

## Stability and properties of stationary state of one dimensional space charge limited current

A. Rokhlenko and J. L. Lebowitz

Citation: *J. Appl. Phys.* **111**, 013301 (2012); doi: 10.1063/1.3665875

View online: <http://dx.doi.org/10.1063/1.3665875>

View Table of Contents: <http://jap.aip.org/resource/1/JAPIAU/v111/i1>

Published by the [American Institute of Physics](#).

---

### Related Articles

The origin and consequences of push-pull breakdown in series connected dielectrics  
*Appl. Phys. Lett.* **99**, 263506 (2011)

Dielectric relaxation and alternating current conductivity of polyvinylidene fluoride doped with lanthanum chloride  
*J. Appl. Phys.* **110**, 114119 (2011)

Analytical re-derivation of space charge limited current in solids using capacitor model  
*J. Appl. Phys.* **110**, 094514 (2011)

Gallium gradients in chalcopyrite thin films: Depth profile analyses of films grown at different temperatures  
*J. Appl. Phys.* **110**, 093509 (2011)

Surface and confinement effects on the dielectric relaxation of a monohydroxy alcohol  
*J. Chem. Phys.* **135**, 174510 (2011)

---

### Additional information on J. Appl. Phys.

Journal Homepage: <http://jap.aip.org/>

Journal Information: [http://jap.aip.org/about/about\\_the\\_journal](http://jap.aip.org/about/about_the_journal)

Top downloads: [http://jap.aip.org/features/most\\_downloaded](http://jap.aip.org/features/most_downloaded)

Information for Authors: <http://jap.aip.org/authors>

### ADVERTISEMENT

**AIP**Advances

*Submit Now*

**Explore AIP's new  
open-access journal**

- **Article-level metrics  
now available**
- **Join the conversation!  
Rate & comment on articles**

# Stability and properties of stationary state of one dimensional space charge limited current

A. Rokhlenko<sup>a)</sup> and J. L. Lebowitz<sup>b)</sup>

*Department of Mathematics, Rutgers University, Piscataway, New Jersey 08854-8019, USA*

(Received 17 August 2011; accepted 31 October 2011; published online 3 January 2012)

A linear stability analysis of the planar one dimensional space charge limited flow shows that the classical Child’s solution is stable in time. Small perturbations of the flow are accompanied by high frequency oscillations that decay exponentially. These oscillations should appear when the flow regime changes rapidly, and they might be detectable experimentally by their electromagnetic radiation. We also study the stationary state of more general space charge limited flow and obtain a relation between the current density and the electric field at the emitter surface for arbitrary emission laws and a fixed non-zero initial speed of electrons. In such a case there is the possibility of creation of electrostatic barriers for the emission and stationary flow regimes in this simple model. © 2012 American Institute of Physics. [doi:10.1063/1.3665875]

## INTRODUCTION

The seminal works<sup>1,2</sup> of Child and Langmuir at the beginning of the 20th century created a benchmark for the subsequent research on space charge limited flow. While many important results have been obtained for different flow geometries and realistic emission laws, see for example Refs. 3–6, the original relations found in Refs. 1 and 2, for the very simple case of a planar one dimensional (1D) geometry are still very relevant. We consider here this simple original 1D setup but include the possibility of the time dependent small deviations of the current and potential from their stationary values. This may be relevant to the fluctuations observed in space charge limited flows perturbed by changes of their physical parameters or geometry.

We also study the more general 1D case when the cathode emissivity depends on the electric field strength at its surface and the electrons are emitted with a finite speed  $v_0$ . The latter is especially important in photoelectronic devices,<sup>7</sup> electron guns and rings that though are far from the 1D systems nevertheless often exhibit<sup>8</sup> behavior that can be explained in the framework of our simple model. We obtain a closed form equation for the stationary current density as a function of the electric field  $F$  at the emitter surface. This field can be finite when the current-field rule is given, but for infinite emissivity  $F$  tends to zero as  $X^{1/3}$  ( $v_0 = 0$ ) or as  $X$  ( $v_0 \neq 0$ ) where  $X$  is the distance to the cathode. There is a regime in the latter case when  $F$  is negative that creates a negative potential near the emitter surface and the virtual cathode. Anyway as soon as the cathode current dependence on the electric field is given, finding the space charge limited current in 1D is reduced to solving a transcendental equation. This generalizes the result of Ref. 9 for  $v_0 = 0$ . The time dependence in the case  $v_0 \neq 0$ , especially when  $v(0)$  of electrons is not uniform, requires further work.

## Setup of the problem and main equations

The non relativistic electron flow is described by the Maxwell equation

$$\Delta\Phi(\vec{R}, T) = -\frac{1}{\epsilon_0}\rho(\vec{R}, T) \quad (1)$$

and the current continuity equation

$$\vec{\nabla}\vec{J}(\vec{R}, T) = -\frac{\partial\rho}{\partial T}(\vec{R}, T), \quad \vec{J} = \rho\vec{v}, \quad (2)$$

where  $\Phi$  is the electric potential,  $\rho$  and  $\vec{J}$  are the charge and current densities, respectively at the space-time point  $(\vec{R}, T)$ ,  $\epsilon_0$  is the vacuum dielectric constant. We assume that the electron velocities  $\vec{v}$  are determined by the local time dependent potential, i.e., that  $v = |\vec{v}|$  can be obtained from the energy conservation law

$$\frac{mv^2(\vec{R}, T)}{2} - e\Phi(\vec{R}, T) = \frac{mv_0^2}{2}$$

to give

$$v(\vec{R}, T) = \sqrt{v_0^2 + 2e\Phi(\vec{R}, T)/m}. \quad (3)$$

Here we have assumed also that all the electrons are emitted with the same speed  $v_0$  from the region where  $\Phi = 0$ . Note that  $e$  is the elementary charge, that for electrons  $\rho < 0$ , and the direction of the current  $\vec{J}$  is opposite to  $\vec{v}$ .

