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Robust Multigrid Method for the Planar Linear Elasticity Problems

Young-Ju Lee · Jinbiao Wu · Jinru Chen

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Abstract We consider the solution of the system of equations that arise from the higher order conforming finite element (Scott-Vogelius element) discretizations of the boundary value problems associated to the differential operator $-\rho^2\Delta - \kappa^2\nabla\text{div}$, where ρ and κ are parameters. Constructed are multigrid methods that are of an optimal convergence property with respect to the mesh, the number of levels, and weights on the two terms in the aforementioned differential operator.

Keywords nearly incompressible elasticity problem · multigrid method · nearly singular problem · subspace correction method

Mathematics Subject Classification (2000) 65N55 · 65F10 · 65N30 · 65N12

1 Introduction

We are interested in the fast and robust solution of the finite element discretizations of the linear elasticity. To be precise, let Ω denote a bounded polygonal domain in $\mathbb{R}^2$. The linear elasticity equation is given through:

$$\begin{cases} -\Delta\mathbf{u} - \frac{1}{1-2\nu}\nabla\text{div}\mathbf{u} = \mathbf{g} \ \text{ in } \Omega, \\ \mathbf{u} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1}$$

Young-Ju Lee
Department of Mathematics, Rutgers, The State University of New Jersey, Busch Campus, NJ 08854-8019, USA
E-mail: leeyoung@math.rutgers.edu

Jinbiao Wu
LMAM and School of Mathematical Sciences, Peking University, Beigjing 100871, China
E-mail: jwu@math.pku.edu.cn

Jinru Chen
School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing, 210097, China
E-mail: jrchen@njnu.edu.cn

where $0<\nu<\frac{1}{2}$ is a material-dependent constant, the so-called Poisson's ratio, which describes the compressibility. The Poisson ratio ν is close to $\frac{1}{2}$, which corresponds to a nearly incompressible material. For simplicity, we have imposed the homogenous Dirichlet boundary condition.

The fast solution method for (1) can also be motivated by the solution to the following steady-state Stokes problem in the velocity-pressure formulation with $\mathbf{f}\in(L^2(\Omega))^2$, namely, to find the vector-valued function $\mathbf{u}\in(H_0^1(\Omega))^2$ and the scalar valued function $p\in L^2(\Omega)$ satisfying

$$\begin{cases} -\Delta\mathbf{u}+\nabla p = \mathbf{f} \ \text{ in } \Omega, \\ \quad\quad \operatorname{div}\mathbf{u} = 0 \ \text{ in } \Omega. \end{cases} \tag{2}$$

There have been a number of works on the solution to the mixed finite element for the equation (2). The Augmented Uzawa method can be effectively applied to solve the equation (2) efficiently [6]. The Augmented Uzawa method is based on the Uzawa method to the following reformulated version of the aforementioned Stokes equation:

$$\begin{cases} -\Delta\mathbf{u}-r\nabla\operatorname{div}\mathbf{u}+\nabla p = \mathbf{f} \ \text{ in } \Omega, \\ \quad\quad\quad\quad\quad\quad \operatorname{div}\mathbf{u} = 0 \ \text{ in } \Omega, \end{cases} \tag{3}$$

where $r>0$ is some parameter that can be adjusted arbitrarily. In particular, the speed of the Augmented Uzawa method will be dramatically improved when $r\gg 0$. The trade off is to make the inversion of the operator $-\Delta-r\nabla\operatorname{div}$ difficult.

The multigrid methods for the non-conforming finite element and mixed finite element discretizations for the nearly incompressible elasticity problem are discussed in [4,11]. The main concern in this paper is to devise a robust multigrid method for the higher order conforming finite element (Scott-Vogelius element) discretizations for the following general parameter-dependent problem: Find $\mathbf{u}\in(H_0^1(\Omega))^2$ satidfying

$$-\rho^2\Delta\mathbf{u}-\kappa^2\nabla\operatorname{div}\mathbf{u}=\mathbf{f}.$$

We present several technicalities in accomplishing our purpose. Our analysis is performed for the higher order finite element discretization analyzed by Scott and Vogelius and heavily rely on the properties of the finite element space, introduced therein. The standard multigrid methods applied to solved the discrete system deteriorates when $\rho^2/\kappa^2\to 0$. By the framework developed in [6], we construct a multigrid method with vertex-based Schwarz method as the smoother. We prove the convergence rate of the multigrid method is independent of the mesh size, number of levels and the two parameters ρ and κ.

We use the standard Sobolev space notation in the paper. If $\Omega'\subset\Omega$ is a polygonal (sub) domain and k is a nonnegative integer, $H^k(\Omega')$, denotes the set of functions with derivatives of order $\leq k$ in $L^2(\Omega')$ and the corresponding norm is denoted by $\|\cdot\|_{k,\Omega'}$. Following [17], we use the notation $x_1 \lesssim y_1$ and $x_2 \gtrsim y_2$ whenever there exist constants C_1 and C_2 independent of the important parameters such that $x_1 \leq C_1 y_1$, and $x_2 \geq C_2 y_2$. Additionally we write $x_3 \simeq y_3$, if $x_3 \lesssim y_3$ and $x_3 \gtrsim y_3$.

The rest of this paper is organized as follows. In Section 2, we introduce abstract framework that treats the multigrid analysis for the parameter dependent problems. The Scott-Vogelius higher order finite element analysis is revisited and analyzed in an appropriate manner in which the multigrid analysis is transparent. The multilevel finite element spaces are constructed and a number of technical lemmas are introduced in Section 3 and Section 4. In Section 5, we design and analyze a robust multigrid method for the linear elasticity equation for which the convergence is independent of the mesh size, number of levels, and the Poisson's ratio. We conclude the paper with some remarks in Section 6.

2 Abstract Convergence Framework for Subspace correction method for Nearly Singular Problems

In this section, we give a brief description of the subspace correction methods to solve the parameter dependent system of equations that includes a class of nearly singular system of equations, [6].

We begin by casting the model equation (1) in a variational (or weak) form as follows: Find $\mathbf{u} \in (H_0^1(\Omega))^2$ such that

$$\int_\Omega \nabla \mathbf{u} \nabla \mathbf{v}\, dx + \frac{1}{1-2\nu} \int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\, dx = \int_\Omega \mathbf{g} \mathbf{v}\, dx, \quad \forall \mathbf{v} \in (H_0^1(\Omega))^2. \tag{4}$$

In order to derive the abstract convergence framework in particular, for the equation (4), and consider more general setting, we introduce some additional spaces and bilinear forms. Let V be a real Hilbert space with an inner product $a(\cdot,\cdot)$ and (energy) norm $\|\cdot\|_a = a(\cdot,\cdot)^{1/2}$. The variational formulation of our interest in this section is as follows:

$$\text{Find } u \in V \text{ such that } a(u,v) = f(v), \quad \forall v \in V, \tag{5}$$

where the bilinear form $a(\cdot,\cdot)$ can be decomposed into the following form:

$$a(u,v) = \rho^2 a_p(u,v) + \kappa^2 a_s(u,v), \quad \forall u,v \in V, \tag{6}$$

where ρ^2 and κ^2 are positive parameters and two bilinear forms $a_s(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$ are assumed to be

(i) a_p is symmetric and positive definite,
(ii) a_s is symmetric and positive semidefinite.

In want follows, $\|\cdot\|_{a_p}$ and $|\cdot|_{a_s}$ denote the corresponding norm and semi-norm, respectively. The model equation (4) can be a specific one if we choose $\kappa^2 = \frac{1}{1-2\nu}$ and $\rho^2 = 1$ with

$$a_p(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}\,dx, \quad \text{and} \quad a_s(\mathbf{u},\mathbf{v}) = \int_\Omega \mathrm{div}\mathbf{u}\,\mathrm{div}\mathbf{v}\,dx.$$

In the following discussion, for any closed subspace $W \subset V$, by $W^\perp$ we denote the orthogonal complement of W with respect to the inner product $a(\cdot,\cdot)$ and by $\mathscr{N}$ the null space of the bilinear form a_s defined through:

$$\mathscr{N} = \{v \in V : a_s(v,w) = 0, \quad \forall w \in V\}. \tag{7}$$

The construction of the general subspace correction methods is based on the space decomposition that the space V is decomposed into a number of subspaces V_k, with $k = 1,\cdots,J$ and the introduction of the local subspace solver in each subspace. In particular, to achieve the parameter independent convergence property of the method, the following formal sets of assumptions should be justified.

A0 There exist a number of closed subspaces V_k with $k = 1,\cdots,J$ such that

$$V = \sum_{k=1}^{J} V_k.$$

A1 The null space $\mathscr{N}$ can be represented by sum of elements in subspaces, namely,

$$\mathscr{N} = \sum_{k=1}^{J} (V_k \cap \mathscr{N}).$$

For each subspace V_k, we define the orthogonal projection $P_k : V \mapsto V_k$ with respect to $a(\cdot,\cdot)$ inner product by

$$a(P_k v, v_k) = a(v, v_k), \quad \forall v \in V, v_k \in V_k. \tag{8}$$

Note that the bilinear form $a(\cdot,\cdot)$ is coercive on V_k for each $k = 1,\cdots,J$, therefore, the operator P_k is well-defined, which will be used as our local subspace solvers. Now, all components of the subspace correction methods are in place. The method of subspace corrections can be stated as follows:

Algorithm 21 (MSSC) *Let $u^0 \in V$ be given.*

```
for $\ell = 1,2,\cdots$
  $u_0^{\ell-1} = u^{\ell-1}$
  for $i = 1,\cdots,J$
```

```
            Let $e_i \in V_i$ solve

                $a(e_i, v_i) = f(v_i) - a(u_{i-1}^{\ell-1}, v_i), \quad \forall v_i \in V_i$        (9)

            $u_i^{\ell-1} = u_{i-1}^{\ell-1} + e_i$
        endfor
        $u^\ell = u_J^{\ell-1}$
    endfor
```

First, we introduce for each $k = 1, \cdots, J$, the local null space $\mathcal{N}_k$ which are defined through :

$$\mathcal{N}_k = \{u_k \in V_k : a_s(u_k, v_k) = 0, \quad \forall v_k \in V_k\}. \tag{10}$$

It is easy to establish the identity: $\mathcal{N}_k = V_k \cap \mathcal{N}$. The orthogonal complement of $\mathcal{N}_k$ restricted to the subspace V_k would be denoted by (with some abuse of notation) $\mathcal{N}_k^\perp$. More precisely,

$$\mathcal{N}_k^\perp = \{u_k \in V_k : a(u_k, c_k) = 0, \quad \forall c_k \in \mathcal{N}_k\}. \tag{11}$$

The space V_k can be written as $V_k = \mathcal{N}_k \oplus \mathcal{N}_k^\perp$ with such a notation. Note that the orthogonality $\oplus$ is given with respect to $a(\cdot,\cdot)$ inner product. Obviously it can be thought to be with respect to $a_p(\cdot,\cdot)$ inner product, which is the symmetric positive part of the inner product $a(\cdot,\cdot)$.

