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ON THE CONVERGENCE OF ITERATIVE METHODS FOR SEMIDEFINITE LINEAR SYSTEMS[*]

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Abstract. Necessary and sufficient conditions for the energy norm convergence of the classical iterative methods for semidefinite linear systems are obtained in this paper. These new conditions generalize the classic notion of the P-regularity introduced by Keller [*J. Soc. Indust. Appl. Math. Ser. B Numer. Anal.*, 2 (1965), pp. 281–290].

Key words. iterative methods, matrix splittings, P-regularity, weak regularity, energy norm convergence

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1. Introduction.

Semidefinite linear systems arise in many applications, as a result of discretization of semidefinite partial differential equations or as an important part of a complex numerical model. Examples include the equations coming from discretizing the Poisson equation with Neumann boundary conditions (see Bochev and Lehoucq [3]), linear elasticity equations with traction free boundary conditions, systems arising in Markov processes [7], and also graph partitioning applications [8, 9]. Some more subtle examples are provided by the linear systems obtained from the generalized finite element method discretizations of partial differential equations (see [23, 21, 22] and [26]). For such problems, using iterative solvers is unavoidable, because direct methods such as Gaussian elimination or the Cholesky decomposition are hard to apply in a straightforward way (since the problem is singular).

We consider the problem of finding a solution $x \in \mathbb{R}^n$ to

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a given symmetric and semidefinite matrix and $b \in \mathbb{R}^n$ is a given vector in the range of A. A stationary linear iterative method to solve (1.1) can be obtained using a splitting of the matrix $A = M - N$,

$$Mx^{\ell} = Nx^{\ell-1} + b. \tag{1.2}$$

Convergence properties of (1.2) for semidefinite problems have been studied by many authors. Classic as well as more recent results on this topic can be found in [1, 15, 4, 6, 19] (and also many references listed therein). We note that all of these convergence results require that the *iterator* M be an invertible matrix (see, e.g., [6]).

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It is easy to see, however, that the iterations (1.2) are well defined even for a singular iterator M as long as the right-hand side falls in the range of M. Such a situation occurs, for example, in multigrid methods (when the coarsest grid problem is singular; see [13]) and balancing domain decomposition methods (nonoverlapping or overlapping; see [14, 12]). In this paper we will study the convergence of (1.2) without assuming that M is invertible. Our main result is given in Theorem 4.4 below. The conditions under which this convergence result holds have been applied to studying more general subspace correction algorithms for variational problems (multigrid and domain decomposition methods being classical examples for such methods). For nonsingular symmetric positive definite problems we refer to [25] and [27] for general theory of subspace correction methods, and for the relations between the conditions given here and the convergence of subspace correction methods for variational semidefinite problems in Hilbert spaces we refer to our recent work [13]. In any event, deriving necessary and sufficient conditions for energy norm convergence without assuming that M is invertible is of theoretical interest in its own right. We further point out that our result here is also new even when M is nonsingular. Related convergence results that use an algebraic framework for Schwarz methods (multiplicative and additive) can be found in Nepomnyaschikh [18], Chang and Sun [5], Marek and Szyld [16], and Nabben and Szyld [17].

The structure of the paper is as follows. In section 2, we give definitions and relevant notation related to matrix splittings. In section 3, we introduce the P-regularity and weak regularity of splittings and their relations to the energy norm convergence. We also provide a simple example of a matrix that is not P-regular but results in a convergent iteration (as defined in (2.3) below). In section 4, we give more refined necessary and sufficient conditions and prove energy norm convergence.

2. Notation and preliminaries.

We first introduce standard notation. For a finite dimensional space V with an inner product $(\cdot,\cdot)$ and any subspace $W \subset V$, $W^{\perp}$ denotes the orthogonal complement of W with respect to the inner product $(\cdot,\cdot)$, and V/W denotes the quotient space; for a given matrix T, the range of T and the null space of T are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. Further, following [25], we write $x_1 \lesssim y_1$ ($x_2 \gtrsim y_2$) whenever there exist generic constants c_1 and c_2 such that $x_1 \le c_1 y_1$ ($x_2 \ge c_2 y_2$).

