

Global Existence, Uniqueness and Optimal Solvers of Discretized Viscoelastic Flow Models

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This paper is devoted to studying the well-posedness and optimal solution techniques for a full discretization scheme, proposed by Lee and Xu (2006), for a large class of viscoelastic flow models. By using some special properties of the scheme such as the energy-norm estimates and positivity preserving for conformation tensor, the global existence and uniqueness is established for the solution of the discrete scheme. Furthermore it is proved that the solution of the discrete scheme at each time step can be obtained by an iterative procedure that only requires $\mathcal{O}(N \log N)$ operations (with N being the number of nodes of the underlying finite element grid).

Keywords: Non-Newtonian models, viscoelastic models, constitutive equations, positivity-preserving schemes, Riccati differential equations, Navier–Stokes equations, Stokes equations

1. Introduction

A complex fluid is described as “a fluid made up of a lot of different kinds of stuff” as described by Gelbart and Ben-Shaul¹⁵. Typically, computational rheology refers to the design, analysis, and implementation of numerical methods to study such complex fluids. Many complex fluids of interest, such as mucus, liquid soap, pudding, toothpaste, or clay, are viscoelastic; they exhibit a combination of both fluid

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and solid characteristics. Many fundamental theoretical questions on mathematical models of complex fluids, such as whether (weak) solutions exist, are still open. The global existence of weak solutions was established by Barrett and Süli³ for the corotational models, under the assumption that the velocity field is regularized. For the non-corotational models, both the velocity and the extra stress fields are assumed to be mollified in the weak formulations in order to obtain the global existence of weak solutions³.

The Oldroyd-B model²⁸ is one of the simplest non-corotational constitutive models that can describe the viscoelastic behavior of dilute polymeric solutions. The existence of global weak solutions of the Oldroyd-B model has not been fully understood yet. Lin, Liu, and Zhang²⁶ established the global existence of classical solutions for the Oldroyd-B model at infinite Weissenberg number with small initial data. A stability result has also been obtained by He and Zhang¹⁷: if the initial data is sufficiently close to the equilibrium, the solution approaches to the equilibrium with a certain decaying rate measured in the L^2 -norm. The main idea behind the existence and stability proof is to take full advantage of the divergence-free condition imposed at the velocity field, which is shown to generate a dissipation mechanism and hence stabilize the equation^{26,17}. This work strongly indicates that incompressibility plays an important role in the stability of the system. It hints at the importance of preserving incompressibility on the discrete level in order to obtain stable numerical schemes.

While global-in-time existence remains illusive for continuous problems, we will establish the global existence of the discrete macroscopic viscoelastic models in this paper. Similar to the theories on the continuous level, the strong divergence-free condition and the positivity of the conformation tensors play critical roles in our analysis. We study the positivity-preserving discretization schemes proposed by Lee and Xu²³ that have been shown to obey the energy norm stability. We establish that the fully discrete systems admit a unique global solution. In turn, we show computational optimality with respect to both relevant physical parameters and spatial/temporal mesh sizes of the proposed numerical algorithm for solving the fully discrete systems. We would like to note that there are few results on the existence of fully discrete solution to the Navier-Stokes equations in the literature^{2,20}.

Throughout this paper, we use the standard notation for Sobolev spaces: $H^k(\Omega)$ denotes the classical Sobolev space of scalar functions on a bounded domain $\Omega \subset \mathbb{R}^d$ whose derivatives up to order k are square integrable, with the full norm $\|\cdot\|_k$ and the corresponding semi-norm $|\cdot|_k$. The symbol $H_0^1(\Omega)$ denotes the subspace of $H^1(\Omega)$ whose trace vanishes on the boundary $\partial\Omega$. We will also discuss the corresponding spaces restricted to the subdomain of Ω . The usual L^∞ -norm and L^2 -norm will be denoted by $\|\cdot\|_\infty$ and $\|\cdot\|_0$, respectively. The symbol $L_0^2(\Omega)$ denotes a subspace of $L^2(\Omega)$ consisting of functions that have a zero average. (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ denote the classical L^2 -inner product and the dual pairing, respectively. The space $L^p(0, T; H^1(\Omega))$ for $1 \leq p < \infty$ is the Hilbert space consisting of functions $f(x, t) : \Omega \times [0, T] \mapsto \mathbb{R}$ such that $(\int_0^T \|f(\cdot, \nu)\|_1^p d\nu)^{1/p} < \infty$. The symbol \mathbb{M} de-

notes the space of matrix-valued functions whose ranges are in $\mathbb{R}^{d \times d}$, and \mathbb{S} denotes the subspace of \mathbb{M} consisting of the symmetric matrices. In addition, \mathbb{S}^+ denotes the subset of \mathbb{S} consisting of positive-definite matrices. Finally, following Xu³⁶, the symbol $A \lesssim B$ means $A \leq CB$ with a constant C independent of space mesh size h and time step k , and $A \approx B$ is an abbreviation of $A \lesssim B \lesssim A$.

The rest of the paper is organized as follows. In §2, we review the basic properties of the flow maps, the generalized Lie derivatives, and the connection between the algebraic Riccati differential equations and the macroscopic constitutive relations for viscoelastic fluids. In §3, we discuss the numerical discretization schemes that preserve the positive-definiteness of the conformation tensor. Finally, in §4, we show the energy estimate, the long-term stability, and the well-posedness of the discrete solution.