We introduce two additional projections with respect to the bilinear forms $a_s(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$ for the discussion that follows. For each $k = 1, \cdots, J$, they will be denoted respectively by $P_{k,s} : V \mapsto \mathcal{N}_k^\perp$ and $P_{k,p} : V \mapsto V_k$ and satisfy the following relations:

$$\begin{aligned} a_s(P_{k,s}v, v_k) &= a_s(v, v_k), \quad \forall v \in V, v_k \in V_k \\ a_p(P_{k,p}v, v_k) &= a_p(v, v_k), \quad \forall v \in V, v_k \in V_k. \end{aligned}$$

By the coercivity of a_s on $\mathcal{N}_k^\perp$, we can prove that the operator $P_{k,s}$ maps $\mathcal{N}$ into $\{0\}$.

In light of the assumptions **A0** and **A1**, the parameter independent estimates can be attained:

Theorem 1 *Under the assumptions* **A0** *and* **A1**, *the estimate of the energy norm for the error transfer operator $E_J = (I - P_J) \cdots (I - P_1)$ can be established as follows:*

$$\|E_J\|_a^2 = 1 - \frac{1}{K}, \tag{12}$$

where

$$
\begin{aligned}
K &= \sup_{v\in V} \inf_{\sum_{k=1}^{J} v_k = v} \frac{\sum_{k=1}^{J} \left\| P_k \sum_{j\geq k} v_j \right\|_a^2}{(v,v)_a} \\
&\lesssim \sup_{v\in \mathscr{N}^{\perp}} \inf_{\sum_{k=1}^{J} v_k = v} \left(\frac{\sum_{k=1}^{J} \left| P_{k,s} \sum_{j\geq k} v_j \right|_{a_s}^2}{(v,v)_{a_s}} + \frac{\sum_{k=1}^{J} \left\| P_{k,p} \sum_{j\geq k} v_j \right\|_{a_p}^2}{(v,v)_{a_p}} \right) \\
&+ \sup_{v_c\in \mathscr{N}} \inf_{\sum_{k=1}^{J} v_{k,c} = v_c} \frac{\sum_{k=1}^{J} \left\| P_{k,p} \sum_{j\geq k} v_{j,c} \right\|_{a_p}^2}{(v_c,v_c)_{a_p}},
\end{aligned}
$$

where v_k *belongs to* V_k *and* $v_{k,c}$*'s are in* $\mathscr{N}_k$ *for each* $k = 1,\cdots,J$.

Proof For any given $v \in V$, consider the orthogonal decomposition of v given as follows

$$
v = w + v_c, \quad w \in \mathscr{N}^{\perp}, \quad v_c \in \mathscr{N}. \tag{13}
$$

The use of the assumption **A1** leads to write v_c as the sum of local null elements. Namely, we can make the following decomposition:

$$
v = \sum_{k=1}^{J} w_k + \sum_{k=1}^{J} v_{k,c} = \sum_{k=1}^{J} v_k, \tag{14}
$$

where $w_k, v_k \in V_k$ and $v_{k,c} \in \mathscr{N}_k$,

$$
\sum_{k=1}^{J} w_k = w \quad \text{and} \quad \sum_{k=1}^{J} v_{k,c} = v_c. \tag{15}
$$

An application of the Cauchy-Schwartz inequality and the relation (14) admit the following inequalities:

$$
\begin{aligned}
\left\| P_k \sum_{j\geq k} v_j \right\|_a^2 &\leq \kappa^2 \left| P_{k,s} \sum_{j\geq k} w_k + v_{k,c} \right|_{a_s}^2 + \rho^2 \left\| P_{k,p} \sum_{j\geq k} w_k + v_{k,c} \right\|_{a_p}^2 \\
&= \kappa^2 \left| P_{k,s} \sum_{j\geq k} w_k \right|_{a_s}^2 + \rho^2 \left\| P_{k,p} \sum_{j\geq k} w_k + v_{k,c} \right\|_{a_p}^2 \\
&\lesssim \kappa^2 \left| P_{k,s} \sum_{j\geq k} w_k \right|_{a_s}^2 + \rho^2 \left\| P_{k,p} \sum_{j\geq k} w_k \right\|_{a_p}^2 \\
&+ \rho^2 \left\| P_{k,p} \sum_{j\geq k} v_{k,c} \right\|_{a_p}^2 .
\end{aligned}
$$

Therefore, we obtain that

$$\sum_{k=1}^{J}\left\|P_k\sum_{j\geq k}v_j\right\|_a^2 \lesssim \left(\kappa^2\sum_{k=1}^{J}\left|P_{k,s}\sum_{j\geq k}w_k\right|_{a_s}^2+\rho^2\sum_{k=1}^{J}\left\|P_{k,p}\sum_{j\geq k}w_k\right\|_{a_p}^2\right)$$
$$+\,\rho^2\sum_{k=1}^{J}\left\|P_{k,p}\sum_{j\geq k}v_{k,c}\right\|_{a_p}^2. \tag{16}$$

The aforementioned inequality can be viewed to hold now for arbitrary decompositions $\{w_k\}_{k=1}^J$ and $\{v_{k,c}\}_{k=1}^J$ of w and v_c respectively satisfying $w_k+v_{k,c}=v_k$ for $k=1,\cdots,J$. Therefore, taking infimum over the decomposition $\{v_k\}_{k=1}^J$, we obtain the desired inequality:

$$\inf_{\sum_{k=1}^J v_k=v}\sum_{k=1}^{J}\left\|P_k\sum_{j\geq k}v_j\right\|_a^2 \lesssim \inf_{\sum_{k=1}^J w_k=w}\left(\kappa^2\sum_{k=1}^{J}\left|P_{k,s}\sum_{j\geq k}w_k\right|_{a_s}^2\right. \tag{17}$$
$$\left.+\rho^2\sum_{k=1}^{J}\left\|P_{k,p}\sum_{j\geq k}w_k\right\|_{a_p}^2\right)$$
$$+\inf_{\sum_{k=1}^J v_{k,c}=v_c}\rho^2\sum_{k=1}^{J}\left\|P_{k,p}\sum_{j\geq k}v_{k,c}\right\|_{a_p}^2,$$

together with a simple observation on the following relation of the denominator,

$$\begin{aligned}\|v\|^2 &= \kappa^2(w+v_c,w+v_c)_{a_s}+\rho^2(w+v_c,w+v_c)_{a_p}\\ &= \kappa^2\|w\|_{a_s}^2+\rho^2\|w\|_{a_p}^2+\rho^2\|v_c\|_{a_p}^2,\end{aligned}$$

we conclude our proof.

Notice that v is restricted to $\mathcal{N}^\perp$ when the supremum is taken and furthermore, although the abstract assumptions in the Theorem 1 are used crucially to get rid of the parameters ρ^2 and κ^2 in the estimate of K, this is in fact, a partial success since there is a "hidden" parameter, which is the mesh size h, that can hinder achieving the robustness of the iterative method more severely. In the actual application, in particular for the linear elasticity equation, using the technique of BPX estimate and under the abstract estimate given in the Theorem 1, we will show that

$$\frac{\left|P_{k,s}\sum_{j\geq k}v_j\right|_{a_s}^2}{|v|_{a_s}^2}\lesssim\frac{\|v\|_{a_p}^2}{|v|_{a_s}^2}.$$

In order to prove that $K=O(1)$, we need to show that $\|v\|_{a_p}\lesssim|v|_{a_s}$, which is irrevant to the assumptions **A0** and **A1** at all and for which it is required that $v\in\mathcal{N}^\perp$, where $\mathcal{N}^\perp$ is the range of the divergence operator as will be seen shortly in the next section.

3 Review on Scott-Vogelius Finite Element Spaces

Let $\mathscr{T}_h = \{\tau_i^h\}_{i=1}$ with $0 < h \leq 1$ be a family of *quasiuniform* triangulations of Ω, parametrized by the mesh size h. To be more precise, for a fixed h, the τ_i^h are disjoint triangles with

$$\mathrm{diam}\tau_i^h \leq h, \quad \text{and} \quad \cup \overline{\tau}_i^h = \overline{\Omega}.$$

