

Comment on “Yang-Lee Zeros for an Urn Model for the Separation of Sand”

In a recent Letter [1], Bena, Coppex, Droz, and Lipowski (BCDL) examine the Yang-Lee zeros of a model for the separation of sand, and show that it matches the results obtained by other methods in [2] about a critical point in the system. They claim this as further evidence of the validity of the Yang-Lee method for nonequilibrium systems. Since the relation between nonequilibrium and equilibrium ensembles is one of great interest we want to point out that the model for which the main results in [1] are obtained is mathematically equivalent to the Weiss-Ising mean-field model with no external field. The phase transition they study is the equilibrium phase transition well known to exist in that system, and the zeros examined are the Fisher zeros [3] of that mean-field system in the complex $z = e^{-2\beta J}$ plane.

The kinetic model used in most of [1,2] is essentially the same as that proposed by Griffiths, Weng, and Langer (GWL) [4] to study the metastable behavior of the mean-field Weiss-Ising model. The master equation (1) in [1] [henceforth (B1)] with the substitution specified in equation (B4), is, with appropriate changes in parameters, identical to that given by (G7) and (G8) [4]. The partition function in (G5) is equivalent to the “effective” partition function in (B5), from which all further results in that Letter are derived. This can be seen by setting $H = 0$ in [4], substituting $n \rightarrow M$, $\beta \rightarrow A/2$ (note that $J = 1$ in [4]), $\alpha \rightarrow \exp[-\frac{1}{2}A(1 + 1/N)]$, and plugging (G3) into (G8), which yields

$$T_{n,n+1} = \frac{N-M}{N} \exp\left(-A \frac{N-M}{N}\right) = F\left(\frac{N-M}{N}\right), \quad (1a)$$

$$T_{n,n-1} = \frac{M}{N} \exp\left(-A \frac{M}{N}\right) = F\left(\frac{M}{N}\right), \quad (1b)$$

where T are the transition rates in [4] and F specifies the transition rates in [1]. Making the same substitutions and letting $z = \exp(-A/N) = \exp(-2\beta)$ as in [1], GWL's partition function (G5) becomes

$$Z = z^{(-N/4)} \sum_M N M z^{M(N-M)}. \quad (2)$$

The factor $z^{(-N/4)}$ (which in any case does not affect the zeros away from the origin) can be eliminated by changing the (arbitrary) dependence of GWL's energy function U on N , making (2) identical to (B5). BCDL conclude by examining the behavior of the zeros of this partition

function; we are not aware of any other attempts at this (in contrast to other forms of the Ising model), but in light of the nature of the system the significance of the findings for the utility of Yang-Lee methods to nonequilibrium systems needs to be reconsidered.

The more general case which BCDL also examines can be seen more directly to be an equilibrium, not merely a steady-state, distribution, since BCDL's p_s , given in equation (B2) satisfies the detailed balance condition, which for this system reduces to

$$F\left(\frac{M+1}{N}\right) p_s(M+1) = F\left(\frac{N-M}{N}\right) p_s(M). \quad (3)$$

It should be noted that this is an equilibrium system in a mathematical sense even though the physical system being modeled is a nonequilibrium one, with a balance between an energy input (shaking of the container) and dissipation (through inelastic collisions). This can be explained by the observation that this model considers only the movement of the grains between the two urns, in which no steady-state currents are possible.

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