2. Constitutive laws in the Riccati form

Macroscopic models for viscoelastic fluids are usually given by three fundamental equations: the momentum balance equation, the continuity equation, and a constitutive equation. The dimensionless form of the momentum balance and continuity equations in these models can be written as follows:

$$\text{Re} \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \mathbf{T} \quad (2.1)$$

and

$$\nabla \cdot \mathbf{u} = 0, \quad (2.2)$$

where \mathbf{u} is the velocity of the fluids, p is the pressure, and \mathbf{T} is the extra stress tensor that can be decomposed into two parts¹⁶ in dilute polymeric fluids as

$$\mathbf{T} = 2\eta_s \mathcal{D}(\mathbf{u}) + \boldsymbol{\tau}. \quad (2.3)$$

Here η_s is the Newtonian viscosity and $\mathcal{D}(\mathbf{u})$ is the symmetric gradient of velocity,

$$\mathcal{D}(\mathbf{u}) = \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2}. \quad (2.4)$$

We remark that $2\eta_s \mathcal{D}(\mathbf{u})$ is the solvent contribution of the stress and the tensor $\boldsymbol{\tau}$ is the polymeric contribution of the stress, which is due to the high-molecular-weight viscoelastic macromolecules. We need to give a constitutive law to relate $\boldsymbol{\tau}$ and the rate of strain and we will consider a class of general constitutive laws in terms of generalized Lie derivatives.

2.1. Generalized Lie derivatives

Consider a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) and a velocity field of flow $\mathbf{u} = (u_i) \in \mathbb{R}^d$. The motion of a particle can be described by the flow map $\phi_{t,s} : \Omega \mapsto \Omega$, such that

$$\frac{\partial}{\partial s} \phi_{t,s}(x) = \mathbf{u}(\phi_{t,s}(x), s), \quad \phi_{t,t}(x) = x. \quad (2.5)$$

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We note that, by $\phi_{t,t}(x) = x$, we mean that the Eulerian coordinate is coincident with the Lagrangian (or material) coordinate at time t . As in classical mechanics, the flow map $\phi_{t,s}$ is assumed to be a diffeomorphism. Moreover, the flow map satisfies the composition rule, i.e., $\phi_{t_1,t_2}\phi_{t,t_1} = \phi_{t,t_2}$, for any $t_1, t_2 \geq 0$.

Using this flow map, with an abuse of notation, we define

$$v(t, s) := v(\phi_{t,s}(x), s) \quad \text{and} \quad v(t, t) = v(t),$$

where v can be any (scalar, vector, or tensor) function. Furthermore, for any $v(x, t)$, we define the material derivative:

$$\frac{Dv}{Dt}(x, t) := \left. \frac{\partial}{\partial s} v(\phi_{t,s}(x), s) \right|_{s=t} = (v_t + (\mathbf{u} \cdot \nabla)v)(x, t). \quad (2.6)$$

Of the different conventions to define the gradient of velocity \mathbf{u} , denoted by $\nabla \mathbf{u}$ (or $\nabla_x \mathbf{u}$), we use the convention that $(\nabla \mathbf{u})_{i,j} = (\partial_j u_i)_{i,j}$.

For any given continuous function $\Phi(t) = \Phi(x, t): \Omega \times (0, +\infty) \mapsto \mathbb{M}$, we define $\mathbf{L}(t, s)$ to be the transition matrix that satisfies the following ordinary differential equation (ODE):

$$\frac{\partial \mathbf{L}(t, s)}{\partial s} = \Phi(t, s) \mathbf{L}(t, s) \quad \text{and} \quad \mathbf{L}(t, t) = \delta. \quad (2.7)$$

We can view this transition matrix $\mathbf{L}(t, s)$ as a generalization of the relative deformation gradient tensor $\mathbf{F}(t, s)$; if $\Phi(t, s) = \nabla \mathbf{u}(t, s)$, $\mathbf{L}(t, s)$ reduces to $\mathbf{F}(t, s)$.

Now we are ready to introduce to the definition and properties of the generalized Lie derivatives.

Definition 2.1 (Generalized Lie Derivatives). We define the generalized Lie derivative of a symmetric tensor with respect to Φ in the Lagrangian frame as follows: for $t, s \geq 0$,

$$\mathcal{L}_{\mathbf{u}, \Phi} \zeta(t, s) := \mathbf{L}(t, s) \frac{\partial (\mathbf{L}(s, t) \zeta(t, s) \mathbf{L}(s, t)^T)}{\partial s} \mathbf{L}(t, s)^T. \quad (2.8)$$

In the Eulerian coordinates, we let $s = t$ and

$$\mathcal{L}_{\mathbf{u}, \Phi} \zeta(t) := \left. \mathbf{L}(t, s) \frac{\partial (\mathbf{L}(s, t) \zeta(t, s) \mathbf{L}(s, t)^T)}{\partial s} \mathbf{L}(t, s)^T \right|_{s=t}. \quad (2.9)$$

The following lemma gives an important property of the generalized Lie derivatives defined above:

Lemma 2.1 (The Generalized Lie Derivative in the Eulerian Frame). For any $\zeta = \zeta(x, t): \Omega \times (0, +\infty) \mapsto \mathbb{S}$, we have

$$\mathcal{L}_{\mathbf{u}, \Phi} \zeta(t) = \frac{D\zeta(t)}{Dt} - \Phi(t) \zeta(t) - \zeta(t) \Phi(t)^T. \quad (2.10)$$