We denote $\{x_i\}_{i=1}^{N_h}$ by the set of vertices of a triangulation $\mathscr{T}_h$ and assume that no vertex of a triangle of $\mathscr{T}_h$ falls in the interior of an internal edge of $\mathscr{T}_h$. For any integer $m \geq 0$, and $\mu = 0$ or 1, we denote $\mathscr{P}_h^{m,\mu}$ by the set of functions in $C^\mu(\Omega)$ that are given by a polynomial of degree $\leq m$ on each of the triangles of $\mathscr{T}_h$. To take into account the homogeneous Dirichlet boundary conditions, we introduce $\mathscr{P}_{0,h}^{m,\mu}$, which denotes the subspace of $\mathscr{P}_h^{m,\mu}$ which vanishes on the boundary of Ω if $\mu = 0$, and denotes the subspace of $\mathscr{P}_{0,h}^{m,\mu}$ for which the function itself vanishes and also its normal derivative vanishes on $\partial\Omega$ if $\mu = 1$. We also introduce the notion of singular vertex. An internal vertex of $\mathscr{T}_h$ is a vertex that lies on Ω and it is called singular if the edges meeting at this vertex fall on two straight lines. Correspondingly, a vertex on $\partial\Omega$ is called a singular boundary vertex of $\mathscr{T}_h$ if all the edges of $\mathscr{T}_h$ meeting at this vertex fall on two straight lines. We are in a position to introduce additional notation. For $m \geq 0$, the space $\mathscr{P}_h^{m,-1}$ denotes functions, ϕ, which are given by a polynomial of degree $\leq m$ on each triangle with no continuity requirement such that for any singular internal vertex x_0 of $\mathscr{T}_h$,

$$\sum_{i=1}^{4} (-1)^i \phi_i(x_0) = 0,$$

where $\phi_i(x_0) = \phi|_{\tau_i}(x_0)$ and $\tau_1, \cdots, \tau_4$ are the triangles meeting at x_0, numbered consecutively. Following Scott and Vogelius, we also introduce $\widetilde{\mathscr{P}}_h^{m,-1}$ which denotes the subspace of $\mathscr{P}_h^{m,-1}$, consisting of functions, ϕ, which additionally satisfy the following two requirements, that is (1) $\int_\Omega \phi\, dx = 0$ and (2) at any singular boundary vertex of $\mathscr{T}_h$, x_0,

$$\sum_{i=1}^{k} (-1)^i \phi_i(x_0) = 0,$$

where $\phi_i(x_0) = \phi|_{\tau_i}(x_0)$, and $\tau_1, \cdots, \tau_k$, are the triangles of $\mathscr{T}_h$ meeting at x_0 with k being any number from 1 to 4, and the triangles are numbered consecutively. For simplicity, we denote $\mathbf{V}_h = \mathscr{P}_{0,h}^{m,0} \times \mathscr{P}_{0,h}^{m,0}$, $S_h = \widetilde{\mathscr{P}}_h^{m,-1}$ and $W_h = \mathscr{P}_{0,h}^{m+1,1}$. Note that a vector field $\mathbf{v}$ in $\mathbf{V}_h$ is uniquely specified by giving its value at a triangular array of $(m+1)(m+2)/2$ points in each triangle for each component v_i of $\mathbf{v}$. We note that oftentimes, we restrict our domain to a subdomain in which we define the corresponding finite element spaces. For this purpose, in case we consider the subset of $\mathscr{T}_h$, denoted by $\mathscr{T}_h'$, and set

$$\Omega' = \text{interior}\left(\bigcup_{\tau \in \mathscr{T}_h'} \overline{\tau} \cap \Omega \right).$$

For the subdomain Ω' of Ω, we define corresponding finite element spaces $\mathscr{P}_{0,h}^{m+1,1}(\Omega')$, $\mathscr{P}_{0,h}^{m,0}(\Omega')$ and $\widetilde{\mathscr{P}}_h^{m,-1}(\Omega')$. The following results are well-known:

Theorem 2 *For $m \geq 4$, the following relation holds true :*

$$W_h \xrightarrow{\mathbf{curl}} \mathbf{V}_h \xrightarrow{\text{div}} S_h \xrightarrow{0} 0,$$

i.e., the sequence is exact or the range of each of the operators in the sequence coincides with the null-space of the following operator.

We further introduce a notation that will be used to measure how close a vertex x_0 in Ω is to being singular, [13]. Let x_0 denote any non-singular vertex of Ω and let θ_i, $1 \leq i \leq k$, be the angles of the triangles τ_i with $1 \leq i \leq k$, meeting at x_0. If x_0 is an internal vertex, we define

$$R(x_0) = \max\{|\theta_i + \theta_j - \pi| : 1 \leq i, j \leq k \quad \text{and} \quad i - j = 1 \bmod k\}. \tag{18}$$

If x_0 is a boundary vertex, $R(x_0)$ is defined in the same way, only deleting the term mod k. We also set

$$R(\Omega) = \min\{R(x_0) : x_0 \text{ is a non-singular vertex of } \Omega\}. \tag{19}$$

The following theorem is the main result on the aforementioned finite element spaces $\mathbf{V}_h$ and S_h obtained in [12]:

Theorem 3 *Let $\mathscr{T}_h$, with $0 < h \leq 1$ be a quasiuniform family of triangulations of the polygonal domain Ω, and let $m \geq 4$. Assume that*

$$R(\Omega) \geq \delta > 0, \quad \delta \text{ independent of } h. \tag{20}$$

Then $\text{div}\mathbf{V}_h = S_h$ and there exists a linear operator

$$\mathscr{G} : S_h \mapsto \mathbf{V}_h \tag{21}$$

such that for all $\psi \in S_h$,

$$\text{div}\mathscr{G}(\psi) = \psi, \quad \text{and} \quad \|\mathscr{G}\psi\|_1 \lesssim m^K \|\psi\|_0, \tag{22}$$

where K is independent of h, m and ψ.

We now define $\mathbf{G}_h : S_h \mapsto \mathbf{V}_h$ as the adjoint of $-\mathrm{div}$ by for $q \in S_h$,

$$(\mathbf{G}_h q, \mathbf{v})_1 = -(q, \mathrm{div}\mathbf{v})_0, \quad \forall \mathbf{v} \in \mathbf{V}_h. \tag{23}$$

We then arrive at the following discrete Helmholtz decomposition due to the Closed Range Theory, [20]:

$$\mathbf{V}_h = \mathscr{N}(\mathrm{div}) \oplus \mathscr{R}(\mathrm{div}^*) = \mathbf{curl} W_h \oplus \mathbf{G}_h S_h, \tag{24}$$

where $\mathscr{N}(\mathrm{div})$ denotes the null space of the divergence operator and $\mathscr{R}(\mathrm{div}^*)$ is the range of the dual of the divergence operator. Note that the decomposition is orthogonal with respect to $(H^1(\Omega))^2$ (semi) inner product $(\mathbf{u},\mathbf{v})_1 = (\nabla\mathbf{u}, \nabla\mathbf{v})_0$ and also $\mathbf{H}(\mathrm{div})$ (semi)inner product, i.e, $(\mathbf{u},\mathbf{v})_{\mathrm{div}} = (\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v})_0$ for $\mathbf{u},\mathbf{v} \in \mathbf{V}_h$. Having introduced the finite element space $\mathbf{V}_h$, we formulate the discrete variational formulation of the equation (4): Find $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$\int_\Omega \nabla\mathbf{u}_h \nabla\mathbf{v}_h \, dx + \frac{1}{1-2\nu} \int_\Omega \mathrm{div}\mathbf{u}_h \, \mathrm{div}\mathbf{v}_h \, dx = \int_\Omega \mathbf{g}\mathbf{v}_h \, dx \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{25}$$

To express the error estimate and emphasize the dependence on the parameter $0 < \nu < \frac{1}{2}$, we denote by $\mathbf{u}^\nu \in (H_0^1(\Omega))^2$, $\mathbf{u}_h^\nu \in \mathbf{V}_h \subset (H_0^1(\Omega))^2$, the solutions to (4) and (25) respectively. Then, for a fixed $0 < \nu < \frac{1}{2}$, the following estimate is a direct consequence of Babuska for $\mathbf{V}_h \subset (H_0^1(\Omega))^2$. Namely,

$$\begin{aligned} &\|\mathbf{u}^\nu - \mathbf{u}_h^\nu\|_1 + \frac{1}{1-2\nu} \|\mathrm{div}(\mathbf{u}^\nu - \mathbf{u}_h^\nu)\|_0 \\ &\qquad \le C_\nu \left(\inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathbf{u}^\nu - \mathbf{v}\|_1 + \inf_{q \in \mathrm{div}\mathbf{V}_h} \left\| \frac{1}{1-2\nu} \mathrm{div}\mathbf{u}^\nu - q \right\|_0 \right), \end{aligned}$$

where C_ν is a positive constant independent of h. In [15, 16, 13], Vogelius showed that the constant C_ν can also be made independent of ν provided the family $\mathbf{V}_h$ is divergence-stable. More precisely, (1) the space $\mathrm{div}\mathbf{V}_h$ is closed in $L^2(\Omega)$ and (2)

$$\sup_{\mathbf{v}\in\mathbf{V}_h/\{0\}} \frac{(\mathrm{div}\mathbf{v}, q)}{\|\mathbf{v}\|_1} \gtrsim \|q\|_0, \quad \forall q \in \mathrm{div}\mathbf{V}_h. \tag{26}$$

The divergence-stability is well-known to be equivalent to the requirement that there exists a uniformly bounded maximal right inverse for the divergence operator on the spaces $\mathbf{V}_h$, i.e., there exists a family of linear operators $\mathscr{G}_h : \mathrm{div}\mathbf{V}_h \mapsto \mathbf{V}_h$ such that

$$\mathrm{div}(\mathscr{G}_h q) = q \quad \text{and} \quad \|\mathscr{G}_h q\|_{1,\Omega} \lesssim \|q\|_{0,\Omega}, \quad \forall q \in \mathrm{div}\mathbf{V}_h. \tag{27}$$

Consequently, the condition of the divergence-stability is crucial in obtaining the optimal convergence property of the finite element space for, in particular, the problem dependent on the parameter. We now introduce the space $\mathscr{N}_h$ defined by

$$\begin{aligned}\mathscr{N}_h &= \{\mathbf{u} \in \mathbf{V}_h : (\mathrm{div}\mathbf{u}, q) = 0, \quad \forall q \in \mathrm{div}\mathbf{V}_h\} \\ &= \{\mathbf{u} \in \mathbf{V}_h : (\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h\}\end{aligned}$$

and $\mathscr{N}_h^{\perp}$ is the $(H^1)^2$ orthogonal complement of $\mathscr{N}_h$. Namely,

$$\mathscr{N}_h^{\perp} = \{\mathbf{u} \in \mathbf{V}_h : (\nabla\mathbf{u}, \nabla\mathbf{v})_0 = 0, \quad \forall \mathbf{v} \in \mathscr{N}_h\}.$$

With this notation, it is well-known that the inf-sup condition (26) is equivalent to the fact that the operator $\mathrm{div} : \mathscr{N}_h^{\perp} \mapsto (\mathrm{div}\mathbf{V}_h)^* = \mathrm{div}\mathbf{V}_h$ is an isomorphism and

$$\|\mathrm{div}\mathbf{v}\|_0 \gtrsim \|\mathbf{v}\|_1, \quad \forall \mathbf{v} \in \mathscr{N}_h^{\perp}, \tag{28}$$

where $(\mathrm{div}\mathbf{V}_h)^*$ is the dual of the space $\mathrm{div}\mathbf{V}_h$. Interested readers can refer to [5]. We would like to note that the inequality (28) provides certain coercivity relation in the range of divergence operator and also since the norm of the maximal right inverse grows algebraically with the degree of the polynomial, the coercivity also deteriorates with the degree of the polynomial. The condition (28) will be used crucially in establishing the theory of the multiplicative Schwartz method whose convergence rate is independent of the parameter $\varepsilon = 1 - 2\nu$ and mesh size as indicated in the remark that follows the Theorem 3. However, our analysis will show that the multigrid convergence may be deteriorating at most algebraically with the degree of polynomials.

