_-})}\,\frac{(T_- - T_+)}{2L+l} \end{aligned} \tag{34}$$

and the entropy production in the reservoirs is given by

$$\sigma_R=-Q\left(\frac{1}{T_-}-\frac{1}{T_+}\right)\geqslant 0 \tag{35}$$

APPENDIX 1

We want to prove the following:

Given $H_-(v)\geqslant 0$, $\int_0^\infty vH_-(v)\,dv=1$ and $\int_0^\infty H_-(v)\,dv<\infty$, there exists $\varepsilon>0$ such that for any $H_+(v)$ such that $\int_0^\infty vH_+(v)\,dv=1$ and $\int_0^\infty |H_+ - H_-|\,dv<\infty$, the stationary solution of (4) and (5) is unique.

Proof. We rewrite Eq. (22) integrating over v and set $2\gamma = z$

$$z = \int_0^\infty \frac{H_+(v)(1+e^z) - H_-(v)(1+e^{-z})\, z^2 v\, dv}{v\left\{z[h_+(1+e^z) - h_-(1+e^{-z})] + (h_+ - h_- - \frac{z(h_+ + h_-)}{\rho q}(e^{-z} - e^z)\right\} + z(e^z - e^{-z})} \tag{A1.1}$$

where ρ is the density of particles.

We consider $H_-(v)$ as fixed and call the integral on the right-hand side of Eq. (A1.1), $I(z, H_+)$. Given now any $\delta > 0$, there will be an $\varepsilon = \varepsilon(\delta) > 0$ such that, for $\int |H_+ - H_-|\, dv < \varepsilon(\delta)$,

$$\frac{I(z, H_+)}{z} < \int_0^\infty \frac{vH_+(v)}{vh_+ + 1}\, dv + \varepsilon\left(\frac{z}{z(h_+ - h_- e^{-z}) + (h_+ - h_-)(e^{-z} - 1)}\right)$$

for any $|z| \geqslant \delta$. The integral on the right-hand side of Eq. (A1.2) is less than 1 and does not depend on H_- and z; hence $|I(z, H_+)/z| < 1$ for $|z| \geqslant \delta$ and $\int |H_+ - H_-|\, dv < \varepsilon(\delta)$ sufficiently small.

On the other hand, for $0 < z < \delta$, it can be shown that for every $H_+(v)$ which verifies the hypotheses, there exists a $d < \infty$ such that

$$\left|\frac{I(z, H_+)}{z} - \frac{c(H_+)}{z}\right| \leqslant d$$

where

$$c(H_+) = \int_0^\infty \frac{(H_+ - H_-)\, v\, dv}{v[(h_+ - h_-)/2] + 1}$$

Hence we can find a δ such that if $\int_0^a |H_+ - H_-|\, dv < \varepsilon$ with ε small enough and $\varepsilon \leqslant \varepsilon(\delta)$, then

(i) if $c(H_+) = 0$ then $\sup_{0 \leqslant z \leqslant \delta} |I(z, H_+)/z| < 1$

(ii) if $c(H_+) > 0$ then $I(z, H_+)/z$ is strictly decreasing for $z \in (0, \delta]$ and goes to infinity when $z \searrow 0$; for $z \in [-\delta, 0)$, $I(z, H_+)/z$ remains negative. If $c(H_+) < 0$, the same holds after changing z to $-z$.

In case (i) the only solution is $z = 0$. In case (ii) there is only one solution for $z \in (0, \delta)$ if $c(H_+) > 0$ and for $z \in (-\delta, 0)$ if $c(H_+) < 0$.

APPENDIX 2

Equation (22) simplifies in the case $h_+ = h_+ = h$. This occurs, for instance, when $H_-(v) = \beta m \exp(-\beta m v^2/2)$, $H_+(v) = \alpha^2 e^{-\alpha v}$, $\alpha = (\beta m \pi/2)^{1/2}$; we then have

$$\hat{c}(v) = \frac{\{H_+(1+e^{2\gamma}) - H_+(1+e^{-2\gamma})\}\, 2\gamma v}{(e^{2\gamma} - e^{-2\gamma})\left\{2\gamma(hv+1) + \dfrac{\gamma}{dq} v\right\}}$$

By integrating over v

$$e^{-2\gamma} = \frac{1 - \displaystyle\int \frac{H_-}{A + 1/v}\, dv}{1 - \displaystyle\int \frac{H_+}{A + 1/v}\, dv}, \qquad A = h + \frac{1}{2qd}$$

For large q, $\lim(qa) = \bar{\gamma}$

$$e^{-2\bar{\gamma}} = \frac{1 - \displaystyle\int \frac{H_-}{h + 1/v}\, dv}{1 - \displaystyle\int \frac{H_+}{h + 1/v}\, dv}$$

In this case $n(s) = \rho - (\gamma/q)\, s$ and $\lim_{q\to\infty} n(s) = \rho$ independent of s.

APPENDIX 3

The solution z of the equation

$$1 = \int_0^\infty \frac{[\alpha e^{-\alpha v^2/2}(1+e^z) - e^{-v^2/2}(1+e^{-z})]\, v\, dv}{\sqrt{\dfrac{\pi}{2}}\, v \left\{[\sqrt{\alpha}(1+e^z) - (1+e^{-z})] + \left(\dfrac{\sqrt{\alpha}-1}{z} - \dfrac{\sqrt{\alpha}+1}{\rho q}\right)(e^{-z} - e^z)\right\} + (e^z - e^z)} = F(z)$$

Table I. $\alpha = 2$, $\rho q = 100$, Function $F(z)$

z	0.1	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.20	0.22	0.25
$F(z)$	1.18	1.11	1.06	1.01	0.96	0.93	0.90	0.87	0.80	0.77	0.72

Table II. Solution z as a function of α

α	1.1	1.5	2	5	10	50	100	150
z	0.01	0.07	0.13	0.29	0.39	0.51	0.49	0.44

obtained by putting, in Eq. (A1.1), $H_+(v) = \beta_+ m\, e^{-\beta_+ mv^2/2}$, $H_- = \beta_- m\, e^{\beta_- mv^2/2}$, has been numerically investigated for values of the parameter $\alpha = \beta_+/\beta_-$ in the range $1.1 \leqslant \alpha \leqslant 150$. The numerical results show that regardless of the assumed value for $\rho q \gg 1$, the function $F(z)$ has, for each z, the typical monotonically decreasing behavior shown in Table I for $\alpha = 2$. Table II shows the behavior of the solution z as a function of α.

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