We consider again the semidefinite system (1.1). Given an initial guess x^0, we rewrite the iteration (1.2) for the solution of (1.1) as follows:

$$M(x^{\ell} - x^{\ell-1}) = b - Ax^{\ell-1}, \quad \ell = 1, 2, \dots. \tag{2.1}$$

Clearly, if $\mathcal{N}(A) \neq \{0\}$, the solution of (1.1) is not unique (but unique in the quotient space $\mathbb{R}^n/\mathcal{N}(A)$). Similarly if $\mathcal{N}(M) \neq \{0\}$, the solution of (2.1) is not unique either. A special solution of (2.1) can be given by

$$x^{\ell} = x^{\ell-1} + M^{\dagger}(b - Ax^{\ell-1}), \tag{2.2}$$

where $M^{\dagger} \equiv (M^T M)^{-1} M^T$ is the Moore–Penrose generalized inverse of M. It is obvious that for an invertible M, $M^{\dagger} = M^{-1}$. The Moore–Penrose inverse for the splitting matrix M has been used, for example, by Joshi [10] in an attempt to generalize the result of Keller [11] for rectangular matrices A. However, since M is assumed to be of full rank in [10] when A is a square matrix, $M^{\dagger}$ is reduced to the usual inverse of M.

Probably the best known and quoted results in the theory of the convergence of iterative methods for singular linear systems are contained in work by Keller [11].

For an invertible iterator M, Keller defines a convergent linear iterative method as a method for which the error propagation matrix T satisfies

$$\lim_{k\to\infty} T^k = P_{\mathcal{N}}, \qquad \text{with} \qquad T := I - M^{-1}A = M^{-1}N, \tag{2.3}$$

where $P_{\mathcal{N}}$ is a projection (not necessarily orthogonal) onto the null space $\mathcal{N}(A)$. Other forms of abstract necessary and sufficient conditions for energy norm convergence are given in [1]. More refined analysis and also relations between various types of such conditions are identified and studied in [24]. In many instances, however, it is difficult to verify a relation such as (2.3). A more practical (but only sufficient) condition, known as the P-regularity condition, introduced in the aforementioned work by Keller, has been used as a criterion in many of the previous studies on the convergence of iterative methods [1, 15, 4] and [6].

3. On P-regular and weak regular splittings. We now recall the convergence concept for the iteration (2.2) given by Keller [11, Theorem 1, p. 282] (again we assume that A is semidefinite).

DEFINITION 3.1. *We say that the scheme* (2.2) *is convergent if, for any initial guess* $x^0 \in R^n$, *we have*

$$\lim_{\ell\to\infty} Ax^{\ell} = b. \tag{3.1}$$

Let x be a solution to (1.1); then it is easy to see that (3.1) is equivalent to

$$\lim_{\ell\to\infty} \|x - x^{\ell}\|_{\mathbb{R}^n/\mathcal{N}(A)} = 0$$

or

$$\lim_{\ell\to\infty} |x - x^{\ell}|_A = 0,$$

where $|x|_A = (Ax, x)^{1/2}$; see, e.g., [1, 11, 15] and references cited therein.

An obvious sufficient condition for the convergence (3.1) is

$$|I - M^{\dagger}A|_A < 1. \tag{3.2}$$

When this condition is satisfied, we will say that (2.2) is *energy norm convergent* or *seminorm convergent*.

One main convergence result for (2.2), derived by Keller [11], can be summarized in the following theorem.

THEOREM 3.2 (see Keller [11]). *The iterative scheme* (2.2) *is energy norm convergent if the splitting* $A = M - N$ *is P-regular, in the sense that*

(K1) M *is invertible, and*

(K2) $M^T + M - A$ *is positive definite.*

For some special singular systems, considered by, e.g., Marek and Szyld [16], the convergence has been studied via the theory of nonnegative matrices, for which the *weak-regularity* condition, proposed in Ortega and Rheinboldt [20], is often used as a sufficient condition. A version of the weak-regularity condition (see, e.g., Berman and Plemmons [1]) is as follows: A splitting $A = M - N$ is called weakly regular if M is invertible and, in addition, both M^{-1} and $M^{-1}N$ are nonnegative matrices. We shall now provide an example to show that neither P-regularity nor weak regularity of the matrix splitting is necessary for the convergence of (2.1).