From now on we study 1D flows between two infinite parallel planes: the cathode, whose potential is 0, is placed at  $X = 0$  while the anode at  $X = D$  has  $\Phi = V$ . For convenience we use dimensionless variables and functions

$$\begin{aligned} x &= X/D, \quad t = T\bar{v}/D, \quad \varphi(x, t) = \Phi(X, T)/V, \\ \text{where } \bar{v} &= \sqrt{v_0^2 + 2eV/m}, \quad \omega = v_0/\sqrt{2eV/m}, \\ j &= -J(X, T)/[\epsilon_0VD^{-2}\sqrt{2eV/m}], \\ \nu(x, t) &= j/\sqrt{\varphi + \omega^2}, \end{aligned} \quad (4)$$

<sup>a)</sup>Electronic mail: rokhlenk@math.rutgers.edu.

<sup>b)</sup>Also at Department of Physics, Rutgers University, Piscataway, New Jersey 08854-8019, USA.

where  $\nu(x, t)$  represents the electron number density  $n(X, T) = (\epsilon_0 V / e D^2) \nu(x, t)$ . Clearly the dimensionless flow density  $j$  is positive because  $J(X, T) < 0$  for electrons.

In the stationary case with infinite cathode emissivity and  $v_0 = 0$  the solution<sup>1,2</sup> of the problem is well known

$$\varphi(x) = x^{4/3}, \quad j_0 = \frac{4}{9}, \quad \nu(x) = \frac{4}{9} x^{-2/3}. \quad (5)$$

It will be used as a starting point in the study of time dependent properties of the perturbed flow.

### Time dependent flow perturbation

We shall study now the time dependent problem when  $\omega = 0$  and the cathode has infinite emissivity. Under our assumptions the electron flow in 1D is governed by two equations

$$\frac{\partial^2 \varphi}{\partial x^2}(x, t) = \rho(x, t), \quad \frac{\partial j}{\partial x}(x, t) = -\frac{\partial \rho}{\partial t}(x, t), \quad (6)$$

where  $j$  is now a function of  $x, t$  and using Eq. (4)  $j(x, t) = \sqrt{\varphi(x, t)} \rho(x, t)$ . The boundary conditions (BC) for Eq. (6)

$$\varphi(0, t) = \frac{\partial \varphi}{\partial x}(0, t) = 0, \quad \varphi(1, t) = 1 \quad (7)$$

are the same as for the stationary flow while the initial conditions will be considered later.

Using the first Eq. (6)  $\rho$  can be replaced by  $\varphi''_{xx}$  and by changing the sequence of differentiation the second Eq. (6) gets the form

$$\frac{\partial}{\partial x} \left[ \frac{\partial^2 \varphi}{\partial x^2}(x, t) \sqrt{\varphi(x, t)} \right] + \frac{\partial}{\partial x} \frac{\partial^2 \varphi}{\partial x \partial t}(x, t) = 0. \quad (8)$$

We integrate Eq. (8) from 0 to  $x$  and obtain

$$\left[ \sqrt{\varphi} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial x \partial t} \right](x, t) = j(0, t) + \frac{\partial^2 \varphi}{\partial x \partial t}(0, t), \quad x = [0, 1] \quad (9)$$

with the BC Eq. (7) for all  $t$ . These imply that the second term in the right side of Eq. (9) vanishes, while the first one is not zero. (Note that when the emission is finite it is possible that  $\varphi''_{xt}(0, t) \neq 0$  and it should be treated as the current perturbation.)

We shall study now the time evolution of small perturbations of the stationary solution (5) in the linear approximation. Taking the perturbed current and potential in the forms

$$j(x, t) = j_0 + i(x, t), \quad \varphi(x, t) = \varphi(x) + \psi(x, t), \quad (10)$$

where  $\varphi(x)$ ,  $j_0$  will come from the solution of the stationary problem while  $i(x, t)$  and  $\psi(x, t)$  are small compared with the corresponding main terms. We substitute Eq. (10) in Eq. (9) and after linearization (i.e., keeping only the terms linear in  $\psi$ ) obtain a single equation

$$\sqrt{\varphi(x)} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial t} + \frac{j_0}{2\varphi(x)} \psi = i(0, t).$$

Using Eq. (5) for the unperturbed potential and current density,  $\varphi(x)$  and  $j_0$ , this equation can be written in the form

$$x^{2/3} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial t} + \frac{2}{9} x^{-4/3} \psi = i(0, t) \equiv I(t). \quad (11)$$

The BC for  $\psi$  are homogeneous:

$$\psi(0, t) = \psi(1, t) = \frac{\partial \psi}{\partial x}(0, t) = 0. \quad (12)$$

The small current perturbation  $i(0, t)$  is an unknown function  $I(t)$ , whose structure is arbitrary for a time being. The properties of  $I(t)$  should be in agreement with physical conditions, and they will be specified to some degree below.