4 Construction of Multilevel Finite Element Spaces and Several Technical Lemmas

In this section, we review and prove a number of important technical lemmas that will be used in our multigrid analysis. Throughout this section, we assume that we have a nested sequence of quasi-uniform triangulations $\mathscr{T}_k = \{\tau_i^k\}$, $1 \le k \le J$ of Ω with characteristic mesh size h_k proportional to γ^{2k} with $\gamma \in (0,1)$. Let $\mathscr{T}_h = \mathscr{T}_J$ and $W_k, \mathbf{V}_k$ and S_k denote the spaces corresponding $W_h, \mathbf{V}_h$ and S_h defined on the triangulations $\mathscr{T}_k$. In what follows, to simplify notation, we omit the subscript h when referring to a fixed finest triangulation, namely, $\mathbf{V} = \mathbf{V}_h = \mathbf{V}_J$, $W = W_h = W_J$, and so forth.

The following relations are evident :

$$\mathbf{V}_1 \subset \cdots \subset \mathbf{V}_k \subset \cdots \subset \mathbf{V}_J.$$

and

$$W_1 \subset \cdots \subset W_k \subset \cdots \subset W_J.$$

The following lemma is simple but crucial in our discussion.

Lemma 1 *For $k = 1, \cdots, J$, we have*

$$\mathbf{curl} W_k = \mathscr{N}_k$$

and

$$\mathscr{N}_1 \subset \cdots \subset \mathscr{N}_k \subset \cdots \subset \mathscr{N}_J.$$

Proof It is clear that by the Theorem 2, $\mathbf{curl} W_k = \mathscr{N}_k$ for $1 \le k \le J$.

Due to the Lemma 1, for each $1 \le k \le J$, the following discrete Helmholtz decomposition holds true:

$$\mathbf{V}_k = \mathbf{curl} W_k \oplus \mathbf{G}_k S_k = \mathscr{N}_k \oplus \mathscr{N}_k^{\perp}, \tag{29}$$

where the orthogonality is with respect to $(H^1)^2$ (*semi*) inner product.

In our multigrid analysis, we make a frequent use of the L^2 projection, Q_k and H^1 projection for each $k = 1, \cdots, J$ defined by

$$(Q_k \mathbf{u}, \mathbf{v}_k) = (\mathbf{u}, \mathbf{v}_k), \quad \forall \mathbf{u} \in \mathbf{V}, \mathbf{v}_k \in \mathbf{V}_k \tag{30}$$

and

$$(P_k^W \phi, \phi_k)_1 = (\phi, \phi_k)_1, \quad \forall \phi \in W, \phi_k \in W_k. \tag{31}$$

We now present a number of technical lemmas.

4.1 Some technical lemmas

In this section, we list several technical lemmas that are crucial for the multigrid analysis. We begin by the discrete L^2 norm equivalence on the space $\mathbf{V}$.

Lemma 2 *The operator P_k^W is stable with respect to $|\cdot|_{1+\alpha}$ semi-norm for $\alpha \in [0,1]$, namely,*

$$|P_k^W \phi|_{1+\alpha} \lesssim |\phi|_{1+\alpha}, \quad \forall \phi \in W.$$

Furthermore, it satisfies the following approximation property :

$$\|(P_k^W - P_{k-1}^W)\phi\| \lesssim h_{k-1} |(P_k^W - P_{k-1}^W)\phi|_1, \quad \forall \phi \in W. \tag{32}$$

Proof From the definition of P_k^W, it is clear that

$$|P_k^W \phi|_1 \le |\phi|_1, \quad \forall \phi \in W.$$

We notice that with $\partial_i = \partial/\partial x_i$ and Q_k^W, the L^2 projection on W_k, for $\phi \in W$,

$$\begin{aligned}
|P_k^W \phi|_2 &\lesssim \sum_i |\partial_i P_k^W \phi|_1 \\
&\le \sum_i \|\partial_i P_k^W \phi - Q_k^W \partial_i \phi\|_1 + \|Q_k^W \partial_i \phi\|_1 \\
&\lesssim \sum_i \|\partial_i P_k^W \phi - Q_k^W \partial_i \phi\|_1 + \|\partial_i \phi\|_1 \\
&\lesssim \sum_i \|\partial_i P_k^W \phi - Q_k^W \partial_i \phi\|_1 + |\phi|_2.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\|\partial_i P_k^W \phi - Q_k^W \partial_i \phi\|_1 &\lesssim h_k^{-1} \|\partial_i P_k^W \phi - Q_k^W \partial_i \phi\| \\
&\lesssim h_k^{-1} \big(\|\partial_i P_k^W \phi - \partial_i Q_k^W \phi\| + \|\partial_i Q_k^W \phi - \partial_i \phi\| \\
&\quad + \|\partial_i \phi - Q_k^W \partial_i \phi\| \big) \\
&\lesssim h_k^{-1} \big(\|P_k^W (I - Q_k^W)\phi\|_1 + \|(I - Q_k^W)\phi\|_1 \\
&\quad + \|(I - Q_k^W)\partial_i \phi\| \big) \\
&\lesssim h_k^{-1} \big(\|(I - Q_k^W)\phi\|_1 + \|(I - Q_k^W)\partial_i \phi\| \big) \\
&\lesssim |\phi|_2.
\end{aligned}$$

Due to the standard arguments from the real interpolation of Sobolev spaces, we complete the stability of the H^1 projection P_k^W. To show the inequality (32), from the fact that $(I - P_{k-1}^W)(P_k^W - P_{k-1}^W) = (P_k^W - P_{k-1}^W)$, we have

$$\begin{aligned}
\|(P_k^W - P_{k-1}^W)\phi\| &= \|(I - P_{k-1}^W)(P_k^W - P_{k-1}^W)\phi\| \\
&\lesssim h_{k-1} |(P_k^W - P_{k-1}^W)\phi|_1.
\end{aligned}$$

This completes the proof.

The following lemma establishes the equivalence between L^2 norm and nodal values of the functions in $\mathbf{V}$.

Lemma 3 *For each $k = 1, \cdots, J$, and for any $v \in \mathscr{P}_h^{m,0}(\tau)$ with $\tau \in \mathscr{T}_k$ and* $\mathrm{diam}\tau = h_\tau$, *the following relation holds true :*

$$\|v\|_{0,\tau}^2 \simeq h_\tau^2 \sum_{i=1}^{(m+1)(m+2)/2} v^2(a_i), \tag{33}$$

where $\{a_i\}$ is the set of nodes that determines v on the triangle τ.

Proof The proof is based on the norm equivalence on the finite dimensional space and a standard scailing argument.

Lemma 4 *For any $q \in W_h$, the following holds true:*

$$\sum_{k=1}^{J} |(P_k^W - P_{k-1}^W)q|_2^2 \lesssim |q|_2^2. \tag{34}$$

Proof Let $\widetilde{P}_k^W = P_k^W - P_{k-1}^W$ and $\bar{q}_i = (\bar{P}_i - \bar{P}_{i-1})q$, where for each $i = 1, \cdots, J$,

$$(\bar{P}_i q, q_i)_2 = (q, q_i)_2, \quad \forall q_i \in W_i.$$

We note that the following inequalities hold true:

$$\begin{aligned} |\widetilde{P}_k^W \bar{q}_i|_2^2 &\lesssim h_k^{-2\alpha} |\widetilde{P}_k^W \bar{q}_i|_{2-\alpha}^2, \quad \text{by inverse inequality} \\ &\lesssim h_k^{-2\alpha} |\bar{q}_i|_{2-\alpha}^2, \quad \text{by the stability of } P_k^W \\ &\lesssim h_k^{-2\alpha} h_{i-1}^{2\alpha} |\bar{q}_i|_2^2 \quad \text{by the approximation property of } \bar{P}_i. \end{aligned}$$

Set $i \wedge j = \min\{i, j\}$. We then obtain that

$$\begin{aligned} \sum_{k=1}^{J} |(P_k^W - P_{k-1}^W)q|_2^2 &= \sum_{k=1}^{J} \sum_{i,j=k}^{J} (\widetilde{P}_k^W \bar{q}_i, \widetilde{P}_k^W \bar{q}_j)_2 \\ &= \sum_{i,j=1}^{J} \sum_{k=1}^{i\wedge j} (\widetilde{P}_k^W \bar{q}_i, \widetilde{P}_k^W \bar{q}_j)_2 \\ &\lesssim \sum_{i,j=1}^{J} \sum_{k=1}^{i\wedge j} h_k^{-2\alpha} h_i^{\alpha} h_j^{\alpha} |\bar{q}_i|_2 |\bar{q}_j|_2 \\ &\lesssim \sum_{i,j=1}^{J} h_{i\wedge j}^{-2\alpha} h_i^{\alpha} h_j^{\alpha} |\bar{q}_i|_2 |\bar{q}_j|_2 \\ &= \sum_{i,j=1}^{J} \gamma^{\alpha|i-j|} |\bar{q}_i|_2 |\bar{q}_j|_2 \lesssim \sum_{k=1}^{J} |\bar{q}_k|_2^2 = |q|_2^2. \end{aligned}$$

The results below are related to the strengthened Cauchy-Schwarz inequality, [17].