Proof. Using the composition rule of \mathbf{L} (see Theorem 1.4.2 in Brockett⁸) and the product rule, we have

$$\begin{aligned} \frac{\partial(\mathbf{L}(s,t)\boldsymbol{\zeta}(t,s)\mathbf{L}(s,t)^T)}{\partial s} &= \frac{\partial\mathbf{L}(s,t)}{\partial s}\boldsymbol{\zeta}(t,s)\mathbf{L}(s,t)^T \\ &\quad + \mathbf{L}(s,t)\frac{\partial\boldsymbol{\zeta}(t,s)}{\partial s}\mathbf{L}(s,t)^T + \mathbf{L}(s,t)\boldsymbol{\zeta}(t,s)\frac{\partial\mathbf{L}(s,t)^T}{\partial s}. \end{aligned}$$

Hence, we can immediately obtain

$$\begin{aligned} \mathcal{L}_{\mathbf{u},\boldsymbol{\Phi}}\boldsymbol{\zeta}(t,s) &= \mathbf{L}(t,s)\frac{\partial(\mathbf{L}(s,t)\boldsymbol{\zeta}(t,s)\mathbf{L}(s,t)^T)}{\partial s}\mathbf{L}(t,s)^T \\ &= \mathbf{L}(t,s)\frac{\partial\mathbf{L}(s,t)}{\partial s}\boldsymbol{\zeta}(t,s) + \frac{\partial\boldsymbol{\zeta}(t,s)}{\partial s} + \boldsymbol{\zeta}(t,s)\frac{\partial\mathbf{L}(s,t)^T}{\partial s}\mathbf{L}(t,s)^T. \end{aligned}$$

We can derive that

$$\mathcal{L}_{\mathbf{u},\boldsymbol{\Phi}}\boldsymbol{\zeta}(t,s) = \frac{\partial\boldsymbol{\zeta}(t,s)}{\partial s} - \boldsymbol{\Phi}(t,s)\boldsymbol{\zeta}(t,s) - \boldsymbol{\zeta}(t,s)\boldsymbol{\Phi}(t,s)^T. \quad (2.11)$$

Now, by letting $s = t$, we get equation (2.10) in this lemma. \square

2.2. An example: the Oldroyd-B model

A basic model for complex fluids that introduces the outstanding challenge for high Weissenberg number regimes is the so-called Oldroyd-B model²⁷. It obeys the following constitutive law:

$$\boldsymbol{\tau} + \text{Wi}(\partial_t\boldsymbol{\tau} + \mathbf{u} \cdot \nabla\boldsymbol{\tau} - \nabla\mathbf{u}\boldsymbol{\tau} - \boldsymbol{\tau}(\nabla\mathbf{u})^T) = 2\eta_p\mathcal{D}(\mathbf{u}), \quad (2.12)$$

where η_p is the polymeric viscosity and the Weissenberg number $\text{Wi} = \lambda\bar{U}/\bar{L}$. Here λ , \bar{U} and \bar{L} are the relaxation time, the characteristic velocity scale, and the characteristic length scale, respectively. When Wi is equal to zero, the stress becomes linearly proportional to the strain rate (i.e., the Newtonian constitutive relation); in this case, the equations (2.1)–(2.3), become the Navier–Stokes equations. The Weissenberg number Wi is thus the characteristic constant that distinguishes the polymeric fluids from the Newtonian fluids. The Oldroyd-B model reduces to the Upper Convected Maxwell (UCM) model when $\eta_s = 0$.

The Oldroyd-B constitutive model (2.12) can be viewed as the simplest nonlinear extension of Maxwell’s idea of formulating a system of ODEs to determine the stress in terms of the velocity gradient and the time derivative. In Lemma 2.1, we have observed that the upper convective Maxwell derivative $\partial_t\boldsymbol{\tau} + \mathbf{u} \cdot \nabla\boldsymbol{\tau} - \nabla\mathbf{u}\boldsymbol{\tau} - \boldsymbol{\tau}(\nabla\mathbf{u})^T$ that appears in (2.12) is equal to $\mathcal{L}_{\mathbf{u},\nabla\mathbf{u}}\boldsymbol{\tau}$. Therefore, the Oldroyd-B constitutive law can be simply written as:

$$\boldsymbol{\tau} + \text{Wi}\mathcal{L}_{\mathbf{u},\nabla\mathbf{u}}\boldsymbol{\tau} = 2\eta_p\mathcal{D}(\mathbf{u}). \quad (2.13)$$

The constitutive laws are frequently written in terms of the conformation tensor

$$\mathbf{c} := \boldsymbol{\tau} + \frac{\eta_p}{\text{Wi}}\boldsymbol{\delta}. \quad (2.14)$$

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From a physical point of view, the conformation tensor can be regarded as a molecular deformation tensor on a continuum level. More precisely, the conformation tensor is the ensemble average of the dyadic product of the end-to-end vector of the dumbbell. Therefore, it is SPD and is often used as a primary variable in viscoelastic flow calculations⁹. Using the conformation tensor, we can easily write (2.13), in a more general form, as:

$$\mathcal{L}_{\mathbf{u}, \nabla \mathbf{u}} \mathbf{c} + \alpha \mathbf{c} = \beta \delta, \quad (2.15)$$

where $\alpha \geq 0$ and $\beta > 0$ may depend on t and/or \mathbf{c} . For instance, the Oldroyd-B model can be recovered from the equation (2.15) by choosing $\alpha = \frac{1}{Wi}$ and $\beta = \frac{\eta_p}{Wi^2}$.

The simple formulation (2.15) can be generalized to represent many existing models. For example, it can represent the well-known Johnson–Segalman model¹⁸, the Phan-Thien and Tanner (PTT) model³⁴, and other models that belong to the Finitely Extensible Nonlinear Elastic (FENE) models^{10,25,30,33}. We refer to Lee, Xu, and Zhang²⁴ for details.