### Solution

It is clear that we are looking for a solution  $\psi(x, t)$  of Eq. (11), which is small compared with  $x^{4/3}$  for all  $x$  including  $x \rightarrow 0$ . This specifies the derivative in Eq. (12) as  $\psi'_x(x, t) = O(x^{1/3})$ . Under this condition, Eq. (11) with the BC Eq. (12) can be solved analytically. Let us look for its solution in the form

$$\psi(x, t) = \Re \sum_{n=0} \psi_n(x, k_n) e^{ik_n t}, \quad I(t) = \Re \sum_{n=0} c_n e^{ik_n t}, \quad (13)$$

where for each term  $\psi_n(x, k_n)$  we require the homogeneous BC identical to Eq. (12) while all  $c_n$  should be small. Substituting Eq. (13) in Eq. (11) and using temporarily a new independent variable  $y = x^{1/3}$  we convert Eq. (11) into a set of independent ordinary differential equations

$$y^2 \frac{d^2 \tilde{\psi}_n}{dy^2} + y(3ik_n y - 2) \frac{d\tilde{\psi}_n}{dy} + 2\tilde{\psi}_n = 9y^4 c_n, \quad 0 \leq y \leq 1, \quad n = 0, 1, \dots, \quad (14)$$

where  $k_n$  plays the role of a parameter. The symbol tilde indicates that  $\tilde{\psi}(x)$  and  $\tilde{\psi}(y)$  are different functions though with the same BC, analogous to Eq. (12).

For solving Eq. (14) we expand  $\tilde{\psi}_n(y, k)$  in a power series in  $y$ , which can be summed up. But as a first step we find a particular solution  $\phi_0(y, k)$  in the following form  $a_1 y^3 + a_2 y^2$ . Direct substitution into Eq. (14) and subsequent elementary computations give  $a_2 = -a_1 / 3ik_n$ ,  $a_1 = c_n / ik_n$ . The series expansion in  $y$  and straightforward manipulations with the series summation yield the first solution of the homogeneous equation corresponding to Eq. (14)

$$\phi_1 = y^2 e^{-3ik_n y}. \quad (15)$$

Equation (15) makes it possible<sup>10</sup> to find the second solution  $\phi_2$  of the homogeneous differential equation by a standard procedure. Thus the general solution of Eq. (14)  $\phi_0(y, k_n) + A\phi_1(y, k_n) + B\phi_2(y, k_n)$  is the sum

$$\tilde{\psi}_n(y, k_n) = \frac{c_n}{ik_n} y^2 \left( y - \frac{1}{3ik_n} \right) + y^2 e^{-3ik_n y} \left( A + B \int \frac{e^{3ik_n y}}{y^2} dy \right), \quad (16)$$

where the arbitrary constants  $A, B$  are to be found by using the BC.

The integral in Eq. (16) should be eliminated by taking  $B = 0$  because it represents a term whose  $x$ -derivative for small  $x$

$$\left(\frac{2}{3}x^{-1/3} - ik_n\right)(x^{-1/3} + ik_n \ln x + \text{Const})$$

diverges as  $x^{-2/3}$  when  $x \rightarrow 0$  for all values of  $k_n$ . Using the original variable  $x$  and the requirement  $\tilde{\psi}(1, k_n) = 0$  we evaluate  $A$  and obtain the final form for the solution of Eq. (14)

$$\psi_n(x, k_n) = -\frac{c_n}{ik_n} x^{2/3} \left[ \left(1 - \frac{1}{3ik_n}\right) e^{3ik_n(1-x^{1/3})} + \frac{1}{3ik_n} - x^{1/3} \right]. \tag{17}$$

This function is equal to zero at  $x = 0$  and  $x = 1$  in agreement with BC Eq. (12), its derivative should decay as  $\phi'(x)$ , i.e., as  $x^{1/3}$ , or faster when  $x \rightarrow 0$ . However, the  $x$ -derivative of  $\psi$  cannot be equal to zero at  $x = 0$  for all values of  $k_n$ , namely, when  $x \rightarrow 0$  one has

$$\frac{\partial \psi_n}{\partial x} \rightarrow -\frac{c_n}{ik_n} \left\{ \left[1 - (1 - 3ik_n)e^{3ik_n}\right] \left(1 - \frac{2x^{-1/3}}{9ik_n}\right) + 2ik_n(1 - 3ik_n)e^{3ik_n}x^{1/3} \right\}.$$

This equation implies that for satisfying Eq. (12) the variable  $k_n$  in Eq. (17) must be within the discrete set of solutions of the following transcendental equation

$$1 - 3ik_n = e^{-3ik_n}. \tag{18}$$

Thus the relevant solution of Eq. (14) is given by Eq. (17) together with Eq. (18).

Clearly Eq. (18) has only one real solution  $k_0 = 0$ . To find all the solutions we denote  $3k_n = \xi + i\eta$ , ( $\xi, \eta$  real), substitute this into Eq. (18), and obtain two relations

$$1 + \xi = e^\xi \cos \eta, \quad \eta = e^\xi \sin \eta.$$

Thus we come to a pair of equations

$$\eta = \ln \frac{\xi}{\sin \xi}, \quad 1 + \ln \frac{\xi}{\sin \xi} = \xi \cot \xi. \tag{19}$$

One can see in Fig. 1 that if  $\xi$  satisfies Eq. (19) then  $-\xi$  is a solution too. Keeping this in mind we consider first only  $\xi \geq 0$  and let  $\xi = \alpha + 2\pi n$ , where  $0 \leq \alpha < \pi$ . Evidently  $\eta \geq 0$  and from the second relation Eq. (19) follows the more restrictive condition  $0 \leq \alpha < \pi/2$ .