Lemma 5 *Assume that $1 \le i \le j \le J$. The following holds true that*

$$(\mathrm{div}\mathbf{u}_j, \mathrm{div}\mathbf{v}_i) \lesssim \gamma^{j-i} h_j^{-1} \|\mathbf{u}_j\|_0 \|\mathrm{div}\mathbf{v}_i\|_0, \quad \forall \mathbf{u}_j \in \mathbf{V}_j, \mathbf{v}_i \in \mathbf{V}_i. \tag{35}$$

Proof Let $\tau \in \mathscr{T}_i$. To show (35), it suffices to show that

$$\int_\tau \mathrm{div}\mathbf{u}_j \, \mathrm{div}\mathbf{v}_i \, d\mathbf{x} \lesssim \gamma^{j-i} h_j^{-1} \|\mathbf{u}_i\|_{0,\tau} \|\mathrm{div}\mathbf{v}_j\|_{0,\tau},$$

where τ indicates that the L^2-norms are defined with respect to the domain τ.

By Green's identity, it follows that

$$\begin{aligned}
\int_\tau \mathrm{div}\mathbf{u}_j \, \mathrm{div}\mathbf{v}_i \, d\mathbf{x} &= -\int_\tau \mathbf{u}_j \cdot \nabla(\mathrm{div}\mathbf{v}_i) \, d\mathbf{x} + \int_{\partial\tau} (\mathbf{u}_j \cdot \mathbf{n}) \mathrm{div}\mathbf{v}_i \, d\mathbf{x} \\
&\le \|\mathbf{u}_j\|_{0,\tau} \|\nabla \mathrm{div}\mathbf{v}_i\|_{0,\tau} \\
&+ \|\mathbf{u}_j\|_{0,\partial\tau} \|\mathrm{div}\mathbf{v}_i\|_{0,\partial\tau} \\
&\lesssim \|\mathbf{u}_j\|_{0,\tau} \left(h_i^{-1} \|\mathrm{div}\mathbf{v}_i\|_{0,\tau}\right) \qquad \text{by the inverse inequality} \\
&+ \left(h_j^{-1/2} \|\mathbf{u}_j\|_{0,\tau}\right) \left(h_i^{-1/2} \|\mathrm{div}\mathbf{v}_i\|_{0,\tau}\right) \qquad \text{since } h_j \le h_i \\
&\le 2(h_i h_j)^{-1/2} \|\mathbf{u}_j\|_{0,\tau} \|\mathrm{div}\mathbf{v}_i\|_{0,\tau} \qquad \text{since } h_j \le h_i \\
&\lesssim \gamma^{j-i} h_j^{-1} \|\mathbf{u}_j\|_{0,\tau} \|\mathrm{div}\mathbf{v}_i\|_{0,\tau},
\end{aligned}$$

where $\mathbf{n}$ is the unit outer normal vector of τ. This completes the proof.

We would like to remark that the identical result is also valid although div is replaced by ∇, namely, for each $1 \le i \le j \le J$, we have that for all $\mathbf{u}_j \in \mathbf{V}_j, \mathbf{v}_i \in \mathbf{V}_i$,

$$(\nabla \mathbf{u}_j, \nabla \mathbf{v}_i) \lesssim \gamma^{j-i} h_j^{-1} \|\mathbf{u}_j\|_0 \|\nabla \mathbf{v}_i\|_0. \tag{36}$$

The identical proof presented for the Lemma 5 can be found in e.g., Xu, [18].

5 Multigrid Convergence Analysis for the Planar Linear Elasticity Equation

In this section, we construct a robust multigrid method for the planar linear elasticity equation with respect to the two weights, mesh size and the number of levels. As indicated in the abstract theory, the space decomposition is crucial in achieving the parameter independent convergence property. We first introduce the space decomposition and several corresponding decomposition lemmas and then we present our main multigrid analysis.

5.1 Space Decompositions and Subspace Corrections (Smoothing)

The construction of the smoother is closely related to the space decomposition. To describe our space decompositions, for $1 \le k \le J$ fixed, we let $\{x_k^\ell\}_\ell$ be vertices of the triangulation

$\mathcal{T}_k$. We denote $\mathcal{T}_k^\ell$ by the set of triangles in $\mathcal{T}_k$ meeting at the vertex x_ℓ and set

$$\Omega_k^\ell = \bigcup_{\tau \in \mathcal{T}_k^\ell} \tau.$$

These form an overlapping covering of Ω and we build the finite element functions on Ω_k^ℓ as follows:

$$\mathbf{V}_k^\ell = \{\mathbf{v}_k \in \mathbf{V}_k : \mathrm{supp}(\mathbf{v}_k) \in \overline{\Omega}_k^\ell\} = \mathscr{P}_{0,k}^{m,0}(\Omega_k^\ell).$$

Correspondingly, we also define $W_k^\ell = \mathscr{P}_{0,k}^{m+1,1}(\Omega_k^\ell)$ and $S_k^\ell = \widetilde{\mathscr{P}}_k^{m-1,-1}(\Omega_k^\ell)$ Note that the finite overlap condition certainly holds. Having the definition of subspace decomposition, i.e.,

$$\mathbf{V} = \sum_{k=1}^J \mathbf{V}_k = \sum_{k=1}^J \sum_{\ell=1}^{N_k} \mathbf{V}_k^\ell,$$

where N_k is the number of vertices for the triangulation $\mathcal{T}_k$, we solve the local problem exactly, by which we define the block Gauss-Seidel method. Namely, for a fixed $\mathbf{V}_k^\ell \subseteq \mathbf{V}_k$, we apply the exact subspace corrections on $\mathbf{V}_k^\ell$, namely the subspace solvers are defined to be

$$a(P_k^\ell \mathbf{u}, \mathbf{v}_k^\ell) = a(\mathbf{u}, \mathbf{v}_k^\ell), \quad \forall \mathbf{u} \in \mathbf{V}, \mathbf{v}_k^\ell \in \mathbf{V}_k^\ell. \tag{37}$$

We note that Theorem 2 can be directly applied to establish that for $m \geq 4$,

$$W_k^\ell \xrightarrow{\mathbf{curl}} \mathbf{V}_k^\ell \xrightarrow{\mathrm{div}} S_k^\ell \xrightarrow{0} 0,$$

Therefore, we can deduce that the following discrete Helmholtz decomposition holds locally :

$$\mathbf{V}_k^\ell = \mathcal{N}_k^\ell \oplus (\mathcal{N}_k^\ell)^\perp, \quad \forall \ell = 1, \cdots, N_k, \tag{38}$$

where $\mathcal{N}_k^\ell = \mathbf{curl} W_k^\ell$.

We now establish various decomposition results, which will be crucially used in the multigrid analysis that follows.

Lemma 6 *For each $k = 1, \cdots, J$ and a given $\mathbf{v} \in \mathbf{V}_k$, there exists a decomposition*

$$\mathbf{v} = \sum_{\ell=1}^{N_k} \mathbf{v}_\ell$$

such that $\mathbf{v}_\ell \in \mathbf{V}_k^\ell$ for each $\ell = 1, \cdots, N_k$ and

$$\sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\mathbf{v}_j\|_{0,\Omega_k^\ell}^2 \lesssim \|\mathbf{v}\|_0^2, \tag{39}$$

where Ω_k^ℓ is the patch of the vertex x_ℓ and $N_k(\ell) = \{j = \ell, \cdots, N_k : \Omega_k^j \cap \Omega_k^\ell \neq \emptyset\}$.

Proof Note that $\mathbf{V}_k = \mathscr{P}_{0,k}^{m,0} \times \mathscr{P}_{0,k}^{m,0}$ and $\mathbf{V}_k^\ell = \mathscr{P}_{0,k}^{m,0}(\Omega_k^\ell) \times \mathscr{P}_{0,k}^{m,0}(\Omega_k^\ell)$. It suffices to prove that for $v \in \mathscr{P}_{0,k}^{m,0}$, there exists a decomposition $v = \sum_{\ell=1}^{N_k} v_\ell$ such that $v_\ell \in \mathscr{P}_{0,k}^{m,0}(\Omega_k^\ell)$ and

$$\sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|v_j\|_{0,\Omega_k^\ell}^2 \lesssim \|v\|_0^2. \tag{40}$$

For any given $\tau \in \mathscr{T}_k$, by the Lemma 3, we can define a norm of $\mathscr{P}^{m,0}(\tau)$ equivalent to $\|\cdot\|_{0,\tau}$,

$$\|w\|_{*,\tau} = h_\tau^2 \sum_{i=1}^{(m+1)(m+2)} w^2(a_i), \quad \forall w \in \mathscr{P}_h^{m,0}(\tau),$$

where $\{a_i\}$ is the set of nodes that determines w on the triangle τ.

For a given ℓ $(1 \le \ell \le N_k)$, we try to define v_ℓ for the desired decomposition. For any $\tau \in \overline{\Omega}_k^\ell$, by the finite element theory, there exist the nodal basis functions ϕ_i, $i = 1, \cdots, (m+1)(m+2)$ such that

$$v|_\tau = \sum_{i=1}^{(m+1)(m+2)} v(a_i)\phi_i.$$

Now we define a positive constant β_i with respect to the nodal basis function ϕ_i as follow: (1) if $a_i \notin \text{interior}(\Omega_k^\ell)$, $\beta_i = 0$; (2) if $a_i \in \text{interior}(\Omega_k^\ell)$, define $S_i = \{j : \text{supp}(\phi_i) \subset \overline{\Omega}_k^j\}$ and $\beta_i = 1/\#(S_i)$. It is easy to verified that $\beta_i \le 1$.

Then we can define $v_\ell \in \mathscr{P}_{0,k}^{m,0}(\Omega_k^\ell)$ such that

$$v_\ell|_\tau = \sum_{i=1}^{(m+1)(m+2)} \beta_i v(a_i)\phi_i.$$

By the definition of β_i, $v = \sum_{\ell=1}^{N_k} v_\ell$, and the inequality (40) follows by the equivalent norm $\|\cdot\|_{*,\tau}$ and the finite overlap property. Then we get the desired decomposition for $v \in \mathscr{P}_{0,k}^{m,0}$.