Example 1. Consider the symmetric positive semidefinite matrix A given below:

$$A = \begin{pmatrix} 1/2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We introduce a splitting $A = M - N$, where

$$M = \begin{pmatrix} -1 & -4 \\ 0 & 4 \end{pmatrix} \qquad \text{and} \qquad M^{-1} = \begin{pmatrix} -1 & -1 \\ 0 & 1/4 \end{pmatrix}.$$

Then

$$\overline{M} = M + M^T - A = \begin{pmatrix} -5/2 & -3 \\ -3 & 6 \end{pmatrix}.$$

This splitting obviously is not P-regular, since $\overline{M}$ is apparently not positive definite. Moreover, the splitting above is also not weakly regular, since M^{-1} has two negative elements; that is, M^{-1} is not nonnegative.

However, it is also obvious that $E = I - M^{-1}A$ is convergent. More precisely,

$$E^\ell = E \quad \forall\, \ell \geq 1 \quad \text{ and } \quad E = P_{\mathcal{N}(A)},$$

where $P_{\mathcal{N}(A)}$ is a projection onto $\mathcal{N}(A)$. Note that the projection $P_{\mathcal{N}(A)}$ is not orthogonal.

We would like to remark that in case $\mathcal{R}(M^\dagger A) \subset \mathcal{R}(A)$, a convergence result can easily be derived in a fashion similar to the positive definite case, since A becomes symmetric and positive definite on $\mathcal{R}(A)$ (see, e.g., [25]). An example for which such a relation holds is the Richardson iteration. However, this is not the case for other classical iterations, such as the Gauss–Seidel or Jacobi iterations and the multigrid and domain decomposition methods.

4. New conditions and analysis.

In this section, we present new conditions that are necessary and sufficient for the energy norm convergence of the iteration (2.2). They are given as follows:

(A1) $\mathcal{R}(A) \subset \mathcal{R}(M)$, or equivalently, $\mathcal{N}(M^T) \subset \mathcal{N}(A)$.

(A2) $M^T + M - A$ is symmetric positive definite on $\mathcal{R}(M^\dagger A)$.

Note that when M is nonsingular, (A1) always holds. If the decomposition $A = M - N$ satisfies the assumption (A1) and additionally $\mathcal{N}(M) \subset \mathcal{N}(A)$, then the splitting is called a *subproper splitting* (see, e.g., Berman and Neumann [2]).

The assumption (A1) is obviously necessary for (2.2) to be well-defined for any initial guess x^0, since $b - Ax^{\ell-1} \in \mathcal{R}(A)$. We note that both (A1) and (A2) are clearly weaker than (K1) and (K2) in Theorem 3.2. The identity (4.1), proven below, obviously holds for M being square and nonsingular. We also note that the relation given by (4.1) can also be found as an assumption on the iterator M in Joshi [10] for the study of iterative methods for problems with rectangular A.

LEMMA 4.1. *Assume that* (A1) *holds. Then*

$$MM^\dagger A = A, \tag{4.1}$$

and we also have the following inequality:

$$\|M^\dagger Ax\| \gtrsim |x|_A \quad \forall x \in \mathbb{R}^n. \tag{4.2}$$

Moreover, for any $x^{\ell-1}$*, the iterate* (2.1) *has a unique solution* $x^\ell \in \mathbb{R}^n/\mathcal{N}(M)$*, which is given by* (2.2).

Proof. By the definition of the Moore–Penrose inverse, it is obvious that $MM^\dagger M = M$. This then implies

$$MM^\dagger y = y \quad \forall y \in \mathcal{R}(M).$$

Now, from the assumption (A1), we obtain that

$$MM^\dagger y = y \quad \forall y \in \mathcal{R}(A) \subset \mathcal{R}(M).$$

This proves that $MM^\dagger A = A$, and hence (4.1) holds.