We see that the roots of Eq. (18) are in the upper half of the complex  $k$ -plane, which implies that the perturbation of  $\varphi$  goes to zero as  $t \rightarrow \infty$  and therefore the Child-Langmuir solution is linearly stable. The case  $\xi = 0$  corresponds to  $k_0 = 0$ , this belongs to the stationary solution  $x^{4/3}$ . Equation (19) can be easily solved numerically for any  $n$ . The results for  $n = 1, 2, 3, 4$  are

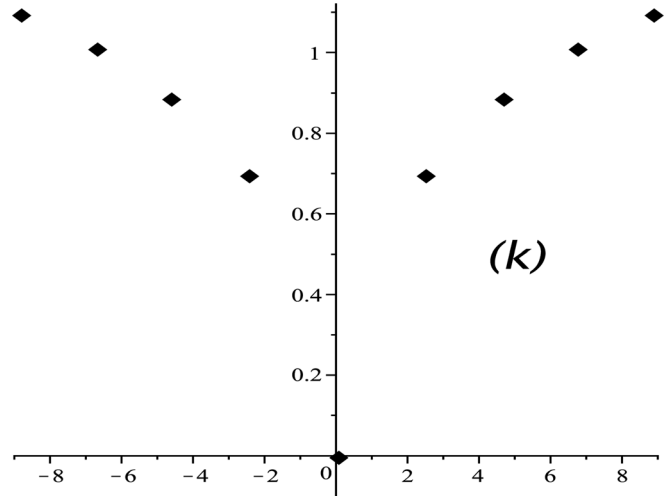


FIG. 1. The first 9 roots of Eq. (18) in complex  $k$ -plane.

$$k_1^\pm = 0.696281i \pm 2.487163, \quad k_2^\pm = 0.888023i \pm 4.626352, \\ k_3^\pm = 1.008766i \pm 6.741278, \quad k_4^\pm = 1.097226i \pm 8.847746,$$

where both signs of  $\xi$  are taken into account. When  $n$  is large  $\alpha_n = \pi/2 - \delta_n$  with a small  $0 < \delta_n \ll 1$  and asymptotically

$$\delta_n \approx \frac{\ln \pi e(2n + 1/2)}{\pi(2n + 1/2)}, \quad \xi_n = \pm[\pi(2n + 1/2) - \delta_n]/3, \\ \eta_n = (1/3) \ln |\xi_n|.$$

This asymptotic formula gives already a decent approximation when  $n > 2$ .

Thus using Eqs. (17), (18) and taking real  $c_n = c_n^+ = c_n^-$  we obtain the solution of Eq. (11)

$$\psi(x, t) = \frac{2x^{2/3}}{3} \Re \sum_{n=1} c_n \frac{e^{ik_n^+ t}}{(k_n^+)^2} (e^{-3ik_n^+ x^{1/3}} - 1 + 3ik_n^+ x^{1/3}). \tag{20}$$

The root of Eq. (18) at  $k_0 = 0$  (which gives a contribution  $\propto x^{4/3}$  to the stationary solution) can be dropped. This form of  $\psi(x, t)$  is directly connected with the current density perturbation Eq. (13), which is

$$i(0, t) = \sum_{n=1} c_n (e^{ik_n^+ t} + e^{ik_n^- t}). \tag{21}$$

Both  $\psi(x, t)$  and  $i(0, t)$  are real.

In part (a) of Fig. 2, plotted for  $t = 0$ , we show the unperturbed potential  $\varphi = x^{3/4}$  in comparison with it perturbed by  $\psi_1$  when  $c_1 = 1/4$  in Eq. (20). Part (b) exhibits  $\psi_1$  and  $\psi_2$  with  $c_1 = c_2 = 1/2$  for illustration.

The initial current perturbation Eq. (21) is not general, its form as well as the form of the potential perturbation Eq. (20) are determined by our method of solution Eq. (13) and necessity to satisfy all properties of the problem. It seems that nevertheless these forms are flexible enough to approximate a wide spectrum of perturbations. Equations (20) and (21) can have any number of terms and all of them