Now we consider the similar decomposition for the C^1 finite element space $W_k = \mathscr{P}_{0,k}^{m+1,1}$. In any given triangle τ of the triangulation $\mathscr{T}_k$, the set of the degrees of freedom consists of the nodal variables of the following four types: (1) the value, the gradient and the second-order derivatives at the vertices (a_1, a_2, a_3) of the triangle; (2) the edge-normal derivative at the distinct points $(a_4, \cdots, a_{m_1})$ in the interior of edges; (3) if $m > 4$, the value at the distinct points $(a_{m_1+1}, \cdots, a_{m_2})$ in the interior of edges; (4)if $m > 4$, the value at the distinct points $(a_{m_2+1}, \cdots, a_{m_3})$ in the interior of the triangle.

Lemma 7 *For each* $k = 1,\cdots,J$*, and for any* $v \in \mathscr{P}^{m+1,1}(\tau)$ *with* $\tau \in \mathscr{T}_k$*, the following relation holds true:*

$$\|v\|_{0,\tau}^2 \simeq \sum_{|\alpha|=0}^{2} h_\tau^{2+2|\alpha|} \sum_{i=1}^{3} |D^\alpha v(a_i)|^2 + h_\tau^4 \sum_{i=4}^{m_1} |\partial_{\mathbf{n}} v(a_i)|^2 + h_\tau^2 \sum_{i=m_1+1}^{m_3} v^2(a_i),$$

where $\partial_{\mathbf{n}}$ *denote the edge-normal derivative.*

Proof By the norm equivalent in the finite dimensional space and scaling argument, we can get that

$$\|v\|_{0,\tau}^2 \simeq \sum_{|\alpha|=0}^{2} h_\tau^{2+2|\alpha|} \sum_{i=1}^{3} |D^\alpha v(a_i)|^2 + h_\tau^4 \sum_{i=4}^{m_1} |Dv(a_i)|^2 + h_\tau^2 \sum_{i=m_1+1}^{m_3} v^2(a_i). \tag{41}$$

Since a_i $(4 \le i \le m_1)$ are points in the interior of the three edges, $|Dv(a_i)|^2 = |\partial_{\mathbf{n}} v(a_i)|^2 + |\partial_{\mathbf{t}} v(a_i)|^2$ where $\partial_{\mathbf{t}}$ denotes the tangent derivative along the edges.

Now considering $\mathscr{P}_h^{m+1,1}(\tau)$ restricted on each edge of τ, we can get that by the norm equivalent in the finite dimensional space and scaling argument,

$$\sum_{i=4}^{m_1} |\partial_{\mathbf{t}} v(a_i)|^2 \lesssim \sum_{|\alpha|=0}^{2} h_\tau^{2|\alpha|-2} \sum_{i=1}^{3} |D^\alpha v(a_i)|^2 + h_\tau^{-2} \sum_{i=m_1+1}^{m_2} v^2(a_i).$$

Combining the above inequality and (41), we get the desired equivalent norm.

Lemma 8 *For each* $k = 1,\cdots,J$*, and any* $q \in W_k$*, there exists a decomposition* $q = \sum_{j=1}^{N_k} q_j$ *such that* $q_j \in W_k^j$ *and*

$$\sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|q_j\|_{0,\Omega_k^\ell}^2 \lesssim \|q\|_0^2, \tag{42}$$

where $N_k(\ell) = \{j = \ell,\cdots,N_k : \Omega_k^j \cap \Omega_k^\ell \neq \emptyset\}$.

Proof Assume that $q \in W_k$ is given, for any j $(1 \le j \le N_k)$, we try to define q_j for the desired decomposition. For any $\tau \in \overline{\Omega}_k^j$, assume that

$$q|_\tau = v_1 + v_2 + v_3 + \sum_{i=4}^{m_1} \partial_{\mathbf{n}} q(a_i)\phi_i + \sum_{i=m_1+1}^{m_3} q(a_i)\phi_i,$$

where ϕ_i $(4 \le i \le m_3)$ are nodal basis functions; v_1, v_2 and v_3 are the related parts of q only decided by the degrees of freedom at the three vertices of the triangle τ, respectively. For simplicity, we assume v_1 is decided by the degrees of freedom at the vertex $x_k^j \in \text{interior}(\Omega_k^j)$.

Now we define a positive constant β_i with respect to the nodal basis function ϕ_i ($4 \leq i \leq m_3$) as follow: (1) if $a_i \notin \text{interior}(\Omega_k^j)$, $\beta_i = 0$; (2) if $a_i \in \text{interior}(\Omega_k^j)$, define $S_i = \{j : \text{supp}(\phi_i) \subset \overline{\Omega}_k^\ell\}$ and $\beta_i = 1/\#(S_i)$. It is easy to verified that $\beta_i \leq 1$.

Then we define $q_j \in W_k^j$ satisfying

$$q_j|_\tau = v_1 + \sum_{i=4}^{m_1} \beta_i \partial_{\mathbf{n}} q(a_i)\phi_i + \sum_{i=m_1+1}^{m_3} \beta_i q(a_i)\phi_i.$$

By the definition of β_i, we get that $q = \sum_{j=1}^{N_k} q_j$. (42) follows by the Lemma 7 and the finite overlap property.

The following lemma is crucial in estimating the estimate K.

Lemma 9 *For each $k = 1, \cdots, J$ and let $\mathbf{v}_c \in \mathscr{N}_k$ be given by $\mathbf{v}_c = \mathbf{curl} q$ with $q = (P_k^W - P_{k-1}^W)\phi$ for some $\phi \in W_k$, then there exists a decomposition*

$$\mathbf{v}_c = \sum_{\ell=1}^{N_k} \mathbf{v}_{c,\ell}$$

such that $\mathbf{v}_{c,\ell} \in \mathscr{N}_k^\ell$ for each $\ell = 1, \cdots, N_k$ and

$$\sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\mathbf{v}_{c,j}\|_{0,\Omega_k^\ell}^2 \lesssim \|\mathbf{v}_c\|_0^2, \tag{43}$$

where Ω_k^ℓ is the patch of the vertex x_ℓ and $N_k(\ell) = \{j = \ell, \cdots, N_k : \Omega_k^j \cap \Omega_k^\ell \neq \emptyset\}$.

Proof For the function $q \in W_k$ such that $\mathbf{v}_c = \mathbf{curl} q$, we apply the Lemma 8 to construct the decomposition of q as follows:

$$q = \sum_{j=1}^{N_k} q_j, \quad \text{and} \quad \sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|q_j\|_{0,\Omega_k^l}^2 \lesssim \|q\|_0^2.$$

For such a decomposition, it is clear to see that

$$\mathbf{v}_c = \mathbf{curl} q = \mathbf{curl} \sum_{j=1}^{N_k} q_j = \sum_{j=1}^{N_k} \mathbf{v}_{c,j},$$

where $\mathbf{v}_{c,j} = \mathbf{curl} q_j$. Note that by the inverse inequality, we have

$$\|\mathbf{curl} q_j\|_0 \lesssim h_k^{-1} \|q_j\|_0.$$

We now observe that due to the finite overlap property, we have

$$\begin{aligned}\sum_{\ell=1}^{N_k}\sum_{j\in N_k(\ell)}\|\mathbf{v}_{c,j}\|^2_{0,\Omega_k^\ell} &= \sum_{\ell=1}^{N_k}\sum_{j\in N_k(\ell)}\|\mathbf{curl}q_j\|^2_{0,\Omega_k^\ell}\\ &\lesssim h_k^{-2}\sum_{\ell=1}^{N_k}\sum_{j\in N_k(\ell)}\|q_j\|^2_{0,\Omega_k^\ell}\\ &\lesssim h_k^{-2}\|q\|_0^2 = h_k^{-2}\|(P_k^W-P_{k-1}^W)\phi\|\\ &\lesssim h_k^2h_{k-1}^{-2}|(P_k^W-P_{k-1}^W)\phi|_1\\ &\lesssim \|\nabla q\|_0^2 = \|\mathbf{curl}q\|_0^2\\ &= \|\mathbf{v}_c\|_0^2.\end{aligned}$$

This completes the proof.

5.2 Validation of Abstract Assumptions and Convergence Estimates

We are in a position to present our main theorem. First, we notice that by construction, we have

$$\mathscr{N} = \sum_{k=1}^{J}(\mathbf{V}_k\cap\mathscr{N}) = \sum_{k=1}^{J}\mathscr{N}_k = \sum_{k=1}^{J}\sum_{\ell=1}^{N_k}\mathscr{N}_k^\ell \tag{44}$$

and hence the assumption **A1** holds true.

Now, we consider to solve the problem (6) based on the following space decomposition:

$$\mathbf{V} = \sum_{k=1}^{J}\sum_{\ell=1}^{N_k}\mathbf{V}_k^\ell.$$

Under the settings outlined above, the norm of the product operator E_J can be written as follows:

$$E_J = (I-T_1)\cdots(I-T_J) = \Pi_{k=1}^{J}\Pi_{\ell=1}^{N_k}(I-P_k^\ell), \tag{45}$$

where P_k^ℓ is the exact solver on $\mathbf{V}_k^\ell$, (see also [17]).

The following can be easily obtained by applying the abstract results given in the previous section and the several technical lemmas in Section 4.1.

Theorem 4 *The following estimate holds true:*

$$\|\mathbf{E}_J\|^2 = 1-\frac{1}{K} < \delta < 1 \tag{46}$$

where δ is a constant independent of the mesh size h and the parameters and

$$
\begin{aligned}
K \lesssim \sup_{\mathbf{v}\in\mathscr{N}^{\perp}} \inf_{\sum_{k=1}^{J}\sum_{i=1}^{N_k}\mathbf{v}_k^i=\mathbf{v}} & \left(\frac{\sum_{k=1}^{J}\sum_{i=1}^{N_k}\left|P_{k,s}^i\sum_{(\ell,j)\geq(k,i)}\mathbf{v}_\ell^j\right|_{a_s}^2}{(\mathbf{v},\mathbf{v})_{a_s}}, \right. \\
& \left. + \frac{\sum_{k=1}^{J}\sum_{i=1}^{N_k}\|P_{k,p}^i\sum_{(\ell,j)\geq(k,i)}\mathbf{v}_\ell^j\|_{a_p}^2}{(\mathbf{v},\mathbf{v})_{a_p}} \right) \\
& + \sup_{\mathbf{v}_c\in\mathscr{N}} \inf_{\sum_{k=1}^{J}\sum_{i=1}^{N_k}\mathbf{v}_{k,c}^i=\mathbf{v}_c} \frac{\sum_{k=1}^{J}\sum_{i=1}^{N_k}\left\|P_{k,p}^i\sum_{(\ell,j)\geq(k,i)}\mathbf{v}_{\ell,c}^j\right\|_{a_p}^2}{(\mathbf{v}_c,\mathbf{v}_c)_{a_p}}.
\end{aligned}
\tag{47}
$$

Note that $\mathbf{v}_k^i\in\mathbf{V}_k^i$, and $\mathbf{v}_{k,c}^i\in\mathscr{N}_k^i$ for each k and i.