It would be of interest to consider the equation (2.15) as it is, which is because the generalized Lie derivative $\mathcal{L}_{\mathbf{u}, \Phi}$ is ubiquitous in general macroscopic equations. We can, in fact, derive the explicit solution²³ in terms of the transition matrix Φ as follows:

Lemma 2.2 (Explicit Solution of the Constitutive Equation). *The solution to the constitutive equation in Riccati form (2.15) satisfies*

$$\begin{aligned} \mathbf{c}(t) = \exp \left(- \int_s^t \alpha(\zeta) d\zeta \right) \Phi(s, t) \mathbf{c}(t, s) \Phi(s, t)^T \\ + \int_s^t \exp \left(- \int_\nu^t \alpha(\zeta) d\zeta \right) \beta(\nu) \Phi(\nu, t) \Phi(\nu, t)^T d\nu. \end{aligned} \quad (2.16)$$

From Lemma 2.2, we can see the conformation tensor \mathbf{c} is positive-definite if $\alpha(t) \geq 0$ and $\beta(t) > 0$. For simplicity as well as clarity in presenting the main ideas of our algorithm and the proof of global existence, we only discuss the Oldroyd-B model in terms of conformation tensor:

$$\left\{ \begin{array}{ll} \operatorname{Re} \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \eta_s \Delta \mathbf{u} - \nabla p + \nabla \cdot \mathbf{c}, & \text{in } \Omega \times \mathbb{R}^+, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \mathcal{L}_{\mathbf{u}, \nabla \mathbf{u}} \mathbf{c} + \alpha \mathbf{c} = \beta \delta, & \text{in } \Omega \times \mathbb{R}^+, \\ \mathbf{u}(x, t) = 0, & \text{in } \partial\Omega \times \mathbb{R}^+, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), \mathbf{c}(x, 0) = \mathbf{c}_0(x), & \text{in } \Omega, \end{array} \right. \quad (2.17)$$

with $\alpha = 1/Wi$ and $\beta = \eta_p/Wi^2$, in a polygonal domain $\Omega \subset \mathbb{R}^d$ here. The extension of these stability and convergence results to more general cases is straightforward.

3. Uniformly stable finite element methods that preserve the positivity of the conformation tensor

In this section, we present our numerical methods for solving the Oldroyd-B model. The momentum equation as well as the constitutive relation in (2.17) have transport nature; and, it is natural to explore the method of characteristics.

3.1. Temporal discretization

The pure Lagrangian approach that follows the initially seeded particles' trajectories in time evolutions and the other of which is to use the fixed grid as the particle positions on the time level at which the numerical solutions are anticipated. This approach has an inherent disadvantage that the grid points can be severely distorted in the long time calculations, which generally leads to degradation of the accuracy¹. The Eulerian–Lagrangian method (ELM) or the semi-Lagrangian method (SLM) begins by establishing the characteristic foot of any given particle at the current time step^{11,29}. In this way, it avoids remeshing required by the pure Lagrangian approaches.

ELM for the momentum equation Let x be the position of any material particle at the current time t . Suppose that a particle moves with the velocity $\mathbf{u}(\phi_{t,s}(x), s)$ at time s . The characteristic foot (or the departure foot) $y = \phi_{t,s}(x)$ of this particle at any previous time $s \leq t$ can be found by solving (2.5). Feng and Shang¹² analyzed volume preserving numerical difference schemes for the first-order ODEs. We give a simple example of volume-preserving schemes in \mathbb{R}^2 :

$$\tilde{y} = x - k\mathbf{u}\left(\frac{\tilde{y} + x}{2}, s\right). \quad (3.1)$$

If $\nabla \cdot \mathbf{u}(s) = 0$ and k is small enough, then the scheme (3.1) is well-defined and volume-preserving, i.e., $\det(\nabla \tilde{y}) = 1$. For three-dimensional problems, constructing volume-preserving schemes is more complicated; see Feng and Shang¹² for details.

We will now apply this idea for the momentum equation of the Oldroyd-B model. We assume that the solution at time level t^{old} is known; that is, $(\mathbf{u}^{\text{old}}, p^{\text{old}}, \mathbf{T}^{\text{old}})$ is given. For any given mesh points at time level t^{new} , we let \tilde{y} be a discrete solution to the flow map equation (2.5). Namely, \tilde{y} is an approximation of the departure foot $y = \phi_{t^{\text{new}}, t^{\text{old}}}(x)$. In the Lagrangian view, the momentum equation (2.1) can be viewed as an ODE and it can be discretized as follows:

$$\text{Re} \left(\frac{\mathbf{u}^{\text{new}} - \mathbf{u}^{\text{old}} \circ \tilde{y}}{k} \right) = \eta_s \Delta \mathbf{u}^{\text{new}} - \nabla p^{\text{new}} + \nabla \cdot \mathbf{T}^{\text{new}}$$

Therefore, we arrive at the following semi-discrete equation:

$$\frac{\text{Re}}{k} \mathbf{u}^{\text{new}} - \eta_s \Delta \mathbf{u}^{\text{new}} + \nabla p^{\text{new}} - \nabla \cdot \mathbf{T}^{\text{new}} = \frac{\text{Re}}{k} \mathbf{u}^{\text{old}} \circ \tilde{y}. \quad (3.2)$$