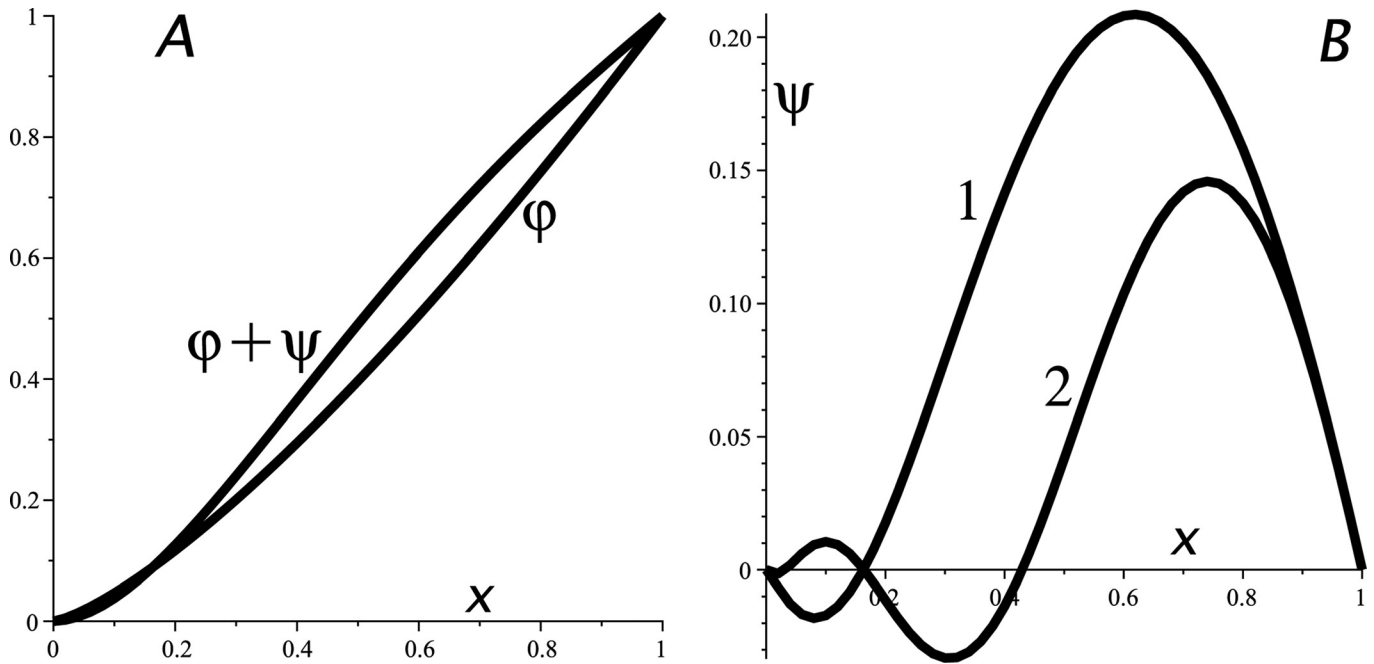


FIG. 2. (a) Potential  $\phi$  perturbed by  $\psi$  when only  $c_1 \neq 0$ . (b) Plots of  $\psi_1$  and  $\psi_2$ .

depict decay of the perturbation because  $\Im k_n > 0$  for all  $n$ . Each of these terms describes also fast oscillations of the current density which gradually die out.

**General stationary 1 D flow**

We now turn to the more general one dimensional space charge limited flow. To obtain the results in a simple closed form, we consider the case where the electrons are emitted with a uniform velocity  $v_0 \neq 0$  but where the cathode emissivity can be infinite or a function of the electric field at the cathode surface. In dimensionless units, the set of Eqs. (1)–(3) for the stationary state can be reduced to a single equation for the potential

$$\frac{d^2 \phi}{dx^2}(x) = \frac{j}{\sqrt{\phi(x) + \omega^2}}, \quad 0 \leq x \leq 1, \quad \phi(0) = 0, \quad \phi(1) = 1, \tag{22}$$

where the scalar flow density  $j$  should be determined in the process of solving Eq. (22) (we study this situation here) or can be a given quantity.<sup>12</sup> After  $\phi$  and  $j$  are computed one can evaluate  $\nu(x)$  using Eq. (4).

The electric field  $\phi'_x(0) = f$  at the emitter surface corresponds to  $\nabla \Phi(0) = fV/D$  (in physical units). Clearly  $f_{max} = 1$  is the cathode field without the space charge. We integrate Eq. (22) by multiplying it with  $d\phi/dx$  and using  $f$  as a parameter that can be evaluated later. The computations are straightforward and give a universal (independent of the emission law) relation between the potential  $\phi(x)$ , space charge limited current density  $j$ , and the surface field  $f$  in the following implicit form

$$\lambda(2\lambda^2 - 3\omega) + (\sqrt{\phi(x) + \omega^2} + 2\omega - 2\lambda^2) \times \sqrt{\sqrt{\phi(x) + \omega^2} - \omega + \lambda^2} = \frac{3}{2}x\sqrt{j}, \tag{23}$$

where  $\lambda = f/2\sqrt{j}$ . Eq. (23) implies that near the emitter  $\phi(x) \rightarrow fx$  for  $x \ll \min\{\omega^2/f, \omega f/2j\}$ . In the case of unlimited emission, when  $f = 0$ ,  $\phi \rightarrow jx^2/2\omega$  for small  $x$  and finite  $\omega$ . If  $\omega = 0$  the result  $\phi = x^{4/3}$  is known.<sup>1,2</sup> Similar to ours the solution of Eq. (22) is given in Ref. 13, where it was studied the virtual cathode formation for a special case with the equipotential electrodes.

Figure 3 shows  $\phi(x)$  for  $x < 0.2$  for the infinite emissivity case. The initial electron velocity makes the growth of the potential much slower and the charge density at  $x = 0$  finite in contrast with the infinite density Eq. (5) when  $v_0 = 0$ . All the curves in Fig. 3 merge when  $x \rightarrow 1$ .