Proof The estimate (47) of K can be obtained by the direct consequence of the abstract results, the Theorem 1. Our main task is to show that $K=O(1)$. For convenience, we denote I,II and III by

$$
\begin{aligned}
\mathsf{I} &= \frac{\sum_{k=1}^{J}\sum_{i=1}^{N_k}\left|P_{k,s}^i\sum_{(\ell,j)\geq(k,i)}\mathbf{v}_\ell^j\right|_{a_s}^2}{(\mathbf{v},\mathbf{v})_{a_s}}, \\
\mathsf{II} &= \frac{\sum_{k=1}^{J}\sum_{i=1}^{N_k}\|P_{k,p}^i\sum_{(\ell,j)\geq(k,i)}\mathbf{v}_\ell^j\|_{a_p}^2}{(\mathbf{v},\mathbf{v})_{a_p}}, \\
\mathsf{III} &= \frac{\sum_{k=1}^{J}\sum_{i=1}^{N_k}\left\|P_{k,p}^i\sum_{(\ell,j)\geq(k,i)}\mathbf{v}_{\ell,c}^j\right\|_{a_p}^2}{(\mathbf{v}_c,\mathbf{v}_c)_{a_p}},
\end{aligned}
$$

respectively. We now estimate I,II and III one by one rigorously.

To estimate I and II, for any given $\mathbf{v}\in\mathscr{N}^{\perp}$, we consider the following decomposition:

$$
\mathbf{v}=\sum_{k=1}^{J}(\mathbf{v}_k-\mathbf{v}_{k-1}), \tag{48}
$$

where $\mathbf{v}_k=Q_k\mathbf{v}$ with $Q_0=0$. We now consider the decomposition of $\mathbf{v}_k-\mathbf{v}_{k-1}\in\mathbf{V}_k$, i.e.,

$$
\mathbf{v}_k-\mathbf{v}_{k-1}=\sum_{j=1}^{N_k}\mathbf{v}_k^j, \tag{49}
$$

such that $\mathbf{v}_k^j\in\mathbf{V}_k^j$ for each $j=1,\cdots,N_k$ and

$$
\sum_{\ell=1}^{N_k}\sum_{j\in N_k(\ell)}\|\mathbf{v}_k^j\|_{0,\Omega_k^\ell}^2\lesssim\|\mathbf{v}_k-\mathbf{v}_{k-1}\|_0^2, \tag{50}
$$

where Ω_k^ℓ is the star (or patch) of the vertex x_ℓ and $N_k(\ell) = \{j \in \{\ell, \cdots, N_k\} : \Omega_k^j \cap \Omega_k^\ell \neq \emptyset\}$. The existence of such decomposition is due to the Lemma 6. The following is the easy consequence:

$$\mathbf{v} = \sum_{k=1}^{J} \mathbf{v}_k - \mathbf{v}_{k-1} = \sum_{k=1}^{J} \sum_{j=1}^{N_k} \mathbf{v}_k^j, \quad \mathbf{v}_k^j \in \mathbf{V}_k^j. \tag{51}$$

We observe that the following relation holds true that

$$\begin{aligned} P_{k,s}^i \sum_{(l,j)\geq(k,i)} \mathbf{v}_l^j &= P_{k,s}^i \sum_{j=i}^{N_k} \mathbf{v}_k^j + P_{k,s}^i \left(\sum_{l=k+1}^{J} \mathbf{v}_l - \mathbf{v}_{l-1} \right) \\ &= P_{k,s}^i \sum_{j=i}^{N_k} \mathbf{v}_k^j + P_{k,s}^i P_{k,s} \left(\sum_{l=k+1}^{J} \mathbf{v}_l - \mathbf{v}_{l-1} \right). \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \sum_{i=1}^{N_k} \left| P_{k,s}^i \sum_{(l,j)\geq(k,i)} \mathbf{v}_l^j \right|_{\mathrm{div}}^2 \lesssim & \left(\sum_{i=1}^{N_k} \left| P_{k,s}^i \sum_{j=i}^{N_k} \mathbf{v}_k^j \right|_{\mathrm{div}}^2 \right. \\ & \left. + \sum_{i=1}^{N_k} \left| P_{k,s}^i P_{k,s} \left(\sum_{l=k+1}^{J} \mathbf{v}_l - \mathbf{v}_{l-1} \right) \right|_{\mathrm{div}}^2 \right). \\ = & \left(\sum_{i=1}^{N_k} \left| P_{k,s}^i \sum_{j=i}^{N_k} \mathbf{v}_k^j \right|_{\mathrm{div}}^2 \right. \\ & \left. + \sum_{i=1}^{N_k} \left| P_{k,s}^i P_{k,s} (\mathbf{v} - \mathbf{v}_k) \right|_{\mathrm{div}}^2 \right). \end{aligned} \tag{52}$$

The first term in the right hand side of the equation (52) can be estimated as follows: Thanks to Lemma 5 and (50), we have

$$\begin{aligned} \sum_{i=1}^{N_k} \left| P_{k,s}^i \sum_{j=i}^{N_k} \mathbf{v}_{k,s}^j \right|_{\mathrm{div}}^2 \lesssim \sum_{i=1}^{N_k} \left| \sum_{j\in N_k(i)} \mathbf{v}_k^j \right|_{\mathrm{div},\Omega_k^i}^2 & \lesssim h_k^{-2} \sum_{i=1}^{N_k} \sum_{j\in N_k(i)} \|\mathbf{v}_k^j\|_{0,\Omega_k^i}^2 \\ & \lesssim h_k^{-2} \|\mathbf{v}_k - \mathbf{v}_{k-1}\|_0^2 = h_k^{-2} h_{k-1}^2 |\mathbf{v}_k - \mathbf{v}_{k-1}|_1^2 \\ & \lesssim |\mathbf{v}_k - \mathbf{v}_{k-1}|_1^2 = |(Q_k - Q_{k-1})\mathbf{v}|_1^2, \end{aligned}$$

where $N_k(i) = \{j = i, \cdots, N_k :\ \Omega_k^j \cap \Omega_k^i \neq \emptyset\}$. The second term in the right hand side of the equation (52) can be similarly estimated. Namely, we observe that due to the finite overlap property,

$$\sum_{i=1}^{N_k} \left|P_{k,s}^i P_{k,s}(\mathbf{v}-\mathbf{v}_k)\right|_{\mathrm{div},\Omega}^2 \lesssim \sum_{i=1}^{N_k} \left|P_{k,s}(\mathbf{v}-\mathbf{v}_k)\right|_{\mathrm{div},\Omega_k^i}^2$$

$$\lesssim \left|P_{k,s}(\mathbf{v}-\mathbf{v}_k)\right|_{\mathrm{div}}^2.$$

We then consider the following estimate:

$$\begin{aligned}
\left|P_{k,s}(\mathbf{v}-\mathbf{v}_k)\right|_{\mathrm{div}}^2 &= (P_{k,s}(\mathbf{v}-\mathbf{v}_k), P_{k,s}(\mathbf{v}-\mathbf{v}_k))_{\mathrm{div}} \\
&= (P_{k,s}(\mathbf{v}-\mathbf{v}_k), \mathbf{v}-\mathbf{v}_k)_{\mathrm{div}} \\
&= \sum_{j=k+1}^{J} (P_{k,s}(\mathbf{v}-\mathbf{v}_k), \mathbf{v}_j-\mathbf{v}_{j-1})_{\mathrm{div}} \\
&\lesssim \sum_{j=k+1}^{J} \gamma^{j-k} h_j^{-1} |P_{k,s}(\mathbf{v}-\mathbf{v}_k)|_{\mathrm{div}} \|\mathbf{v}_j-\mathbf{v}_{j-1}\|_0 \quad \text{by Lemma 5} \\
&\lesssim \sum_{j=k+1}^{J} \gamma^{j-k} |P_{k,s}(\mathbf{v}-\mathbf{v}_k)|_{\mathrm{div}} |(Q_j-Q_{j-1})\mathbf{v}|_1.
\end{aligned}$$

Using the elementary inequality

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j} \gamma^{j-k} a_j b_k \le C_\gamma \left(\sum_{j=1}^{\infty} a_j^2\right)^{1/2} \left(\sum_{k=1}^{\infty} b_k^2\right)^{1/2}, \tag{53}$$

it follows that

$$\sum_{k=1}^{J} \left|P_{k,s}(\mathbf{v}-\mathbf{v}_k)\right|_{\mathrm{div}}^2 \lesssim \sum_{k=1}^{J} |(Q_k-Q_{k-1})\mathbf{v}|_1^2.$$

Consequently, by the well-known BPX estimate, [3] and Xu [17], we have

$$\sum_{k=1}^{J} \left(\sum_{i=1}^{N_k} |P_{k,s}^i \sum_{(l,j)\ge(k,i)} \mathbf{v}_l^j|_{\mathrm{div}}^2\right) \lesssim \sum_{k=1}^{J} |(Q_k-Q_{k-1})\mathbf{v}_k|_1^2$$

$$\lesssim \|\mathbf{v}\|_1^2 \lesssim |\mathbf{v}|_{\mathrm{div}}^2.$$

The last inequality is due to the fact that $\mathbf{v} \in \mathscr{N}^\perp$. We now focus on the second estimate that

$$\sum_{k=1}^{J}\sum_{i=1}^{N_k} \left\| P_{k,p}^i \sum_{(\ell,j)\ge(k,i)} \mathbf{v}_\ell^j \right\|_1^2 \lesssim \|\mathbf{v}\|_1^2.$$