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ELM for the Oldroyd-B constitutive equation The particle-following approach (3.2) can be naturally applied to approximate generalized Lie derivatives as well. We consider the numerical discretization of the generalized Lie derivative $\mathcal{L}_{\mathbf{u}, \nabla \mathbf{u}} \boldsymbol{\zeta}$ at time t^{new} . By Definition 2.1, we can employ the first-order difference approximation for the time derivative to obtain

$$\begin{aligned} & \frac{\mathbf{F}(s, t) \boldsymbol{\zeta}(s, t) \mathbf{F}(s, t)^T - \mathbf{F}(s - k, t) \boldsymbol{\zeta}(s - k, t) \mathbf{F}(s - k, t)^T}{k} \Big|_{s=t} \\ &= \frac{\boldsymbol{\zeta}(t, t) - \mathbf{F}(t - k, t) \boldsymbol{\zeta}(t - k, t) \mathbf{F}(t - k, t)^T}{k}. \end{aligned} \quad (3.3)$$

Let $\tilde{\mathbf{F}}$ be an approximate solution to the ODE for the transition matrix, namely,

$$\frac{\partial \mathbf{F}(s, t)}{\partial t} = \nabla \mathbf{u}(s, t) \mathbf{F}(s, t) \quad \text{and} \quad \mathbf{F}(s, s) = \boldsymbol{\delta}. \quad (3.4)$$

For example, we can apply the implicit Euler method:

$$\tilde{\mathbf{F}} - \boldsymbol{\delta} = k \nabla \mathbf{u}(t^{\text{new}}) \tilde{\mathbf{F}} \quad \implies \quad \tilde{\mathbf{F}} = (\boldsymbol{\delta} - k \nabla \mathbf{u}(t^{\text{new}}))^{-1}. \quad (3.5)$$

Using (3.5) for approximating the transition matrix $\tilde{\mathbf{F}}$ and the approximate solution $\tilde{\mathbf{y}}$ to the flow map equation, we derive a first-order discretization of the generalized Lie derivative as follows:

$$\mathcal{L}_{\mathbf{u}, \nabla \mathbf{u}} \boldsymbol{\zeta}(t^{\text{new}}) \approx \frac{\boldsymbol{\zeta}^{\text{new}} - \tilde{\mathbf{F}}(\boldsymbol{\zeta}^{\text{old}} \circ \tilde{\mathbf{y}}) \tilde{\mathbf{F}}^T}{k}. \quad (3.6)$$

A second-order discretization was also proposed in Lee and Xu²³.

The approximation (3.6) can be easily shown to satisfy the desired property that the conformation tensor is positive-definite in the semi-discrete level when applied to approximate the general Riccati form of the constitutive law (2.15).

Lemma 3.1 (Positivity-Preserving Semi-Discretization). *Consider the semi-discrete scheme (3.6) for the model equation (2.15) with positive parameters α and β , namely,*

$$\frac{\mathbf{c}^{\text{new}} - \tilde{\mathbf{F}}(\mathbf{c}^{\text{old}} \circ \tilde{\mathbf{y}}) \tilde{\mathbf{F}}^T}{k} + \alpha \mathbf{c}^{\text{new}} = \beta \boldsymbol{\delta}. \quad (3.7)$$

If \mathbf{c}^{old} is positive-definite, then the numerical scheme preserves positivity, namely, \mathbf{c}^{new} is still positive definite.

Proof. We can solve the equation (3.7) in terms of \mathbf{c}^{new} to obtain

$$(1 + k\alpha) \mathbf{c}^{\text{new}} = \tilde{\mathbf{F}}(\mathbf{c}^{\text{old}} \circ \tilde{\mathbf{y}}) \tilde{\mathbf{F}}^T + k\beta \boldsymbol{\delta}. \quad (3.8)$$

As an immediate consequence, if \mathbf{c}^{old} is positive-definite, then so is \mathbf{c}^{new} . \square

3.2. Spatial discretization

The Eulerian–Lagrangian framework has been shown to preserve the positivity of the conformation tensor. The main goal of this subsection is to introduce spatial discretizations so that positivity can be realized in the fully discrete sense as well. It is worth noting that the Eulerian–Lagrangian approach follows the particle trajectory and the interpolated solution may not be positive even if the solution is positive at mesh points.

Stokes-like saddle point problems We begin by introducing the equations that will be discretized in space. After applying the ELM to the model problem (2.1), (2.2), and (2.15), we obtain the following semi-discrete problem: Find $(\mathbf{u}^{\text{new}}, p^{\text{new}}, \mathbf{c}^{\text{new}})$, such that

$$\frac{\text{Re}}{k} \mathbf{u}^{\text{new}} - \eta_s \Delta \mathbf{u}^{\text{new}} + \nabla p^{\text{new}} = \frac{\text{Re}}{k} \mathbf{u}^{\text{old}} \circ \tilde{\mathbf{y}} + \nabla \cdot \mathbf{c}^{\text{new}} \quad (3.9)$$

$$\nabla \cdot \mathbf{u}^{\text{new}} = 0 \quad (3.10)$$

$$(1 + k\alpha) \mathbf{c}^{\text{new}} = \tilde{\mathbf{F}}(\mathbf{c}^{\text{old}} \circ \tilde{\mathbf{y}}) \tilde{\mathbf{F}}^T + \beta \boldsymbol{\delta}. \quad (3.11)$$

We note that equation (3.9) is nonlinear and coupled together through the conformation tensor with (3.11); the nonlinearity also lies in the computation of $\tilde{\mathbf{y}}$ as well as $\tilde{\mathbf{F}}$. The finite element spaces for the unknowns \mathbf{u} , p , and \mathbf{c} should be constructed carefully based on the stability conditions to keep the solution from blowing up as the space and time mesh size reduce.