When the current-field law at the cathode  $j = j(f)$  is known Eq. (23) allows one to evaluate the current  $j$ . This is

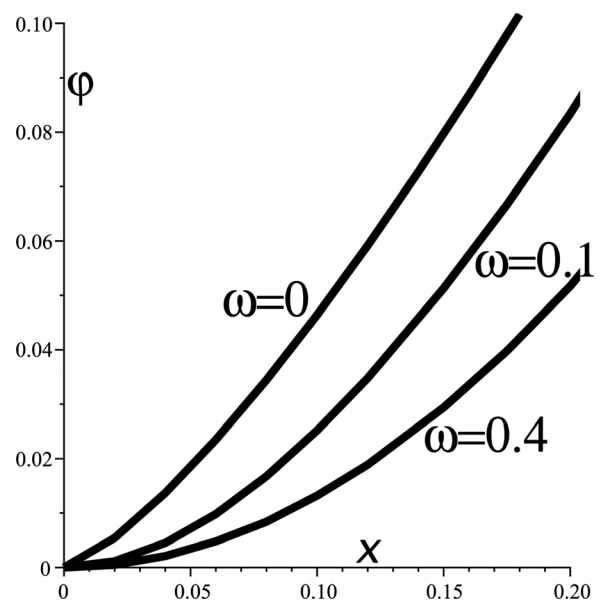


FIG. 3. Potential near cathode surface for different  $\omega$ .

done by noting that at  $x = 1$  (where  $\varphi = 1$ ) Eq. (23) becomes an algebraic equation for  $j$  and  $f$  valid for any law of the electron emission

$$3f^2(1-f) + 9j^2 + 2j[9f\omega - 6\omega - 2(1+\omega^2)^{3/2} + 2\omega^3] = 0, \quad (24)$$

which should be solved together with the relation  $j = j(f)$ . We show in Fig. 4 the solutions of Eq. (24) for three different  $\omega$ . The graph of emission law  $j(f)$  meets the corresponding plot of Fig. 4 and the point of intersection determines both  $j$  and  $f$  for the given setup. This generalizes results of Refs. 9, 11, where  $\omega = 0$ , i.e., electron are emitted with zero velocity. In the case, where  $j$  is proportional to  $f$  or  $f^2$ , the analytic solutions were found in Ref. 9. These solutions are implicit not only for  $\varphi(x)$  but for  $j$  too, which makes the study of their time evolution much more difficult even for these simple emission laws.

In the case of infinite cathode emissivity, which clearly makes  $f = 0$  and provides the BC Eq. (7), the solution for  $j$  of Eq. (24) is

$$j = \frac{4}{9}[(1+\omega^2)^{3/2} - \omega^3 + 3\omega]. \quad (25)$$

In particular, when  $\omega = 0$ ,  $j = j_0 = 4/9$  as it should. Our result Eq. (25) has not the same  $\omega$  dependence as the current density found in Ref. 14 where the author used a different BC at  $x = 0$ . Equation (25) represents the current density that grows very sharply with  $\omega$ , its simple linear approximation  $j \approx (4/9)(1 + 3.83\omega)$  differs from the exact one by less than 7% for  $\omega < 1$ . Note that in the case  $\omega = 1$  (when the electron kinetic energy at the cathode is 1eV) the current density is  $\sim 2.15$ , i.e., almost 5 times larger than the Child-Langmuir limit  $j = j_0$  for  $v_0 = 0$ . The current  $j$  tends to  $2\omega$  when  $\omega \rightarrow \infty$ . A simple computation shows that the total

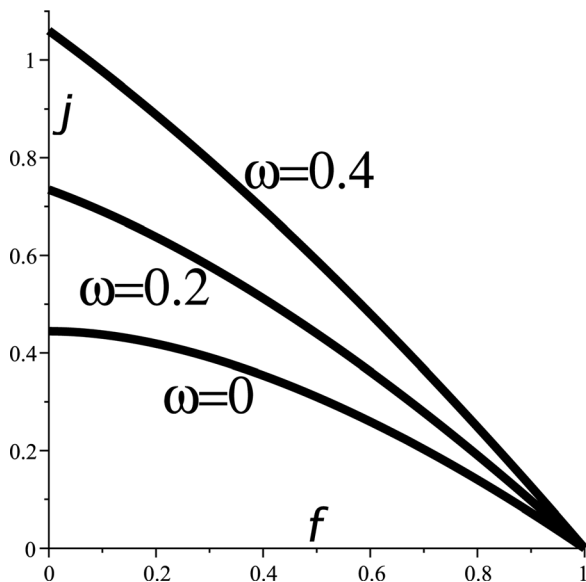


FIG. 4. Current density vs cathode electric field for different  $\omega$ .

space charge  $\int_0^1 \nu(x)dx = 4(1 - \omega^2 + \omega\sqrt{1 + \omega^2})/3$  grows when  $\omega$  increases but only by a factor 3/2 when  $\omega$  runs from zero to  $\infty$ .

The solution Eqs. (24) and (25) are based on the assumption of infinite cathode emissivity, which might not necessary hold if  $v_0 > 0$  even for emitters for which this is a good approximation when  $v_0 = 0$ . We show below that there is a regime when non-zero initial velocity makes the cathode field negative, but paradoxically the stationary current density can exceed Eq. (25). The problem of stability of these solutions is open.

In reality the initial velocities  $v_0$  are not uniform but distributed by some law,<sup>3,6</sup> and when taking  $v_0$  as an average is not acceptable the result of considering each velocity separately and then integrating Eq. (5) is incorrect because the actual electron density is the sum over all the velocity spectrum. This means that our technique should be modified for practical applications.