Similarly to the previous case, the following holds true that

$$\begin{aligned}\sum_{i=1}^{N_k}\left\|P_{k,p}^i\sum_{(l,j)\geq(k,i)}\mathbf{v}_l^j\right\|_1^2 &= \sum_{i=1}^{N_k}\left\|P_{k,p}^i\sum_{j=i}^{N_k}\mathbf{v}_k^j+P_{k,p}^iP_{k,p}\left(\sum_{l=k+1}^{J}\mathbf{v}_l-\mathbf{v}_{l-1}\right)\right\|_1^2\\ &\lesssim\left(\sum_{i=1}^{N_k}\left\|P_{k,p}^i\sum_{j=i}^{N_k}\mathbf{v}_k^j\right\|_1^2\right.\\ &\quad\left.+\sum_{i=1}^{N_k}\left\|P_{k,p}^iP_{k,p}\left(\sum_{l=k+1}^{J}\mathbf{v}_l-\mathbf{v}_{l-1}\right)\right\|_1^2\right).\end{aligned}$$

We then observe that

$$\sum_{i=1}^{N_k}\left\|P_{k,p}^i\sum_{j=i}^{N_k}\mathbf{v}_k^j\right\|_1^2\lesssim\sum_{i=1}^{N_k}\left\|\sum_{j\in N_k(i)}\mathbf{v}_k^j\right\|_{1,\Omega_k^i}^2.$$

The application of the inverse inequality and the property of the decomposition based on the Lemma 6, we obtain that

$$\sum_{i=1}^{N_k}\left\|P_{k,p}^i\sum_{j=i}^{N_k}\mathbf{v}_k^j\right\|_1^2\lesssim|\mathbf{v}_k-\mathbf{v}_{k-1}|_1^2=|(Q_k-Q_{k-1})\mathbf{v}|_1^2.$$

The following inequality is due to the strengthened Cauchy-Schwarz inequality:

$$\begin{aligned}\|P_{k,p}(\mathbf{v}-\mathbf{v}_k)\|_{1,\Omega}^2 &\lesssim (P_{k,p}(\mathbf{v}-\mathbf{v}_k),P_{k,p}(\mathbf{v}-\mathbf{v}_k))_1\\ &=\sum_{j=k+1}^{J}(P_{k,p}(\mathbf{v}-\mathbf{v}_k),\mathbf{v}_j-\mathbf{v}_{j-1})_1\\ &\lesssim\sum_{j=k+1}^{J}\gamma^{j-k}h_j^{-1}\|P_{k,p}(\mathbf{v}-\mathbf{v}_k)\|_1\|\mathbf{v}_j-\mathbf{v}_{j-1}\|_0 \quad \text{by (36)}\\ &\lesssim\sum_{j=k+1}^{J}\gamma^{j-k}\|P_{k,p}(\mathbf{v}-\mathbf{v}_k)\|_1|(Q_j-Q_{j-1})\mathbf{v}|_1.\end{aligned}$$

We then conclude that

$$\sum_{k=1}^{J}\|P_{k,p}(\mathbf{v}-\mathbf{v}_k)\|_1^2\lesssim\sum_{k=1}^{J}|(Q_j-Q_{j-1})\mathbf{v}|_1^2.$$

We are then led to the desired estimate:

$$\sum_{k=1}^{J}\left(\sum_{i=1}^{N_k}\|P_{k,p}^{i}\sum_{(l,j)\geq(k,i)}\mathbf{v}_l^j\|_1^2\right)\lesssim\sum_{k=1}^{J}|(Q_k-Q_{k-1})\mathbf{v}|_1^2$$
$$\lesssim\|\mathbf{v}\|_1^2.$$

Finally, we show:

$$\sup_{\mathbf{v}_c\in\mathscr{N}}\ \inf_{\sum_{k=1}^{J}\sum_{i=1}^{N_k}\mathbf{v}_{k,c}=\mathbf{v}_c}\frac{\sum_{k=1}^{J}\sum_{i=1}^{N_k}\left\|P_{k,p}^{i}\sum_{(\ell,j)\geq(k,j)}\mathbf{v}_{\ell,c}^{j}\right\|_{a_p}^2}{(\mathbf{v}_c,\mathbf{v}_c)_{a_p}}\lesssim O(1). \tag{54}$$

For any $\mathbf{v}_c\in\mathscr{N}$, let $q\in W$ be such that $\mathbf{v}_c=\mathbf{curl}q$. We then consider the following decomposition that

$$\mathbf{v}_c=\sum_{k=1}^{J}(\mathbf{v}_{k,c}-\mathbf{v}_{k-1,c}),\qquad\text{with }\mathbf{v}_{k,c}=\mathbf{curl}P_k^W q.$$

First of all, due to the fact that $\mathbf{v}_{k,c}-\mathbf{v}_{k-1,c}=\mathbf{curl}(P_k^W-P_{k-1}^W)q$ and the Lemma 9, we can consider the following further decomposition:

$$\mathbf{v}_{k,c}-\mathbf{v}_{k-1,c}=\sum_{i=1}^{N_k}\mathbf{v}_{k,c}^{\ell}, \tag{55}$$

such that $\mathbf{v}_{k,c}^{\ell}\in\mathscr{N}_k^{\ell}$ for each $\ell=1,\cdots,N_k$ and the following estimate holds true that

$$\sum_{i=1}^{N_k}\sum_{j\in N_k(i)}\|\mathbf{v}_{k,c}^{j}\|_{0,\Omega_k^i}^2\lesssim\|\mathbf{v}_{k,c}-\mathbf{v}_{k-1,c}\|_0^2, \tag{56}$$

The following identity is straightforward to obtain :

$$P_{k,p}^{i}\sum_{(l,j)\geq(k,i)}\mathbf{v}_{l,c}^{j}=P_{k,p}^{i}\sum_{j=i}^{N_k}\mathbf{v}_{k,c}^{j}+P_{k,p}^{i}\left(\sum_{l=k+1}^{J}\mathbf{v}_{l,c}-\mathbf{v}_{l-1,c}\right)$$
$$=P_{k,p}^{i}\sum_{j=i}^{N_k}\mathbf{v}_{k,c}^{j}+P_{k,p}^{i}P_{k,p}\left(\sum_{l=k+1}^{J}\mathbf{v}_{l,c}-\mathbf{v}_{l-1,c}\right).$$

As before, we have

$$\sum_{i=1}^{N_k}\left\|P_{k,p}^{i}\sum_{(l,j)\geq(k,i)}\mathbf{v}_{l,c}^{j}\right\|_1^2\lesssim\left(\sum_{i=1}^{N_k}\left\|P_{k,p}^{i}\sum_{j=i}^{N_k}\mathbf{v}_{k,c}^{j}\right\|_1^2\right. \tag{57}$$
$$\left.+\sum_{i=1}^{N_k}\left\|P_{k,p}^{i}P_{k,p}\left(\mathbf{v}-\mathbf{v}_{k,c}\right)\right\|_1^2\right).$$

The first term in the right hand side of the equation (57) can be estimated as follows: Thanks to (56), we have

$$\begin{aligned}\sum_{i=1}^{N_k}\left\|P^i_{k,p}\sum_{j=i}^{N_k}\mathbf{v}^j_{k,c}\right\|_1^2 &\lesssim \sum_{i=1}^{N_k}\left\|\sum_{j\in N_k(i)}\mathbf{v}^j_{k,c}\right\|^2_{1,\Omega^i_k}\\ &\lesssim h_k^{-2}\sum_{i=1}^{N_k}\sum_{j\in N_k(i)}\|\mathbf{v}^j_{k,c}\|^2_{0,\Omega^i_k}\lesssim h_k^{-2}\left\|\mathbf{v}_{k,c}-\mathbf{v}_{k-1,c}\right\|_0^2.\end{aligned}$$

The second term in the right hand side of the equation (57) can be similarly estimated. Namely, we observe that due to the finite overlap property,

$$\sum_{i=1}^{N_k}\left\|P^i_{k,p}P_{k,p}(\mathbf{v}_c-\mathbf{v}_{k,c})\right\|^2_{1,\Omega}\lesssim\left\|P_{k,p}(\mathbf{v}_c-\mathbf{v}_{k,c})\right\|_1^2.$$

We then consider the following estimate:

$$\begin{aligned}\left\|P_{k,p}(\mathbf{v}_c-\mathbf{v}_{k,c})\right\|_1^2 &= (P_{k,p}(\mathbf{v}_c-\mathbf{v}_{k,c}),P_{k,p}(\mathbf{v}_c-\mathbf{v}_{k,c}))_1\\ &= \sum_{j=k+1}^{J}(P_{k,p}(\mathbf{v}_c-\mathbf{v}_{k,c}),(\mathbf{v}_{j,c}-\mathbf{v}_{j-1,c}))_1\\ &\lesssim \sum_{j=k+1}^{J}\gamma^{j-k}h_j^{-1}\|P_{k,p}(\mathbf{v}_c-\mathbf{v}_{k,c})\|_1\|\mathbf{v}_{j,c}-\mathbf{v}_{j-1,c}\|_0 \quad \text{by (36).}\end{aligned}$$

Therefore, we obtain that

$$\sum_{k=1}^{J}\left\|P_{k,p}(\mathbf{v}_c-\mathbf{v}_{k,c})\right\|_1^2\lesssim\sum_{k=1}^{J}|\mathbf{v}_{k,c}-\mathbf{v}_{k-1,c}|_1^2.$$

Then the proof is completed by the Lemma 4.

6 Conclusion

In this paper, we review the Scott-Vogelius higher order finite element discretization and apply the finite element to discretize the linear elasticity equations. We construct a robust multigrid method with vertex-based Schwarz method as the smoother, for the solution of the discrete system of equations. By the properties of the Scott-Vogelius elements, our methods transparently analyze the multigrid method for the parameter dependent problems like the linear elasticity. The techniques can also be applied other more general equations that include $\mathbf{H}(\mathbf{curl})$ and $\mathbf{H}(\mathrm{div})$ [1] and other relevant problems.

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