To prove (4.2) we observe that

$$\begin{aligned} |x|_A^2 = (Ax, x) &\leq \|Ax\| \|x\|_{\mathbb{R}^n/\mathcal{N}(A)} \\ &\lesssim \|MM^\dagger Ax\| |x|_A \quad \text{ by (4.1)} \\ &\lesssim \|M^\dagger Ax\| |x|_A, \quad \text{ since } M \text{ is bounded.} \end{aligned}$$

This proves the inequality (4.2).

To complete the proof of the lemma, we first use the fact that (A1) implies that (2.1) is solvable; i.e., for any $x^{\ell-1}$ there exists x^ℓ such that $M(x^\ell - x^{\ell-1}) = b - Ax^{\ell-1}$. However, since $\mathcal{N}(M)$ is not empty, the x^ℓ is determined uniquely only in the space $\mathbb{R}^n/\mathcal{N}(M)$ for any $x^{\ell-1}$. Furthermore, the iterate x^ℓ that is obtained from (2.2) can be a solution to (2.1) for any given $x^{\ell-1}$. We observe that since $x^\ell - x^{\ell-1} = M^\dagger(b - Ax^{\ell-1})$,

$$M(x^\ell - x^{\ell-1}) = MM^\dagger A(x - x^{\ell-1}) = A(x - x^{\ell-1}), \quad \text{ by (4.1).}$$

This completes the proof. ☐

The next lemma is related to assumption (A2) and has a well-known analogue when A is symmetric and positive definite. It implies that the P-regularity is necessary and sufficient for the energy norm convergence when A is nonsingular (see, e.g., Young [28]). For semidefinite A the result is as follows.

LEMMA 4.2. *Under the assumption* (A1)*, the following identity holds for all* $x \in \mathbb{R}^n$*:*

$$|x|_A^2 - |(I - M^\dagger A)x|_A^2 = ((M^T + M - A)M^\dagger Ax, M^\dagger Ax). \tag{4.3}$$

Proof. A direct calculation shows that for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned} |x|_A^2 - |(I - M^\dagger A)x|_A^2 &= (Ax, M^\dagger Ax + (M^\dagger)^T Ax - (M^\dagger)^T AM^\dagger Ax) \\ &= (Ax, (M^\dagger + (M^\dagger)^T - (M^\dagger)^T AM^\dagger)Ax). \end{aligned} \tag{4.4}$$

The desired result then follows by transposition of both sides of (4.1), which is $A(M^\dagger)^T M^T = A$. This completes the proof. ☐

From the identity (4.4), we conclude that the assumption (A2) is sufficient for the energy norm convergence. To prove that it is also necessary, we now introduce a pair of conditions (A2a) and (A2b). Together they are equivalent to (A2), as seen in the next lemma.

LEMMA 4.3. *If the assumption* (A1) *holds, then the* (A2) *is equivalent to the following two conditions:*

(A2a) $\exists \omega \in (0,2)$ *such that* $(M^\dagger Ax, M^\dagger Ax)_A \le \omega(M^\dagger Ax, Ax)$ $\forall x \in \mathbb{R}^n$.

(A2b) $\exists \alpha > 0$ *such that* $(M^\dagger Ax, M^\dagger Ax)_A \ge \alpha(M^\dagger Ax, M^\dagger Ax)$ $\forall x \in \mathbb{R}^n$.

Proof. We first show that (A2a) and (A2b) imply (A2). We begin by rewriting (4.4) in the following form:

$$\begin{aligned} |x|_A^2 - |(I - M^\dagger A)x|_A^2 &= \big((M^T + M - A)M^\dagger Ax, M^\dagger Ax\big) \\ &= 2(Ax, M^\dagger Ax) - (M^\dagger Ax, M^\dagger Ax)_A. \end{aligned} \tag{4.5}$$

We then conclude that (A2) is equivalent to the existence of a constant $\delta > 0$ such that

$$2(y, M^\dagger y) \ge \delta(M^\dagger y, M^\dagger y) + (M^\dagger y, M^\dagger y)_A \quad \forall y \in \mathcal{R}(A). \tag{4.6}$$

From (A2a) and (A2b) we have that

$$\begin{aligned} 2(y, M^\dagger y) &\ge \frac{2}{\omega}(M^\dagger y, M^\dagger y)_A \\ &= (M^\dagger y, M^\dagger y)_A + \left(\frac{2}{\omega} - 1\right)(M^\dagger y, M^\dagger y)_A \\ &\ge (M^\dagger y, M^\dagger y)_A + \left(\frac{2}{\omega} - 1\right)\alpha(M^\dagger y, M^\dagger y). \end{aligned}$$

This means that (4.6) holds with $\delta = (2/\omega - 1)\alpha$, which proves (A2).