### Possibility of non-monotone potential when $v_0 > 0$

In all stationary flows considered above the potential  $\varphi(x)$  was a strictly increasing function and  $\varphi'(0) \geq 0$ , even when  $\omega > 0$ . But physically in the latter case one can assume that fast electrons might overcome the repulsive force when  $\varphi'(0) < 0$  and be emitted anyway. Like in Ref. 7 we construct a simple model of this process by starting from Eq. (22) whose first integration yields

$$\left[\frac{d\varphi}{dx}(x)\right]^2 = \left[\frac{d\varphi}{dx}(0)\right]^2 + 4j(\sqrt{\varphi(x) + \omega^2} - \omega).$$

We assume that  $\varphi(x)$  is decreasing for  $0 < x < x_1$ , i.e.,  $\varphi'(x) < 0$  there while  $\varphi'(x_1) = 0$ . Clearly  $\varphi(x) < 0$  on some interval that includes  $(0, x_1)$ . Thus we come to

$$\frac{d\varphi}{dx}(x) = \text{sgn}(x - x_1)\sqrt{4j(\sqrt{\varphi(x) + \omega^2} - \omega + \lambda^2)}, \quad (26)$$

$$\lambda^2 = f^2/4j,$$

where  $f = -2\lambda\sqrt{j}$ , ( $\lambda \geq 0$ ) (using the same notation  $f$  for  $\varphi'(0)$ ) and always  $\varphi(x) + \omega^2 \geq 0$ . As  $j \geq 0$  the dimensionless electron velocity  $+\sqrt{\varphi(x) + \omega^2} = j/\nu(x) \geq 0$  is positive too. Therefore the situation  $\varphi(x) + \omega^2 < 0$ , which corresponds to the velocity  $-\sqrt{-\varphi(x) - \omega^2}$  directed toward the cathode, is impossible here. At  $x = x_1$  the square root in Eq. (26) becomes zero and  $\varphi(x_1) < 0$  is the lowest point of the potential. We integrate Eq. (26) and obtain for  $x \leq x_1$

$$\frac{3}{2}x\sqrt{j} = \lambda(3\omega - 2\lambda^2) - (\sqrt{\varphi(x) + \omega^2} + 2\omega - 2\lambda^2)\sqrt{\sqrt{\varphi(x) + \omega^2} - \omega + \lambda^2}.$$

Thus the minimum of  $\varphi$  occurs when Eq. (26) is zero at the point

$$x_1 = \frac{2\lambda}{3\sqrt{j}}(3\omega - 2\lambda^2), \quad \varphi(x_1) = -\lambda^2(2\omega - \lambda^2). \quad (27)$$

Equation (26) implies also the necessary condition for realization of  $\varphi'(x_1) = 0$

$$\lambda^2 < \omega. \tag{28}$$

When  $\omega \ll 1$  both  $x_1$  and the potential barrier  $\varphi(x_1)$  in Eq. (27) are small too.

At the interval  $x_1 < x \leq 1$  the integration of Eq. (26) gives  $\varphi(x)$  implicitly via the following form

$$\frac{3}{2}(x - x_1)\sqrt{j} = (\varphi(x) + \omega)2 + 2\omega - 2\lambda^2 \times \sqrt{\varphi(x) + \omega^2 - \omega + \lambda^2}, \tag{29}$$

which at  $x = 1$  becomes an algebraic equation for  $j$ ,  $\omega$ , and  $f$ . We come again after algebraic transformations to Eq. (24) where  $f = -\lambda < 0$ . The emission rule should give one more mutual dependence for these physical parameters and thus close the set of all necessary equations. If a solution of this set satisfies Eq. (28) the non-monotone potential will be realized in 1 D, see also Refs. 14 and 13 for a different setting.

In Fig. 5 we plot  $\varphi(x)$  by taking the emission rule in the simple form  $\lambda^2 = \omega/2$ , which is not really plausible, but satisfies eq. (28) and is used here for illustration purpose only.

In this case Eq. (27) gives  $x_1$  in terms of  $\omega$  and  $j$  while the solution of Eq. (29) at  $x = \varphi = 1$  yields  $j$  in terms of  $\omega$  only. Thus we have all the parameters to find  $\varphi$  for all  $x$ . Compared with Eq. (25) for zero cathode field the current density here is higher for all  $\omega$ . The shape of potential shown in Fig. 5 for a very special situation is known and studied by many authors, for instance in Refs. 2 and 13, for the Maxwellian distribution of  $\omega$  (more relevant in low voltage regimes). In such cases the slow electrons almost do not contribute to the space charge because they cannot leave the cathode and visually the shape in Fig. 5 is not much different from ones shown in other works for more general setups.

The electron tunneling clearly is dependent on the height of the additional electrostatic barrier. Point  $x_1$  is the location of the so-called virtual cathode,<sup>4,6,13</sup> where the

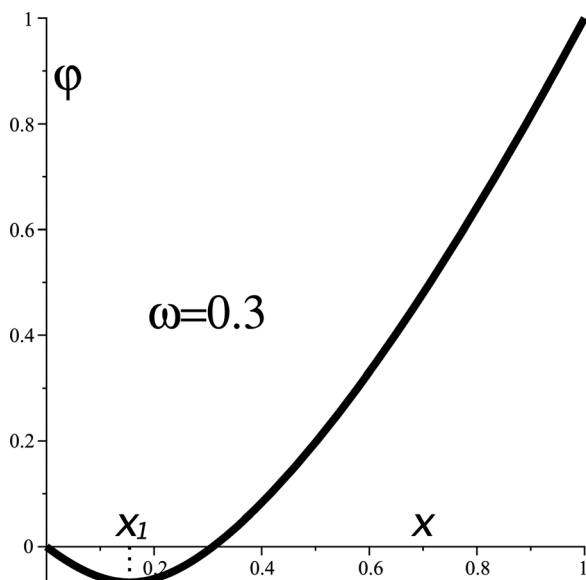


FIG. 5. Electric potential for a hypothetical emission rule  $f = -\sqrt{2\omega j}$ .

potential reaches its minimum and the electric field is zero like at the real cathode in the Child-Langmuir system. There can be an interesting effect: two (or more) different stationary states of the same systems, which can be a source of their unstable behavior, see Ref. 13. But all in all a deeper study needs a reliable model of  $j(f)$  dependence.