Stable discretizations of the generalized Stokes equation We assume that the domain $\Omega \subset \mathbb{R}^d$ has been partitioned into triangular/tetrahedral elements $\mathcal{T}_h = \{E\}$ and that the conforming and quasi-uniform partition \mathcal{T}_h satisfies

$$\bar{\Omega} = \bigcup_{E \in \mathcal{T}_h} \bar{E}. \quad (3.12)$$

We use the standard notation that $(\cdot : \cdot)$ acting on two matrix-valued functions $\mathbf{A} = (a_{i,j}) \in \mathbb{M}$ and $\mathbf{B} = (b_{i,j}) \in \mathbb{M}$ denotes

$$(\mathbf{A} : \mathbf{B}) := \int_{\Omega} \text{tr}(\mathbf{A}\mathbf{B}) \, dx = \int_{\Omega} \sum_{i,j=1}^d a_{i,j} b_{i,j} \, dx, \quad (3.13)$$

where $\text{tr} : \mathbb{M} \mapsto \mathbb{R}$ is the standard trace operator of a matrix.

On \mathcal{T}_h , we choose appropriate approximation spaces \mathbf{V}_h and W_h for the primitive variables, \mathbf{u} and p , respectively and consider a discrete weak formulation that is formulated by making the appropriate choice of space $\mathbf{V}_h \subset (H_0^1(\Omega))^d$ for the velocity and $W_h \subset L_0^2(\Omega)$ for the pressure: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$, such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \langle \mathbf{g}, \mathbf{v}_h \rangle & \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h, q_h) = 0 & \forall q_h \in W_h, \end{cases} \quad (3.14)$$

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where $\mathbf{g} = \nabla \cdot \mathbf{c}^{\text{new}}$ and $a : \mathbf{V}_h \times \mathbf{V}_h \mapsto \mathbb{R}$ and $b : \mathbf{V}_h \times W_h \mapsto \mathbb{R}$ are bilinear forms: for any $\mathbf{u}, \mathbf{v} \in \mathbf{V}_h$ and $q \in W_h$,

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}) + k \frac{\eta_s}{\text{Re}} (\nabla \mathbf{u} : \nabla \mathbf{v}) \quad \text{and} \quad b(\mathbf{u}, q) = (\nabla \cdot \mathbf{u}, q).$$

As demonstrated by Xie, Xu, and Xue³⁵, the uniform well-posedness and error analysis for the finite element pairs $\mathbf{V}_h \times W_h$ can be achieved if they satisfy the following two conditions :

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \gtrsim \|q_h\|_0 \quad \forall q_h \in W_h, \quad (3.15)$$

and

$$\nabla \cdot \mathbf{V}_h \subseteq W_h. \quad (3.16)$$

It is noteworthy that there are abundant choices of mixed finite elements that satisfy (3.15), i.e., the Brezzi condition^{6,7}. However, to construct conforming finite elements that satisfy both (3.15) and (3.16) is more difficult; and, such an example has been provided by Scott and Vogelius³². It uses the piecewise continuous polynomials on triangles of a degree up to 4 to approximate the velocity field and uses the piecewise discontinuous polynomials of a degree up to 3 for the pressure. This element is stable in the sense that it satisfies the Brezzi condition if the triangulation is singular-point^a free.

Approximation Space for Stress Our guiding principle in choosing the approximation space for the stress fields is to preserve the positivity of the conformation tensor. The main bottleneck in this task is to evaluate the conformation tensor at any points which may not necessarily be the mesh points. This can be reinterpreted as constructing an interpolation operator $\Pi_h^S : \mathbb{M} \mapsto \mathbb{M}$, which preserves positivity in the following sense:

$$\boldsymbol{\sigma} > 0 \implies \Pi_h^S(\boldsymbol{\sigma}) > 0, \quad \forall \boldsymbol{\sigma} \in \mathbb{M} \quad (3.17)$$

where $\boldsymbol{\sigma} > 0$ means $\boldsymbol{\sigma}$ is positive-definite.

The simplest finite element space that preserves the positivity of the scalar functions is, of course, the space of the piecewise constant functions. In this case, the existence of the positivity-preserving interpolation operator Π_h is obvious. For example, we can take, on each element $E \in \mathcal{T}_h$,

$$\Pi_h(g)(x) := \frac{1}{|E|} \int_E g \, dx \quad \forall x \in E, \quad (3.18)$$

where $|E|$ is the area of E . It is easy to see that

$$\|\Pi_h(g)\|_\infty = \max_E \frac{1}{|E|} \int_E g \leq \max_E |E|^{-1/2} \|g\|_0, \quad (3.19)$$

$$\|\Pi_h(g)\|_{L^1} = \sum_E \int_E |\Pi_h(g)| \leq \sum_E \int_E |g| = \|g\|_{L^1}. \quad (3.20)$$

^aA vertex is called singular if all edges meeting at the vertex fall into two crossing straight lines.

Notice that these two inequalities are sharp. To see this, we can simply take g to be 1 on an element E and 0 elsewhere.

3.3. Linearization

In this section, we will conclude our discussion on discretization by combining time and space discretizations. We choose the approximation spaces $\mathbf{V}_h \times W_h \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ so that they satisfy both the inf-sup condition and the strong divergence-free condition. In addition, we choose \mathcal{S}_h to be the symmetric tensor space whose entries belong to the piecewise constant polynomial spaces.

Let tol be the tolerance for the nonlinear iteration. If \mathbf{u}_h^n, p_h^n and \mathbf{c}_h^n at the time level t^n are available, then we have the following time-marching algorithm:

Algorithm 1 (Fixed-Point Iteration).