To prove the reverse implication, we assume that (A2) is satisfied. Then (A2a) can be obtained from (4.6),

$$2(y, M^\dagger y) \ge \left(\frac{\delta}{\|A\|} + 1\right)(M^\dagger y, M^\dagger y)_A \quad \forall y \in \mathcal{R}(A),$$

and (A2b) can be obtained by using (4.5), the Cauchy–Schwarz inequality, and (4.2), as follows:

$$\|M^\dagger Ax\|^2 \lesssim 2(Ax, M^\dagger Ax) \le 2|x|_A |M^\dagger Ax|_A \lesssim \|M^\dagger Ax\| |M^\dagger Ax|_A.$$

This completes the proof. ∎

We would also like to remark that (A2b) is equivalent to the following relation:

$$\mathcal{R}(M^\dagger A) \cap \mathcal{N}(A) = \{0\}. \tag{4.7}$$

When M is invertible, (4.7) has been considered as a part of necessary and sufficient condition for convergence in a work by Szyld [24].

The following theorem is the main result in this paper.

THEOREM 4.4. *The iterative scheme* (2.2) *is energy norm convergent if and only if both* (A1) *and* (A2) *(or equivalently* (A1), (A2a), *and* (A2b)*) are satisfied.*

Proof. The identity (4.4), the assumption (A2), and (4.2) in Lemma 4.2 imply that

$$|x|_A^2 - |(I - M^\dagger A)x|_A^2 \gtrsim \|M^\dagger Ax\|^2 \gtrsim |x|_A^2.$$

This shows that $I - M^\dagger A$ is a contraction in the $|\cdot|_A$ seminorm; i.e.,

$$|(I - M^\dagger A)x|_A \le \delta|x|_A \quad \text{for some } \delta \in [0, 1).$$

We shall now prove that (A2) is also a necessary condition. Let us assume that (A2) does not hold. Then there exists $x \in \mathbb{R}^n$ such that

$$|x|_A^2 - |(I - M^\dagger A)x|_A^2 \leq 0. \tag{4.8}$$

This contradicts the fact that $|I - M^\dagger A|_A < 1$. This completes the proof. □

When A is symmetric and positive definite, the assumption (A2a) is the well-known necessary and sufficient condition for the energy norm convergence of the iterative method (2.2). However, as we shall see, for A being only positive semidefinite, (A2a) alone is not sufficient for convergence. For example, whenever $\mathcal{R}(M^\dagger A) = \mathcal{N}(A)$, (A2a) holds true, but it is not difficult to see that in general there will not be energy norm convergence unless (A2b) is added to the set of assumptions.

Example 2: (A2b) *is necessary for the convergence.* Let

$$A = \begin{pmatrix} 1/2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{4.9}$$

We introduce a splitting $A = M - N$, where M is given by

$$M^{-1} = \begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix}.$$

A simple algebraic manipulation yields

$$\mathcal{R}(M^{-1}A) = \mathcal{N}(A).$$

It is then straightforward to see that

$$|E|_A^2 = |I - M^{-1}A|_A^2 = 1.$$

This means that the iteration is not convergent.

As we have seen before, the splitting given in Example 1 above is neither P-regular nor weak-regular. We now revisit the example to show that it in fact satisfies (A2a) and (A2b).

Example 1 *revisited.* Consider the splitting given in Example 1. We have that

$$\mathcal{N}(A)^\perp = \text{span}\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\}, \qquad \mathcal{N}(A) = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\},$$

and

$$\left(\bar{M}\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = \frac{163}{2}.$$

This implies that the splitting $A = M - N$ satisfies (A2a). We note that since $\mathcal{R}(M^{-1}A) \cap \mathcal{N}(A) = \{0\}$, (A2b) holds true.

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