The temporal stability of more general flows is not studied here. The analysis used in the present work requires the stationary solution of Eq. (22) in the explicit form, which is not available when  $\omega \neq 0$  and/or  $f \neq 0$ . We plan to investigate this in future work.

### DISCUSSION

The form of Eqs. (21) and (22) in the physical units of time requires a factor  $\bar{v}/D$  (see Eq. (4)) in front of  $T$ . Thus using Eqs. (22), (4), and (10) we obtain the time dependent flow density at the emitter

$$J(0, T) = \epsilon_0 \frac{V^{3/2}}{D^2} \sqrt{\frac{2e}{m}} \left[ \frac{4}{9} + 2 \sum_{n=1} c_n \Re \exp(ik_n^+ T/\tau) \right], \quad T > 0,$$

where  $\tau = D/\bar{v}$  is roughly the time of flight needed for an electron to cross the diode. The root  $k_1$  corresponds to  $T_1 \approx 3\tau$  for decreasing  $i(0, t)$  by an order. The period of oscillations represented by  $2\pi\tau/\Re k_1^+$  is shorter a little. These effects can be observed only in special experiments because, say for a diode with  $D = 2\text{cm}$  and  $V = 1\text{kV}$ , electrons need only  $\tau \sim 10^{-9}$  s to reach the anode. Usual observations show a quite stable current behavior in thermionic processes (more or less relevant to that considered here) consistent with the stability proven here. In the case of a sudden change of the flow regime (like the one studied in Ref. 15) there might be observable transient current oscillations described above. They would be manifested by the electromagnetic radiation polarized parallel to the electron flow. For its detection the voltage, applied to the vacuum diode, should be modulated by rectangular pulses, and an antenna of a wideband receiver in the GHz range should be placed near the diode. The detected signal probably will look as a high frequency noise because of the presence of terms with different  $k_n$  and random  $c_n$ , but the frequencies of oscillations are determined only by the system geometry and voltage. For increasing the detection sensitivity the tube can be “tuned” by changing the applied voltage for adjusting one of the frequencies  $\Re(k_n)/2\pi\tau$  to the receiver range.

Our study of the stationary space charge limited flow gives a simple recipe for evaluation of the stationary current in systems, which can be treated as one dimensional, for any emission law and arbitrary initial speed of emitted particles. We have considered the possibility of a non-monotone potential and creation of an additional electrostatic barrier for the emission with finite initial speeds of electrons.

### ACKNOWLEDGMENTS

Work was supported in part by AFOSR Grant No. F49620-01-0154. J.L.L. thanks John Luginsland and Kevin Jensen for useful discussions.

- <sup>1</sup>C. D. Child, *Phys. Rev.* **32**, 492 (1911).
- <sup>2</sup>I. Langmuir, *Phys. Rev.* **2**, 450 (1913); I. Langmuir and K. B. Blodgett, *ibid.* **22**, 347 (1923); *ibid.* **24**, 49 (1924).
- <sup>3</sup>I. Langmuir, *Phys. Rev.* **21**, 419 (1923).
- <sup>4</sup>S. Y. Liu and R. A. Dougal, *J. Appl. Phys.* **78**, 5919 (1995).
- <sup>5</sup>L. Page and N. I. Adams, Jr., *Phys. Rev.* **76**, 381 (1949).
- <sup>6</sup>C. B. Wheeler, *J. Phys. A* **6**, 1439 (1973).
- <sup>7</sup>C. K. Birdsall and W. B. Bridges *Electron Dynamics of Diode Regions* (Academic, New York, 1966).
- <sup>8</sup>A. Valfells, D. W. Feldman, M. Virgo, and P. G. O'Shea, *Phys. Plasmas* **9**, 2377 (2002).
- <sup>9</sup>A. Rokhlenko, K. L. Jensen, and J. L. Lebowitz, *J. Appl. Phys.* **107**, 014904 (2010).
- <sup>10</sup>E. Kamke, *Differential Gleichungen* (B. G. Teubner, Leipzig, 1977), Vol. 1.
- <sup>11</sup>R. G. Forbes, *J. Appl. Phys.* **104**, 084303 (2008).
- <sup>12</sup>R. R. Puri, D. Biswas, and R. Kumar, *Phys. Plasmas* **11**, 1178 (2004).
- <sup>13</sup>D. H. Dowell, S. Joly, and A. Loulergue, *AIP Conf Proc.* **398**, 793 (1997).
- <sup>14</sup>G. Jaffe, *Phys. Rev.* **65**, 91 (1944).
- <sup>15</sup>A. Kadish, W. Peter, and M. E. Jones, *Appl. Phys. Lett.* **47**, 115 (1985).