0: Given \mathbf{u}_h^n, p_h^n and \mathbf{c}_h^n . Set

$$\mathbf{u}_h^{n,0} := \mathbf{u}_h^n, \quad p_h^{n,0} := p_h^n, \quad \text{and} \quad \mathbf{c}_h^{n,0} := \mathbf{c}_h^n.$$

1: For any particle x , compute the departure feet

$$\tilde{y}^n = x - k \mathbf{u}_h^{n,0} \left(\frac{\tilde{y}^n + x}{2} \right).$$

2: For $\ell = 0, 1, 2, \dots$, do

(1) Solve the Stokes-type system

$$\begin{cases} \text{Re} \mathbf{u}_h^{n,\ell+1} - k \Delta_h \mathbf{u}_h^{n,\ell+1} + k \nabla_h p_h^{n,\ell+1} = \text{Re} \Pi_h^V (\mathbf{u}_h^{n,0} \circ \tilde{y}^n) + k \nabla_h \cdot \mathbf{c}_h^{n,\ell} + k \mathbf{f}, \\ \nabla \cdot \mathbf{u}_h^{n,\ell+1} = 0. \end{cases}$$

(2) Update the conformation tensor

$$(1 + k\alpha) \mathbf{c}_h^{n,\ell+1} = \mathbf{F}_h^{n,\ell+1} \Pi_h^S (\mathbf{c}_h^{n,0} \circ \tilde{y}^n) (\mathbf{F}_h^{n,\ell+1})^T + k\beta \boldsymbol{\delta},$$

$$\text{where } \mathbf{F}_h^{n,\ell+1} := (\boldsymbol{\delta} - k \Pi_h^S (\nabla \mathbf{u}_h^{n,\ell+1}))^{-1}.$$

(3) If $\|\mathbf{u}_h^{n,\ell+1} - \mathbf{u}_h^{n,\ell}\|_1 \leq \text{tol}$ and $\|p_h^{n,\ell+1} - p_h^{n,\ell}\|_0 \leq \text{tol}$, then break.

3: Update solution: $\mathbf{u}_h^{n+1} := \mathbf{u}_h^{n,\ell+1}$, $p_h^{n+1} := p_h^{n,\ell+1}$, and $\mathbf{c}_h^{n+1} := \mathbf{c}_h^{n,\ell+1}$.

Algorithm 1 is a single-step time-marching algorithm. Once the initial condition $(\mathbf{u}_h^0, p_h^0, \mathbf{c}_h^0)$ is given, we can proceed to the evolution process.

4. Global existence and uniqueness of discrete solutions

In this section, we show that the discretization scheme (Algorithm 1) is stable. The stability will then be used to establish the existence of the discrete solutions in time evolution.

In our analysis, we shall assume that the flow map $\phi_{t,s} : \Omega \mapsto \Omega$ can be obtained so that the approximate flow map \tilde{y} satisfies the following identity:

$$\int_{\Omega} g \circ \tilde{y} dx = \int_{\Omega} g dx \quad \forall g \in L^1(\Omega). \quad (4.1)$$

The identity (4.1) is the key to deriving uniform energy estimates for the solution to the discrete model equations (2.17). Recall that the discrete scheme (3.1) preserves volume in \mathbb{R}^2 . We can easily show that this scheme satisfies (4.1) by changing variables.

4.1. Discrete energy estimate

The discrete model equations (2.17) are fully nonlinear and the well-posedness of this model is not trivial. In this section, we will show the main result of this paper that the solution to the discrete problem exists for sufficiently small time step size k ; furthermore, the discrete solution is unique. Theoretically, the restriction of k is only given by the mesh size h . These will, in turn, confirm that Algorithm 1 is well-defined.

Before introducing the main existence result, we would like to make a few comments: (1) We note that the notion we used here as the well-posedness for the solution to the discrete models should be distinguished from the one introduced by Kreiss¹⁹. In particular, our result does not necessarily imply stability with respect to the perturbation of the data. (2) Our analysis fully exploits the finite dimensionality of the solution space; therefore, technically, it will be difficult to extend this analysis to the existence analysis for the continuous level.

Our proof is based on the induction argument. Specifically, we will assume that at time level t^n , the discrete solutions \mathbf{u}_h^n and \mathbf{c}_h^n are well-defined and generate a sequence of iterates according to Algorithm 1 and show that the nonlinear iteration converges and defines \mathbf{u}_h^{n+1} and \mathbf{c}_h^{n+1} . More precisely, we will show that the solutions at the time level t^{n+1} can be obtained uniquely by the Algorithm 1. We note that if \mathbf{u}_h^n and \mathbf{c}_h^n at the time level t^n satisfy the uniform bounds

$$\|\mathbf{u}_h^n\|_0 \lesssim 1 \quad \text{and} \quad \|\mathbf{c}_h^n\|_{L^1} \lesssim 1, \quad (4.2)$$

the fixed-point iteration (Algorithm 1) converges. This assumption is valid due to the discrete energy estimates established by Lee and Xu²³:

Lemma 4.1 (Discrete Energy Estimate). *If $W_i < \infty$ and $n \geq 1$, then*

$$\text{Re}\|\mathbf{u}_h^n\|_0^2 + \|\mathbf{c}_h^n\|_{L^1} \leq c_1 e^{-C_1 t^n} (\text{Re}\|\mathbf{u}_h^0\|_0^2 + \|\mathbf{c}_h^0\|_{L^1}) + c_2, \quad (4.3)$$

$$2\eta_s \sum_{\ell=1}^n k \|\mathcal{D}(\mathbf{u}_h^\ell)\|_0^2 \leq \text{Re}\|\mathbf{u}_h^0\|_0^2 + \|\mathbf{c}_h^0\|_{L^1} + c_2 t^n. \quad (4.4)$$

Here c_1 and c_2 are generic constants.

For the sake of completeness, we include a proof to this lemma here.

Proof. From Step 2 in Algorithm 1, we have the following relation:

$$(1 + k\alpha) \mathbf{c}_h^{n+1} = (\boldsymbol{\delta} - k\Pi_h^S(\nabla \mathbf{u}_h^{n+1}))^{-1} \Pi_h^S(\mathbf{c}_h^n \circ \tilde{\mathbf{y}}^n) (\boldsymbol{\delta} - k\Pi_h^S(\nabla \mathbf{u}_h^{n+1}))^{-T} + k\beta \boldsymbol{\delta}. \quad (4.5)$$

We first multiply $\boldsymbol{\delta} - k\Pi_h^S(\nabla \mathbf{u}_h^{n+1})$ to the left and $\boldsymbol{\delta} - k\Pi_h^S(\nabla \mathbf{u}_h^{n+1})^T$ to the right of the equation (4.5) to obtain that

$$(1 + k\alpha) (\boldsymbol{\delta} - k\Pi_h^S(\nabla \mathbf{u}_h^{n+1})) \mathbf{c}_h^{n+1} (\boldsymbol{\delta} - k\Pi_h^S(\nabla \mathbf{u}_h^{n+1}))^T = \Pi_h^S(\mathbf{c}_h^n \circ \tilde{\mathbf{y}}^n) + k\beta (\boldsymbol{\delta} - k\Pi_h^S(\nabla \mathbf{u}_h^{n+1})) (\boldsymbol{\delta} - k\Pi_h^S(\nabla \mathbf{u}_h^{n+1}))^T. \quad (4.6)$$

Hence, by taking trace and then integration on both sides of the equation above, we get that

$$\begin{aligned} & k(1 + k\alpha) \int_{\Omega} \text{tr} (\Pi_h^S(\nabla \mathbf{u}_h^{n+1}) \mathbf{c}_h^{n+1} + \mathbf{c}_h^{n+1} \Pi_h^S(\nabla \mathbf{u}_h^{n+1})^T) dx \\ &= (1 + k\alpha) \|\mathbf{c}_h^{n+1}\|_{L^1} - \|\mathbf{c}_h^n \circ \tilde{\mathbf{y}}^n\|_{L^1} - d\beta|\Omega|k \\ &+ k^2\beta \int_{\Omega} \text{tr} (\Pi_h^S(\nabla \mathbf{u}_h^{n+1}) + \Pi_h^S(\nabla \mathbf{u}_h^{n+1})^T) dx \\ &+ k^2(1 + k\alpha) \int_{\Omega} \text{tr} \left(\Pi_h^S(\nabla \mathbf{u}_h^{n+1}) \left(\mathbf{c}_h^{n+1} - \frac{k\beta}{1 + k\alpha} \boldsymbol{\delta} \right) \Pi_h^S(\nabla \mathbf{u}_h^{n+1})^T \right) dx. \end{aligned} \quad (4.7)$$

We note that from (4.5), the approximate conformation tensor \mathbf{c}_h^{n+1} satisfies

$$(1 + k\alpha) \mathbf{c}_h^{n+1} - k\beta \boldsymbol{\delta} \geq 0 \quad \forall n \geq 1, \quad (4.8)$$

if the initial condition $\mathbf{c}_h^0 \geq 0$. Since \mathbf{c}_h^{n+1} is symmetric, by the discrete divergence-free condition, we can easily see that

$$\begin{aligned} & \int_{\Omega} \text{tr} (\Pi_h^S(\nabla \mathbf{u}_h^{n+1}) + \Pi_h^S(\nabla \mathbf{u}_h^{n+1})^T) dx \\ &= \int_{\Omega} \text{tr} (\nabla \mathbf{u}_h^{n+1} + (\nabla \mathbf{u}_h^{n+1})^T) dx = 2 \int_{\Omega} \nabla \cdot \mathbf{u}_h^{n+1} dx = 0 \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} & \int_{\Omega} \text{tr} (\Pi_h^S(\nabla \mathbf{u}_h^{n+1}) \mathbf{c}_h^{n+1}) dx = \int_{\Omega} \text{tr} (\mathbf{c}_h^{n+1} \Pi_h^S(\nabla \mathbf{u}_h^{n+1})^T) dx \\ &= (\mathbf{c}_h^{n+1} : \Pi_h^S(\mathcal{D}(\mathbf{u}_h^{n+1}))) = (\mathbf{c}_h^{n+1} : \mathcal{D}(\mathbf{u}_h^{n+1})). \end{aligned} \quad (4.10)$$

Finally, based on the volume-preserving property of $\tilde{\mathbf{y}}^n$, we have $\|\mathbf{c}_h^n \circ \tilde{\mathbf{y}}^n\|_{L^1} = \|\mathbf{c}_h^n\|_{L^1}$. Taking the facts (4.8), (4.9), and (4.10) into account, we can derive the following inequality from (4.7):

$$(\mathbf{c}_h^{n+1} : \mathcal{D}(\mathbf{u}_h^{n+1})) \geq \frac{1}{2k} \|\mathbf{c}_h^{n+1}\|_{L^1} - \frac{1}{2k(1 + k\alpha)} \|\mathbf{c}_h^n\|_{L^1} - \frac{d\beta|\Omega|}{2(1 + k\alpha)}. \quad (4.11)$$

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We now consider the momentum equation (2.17). Using the energy method, together with the discrete divergence-free condition and (4.11), we can obtain

$$\begin{aligned}
 & \frac{\operatorname{Re}}{k} \|\mathbf{u}_h^{n+1}\|_0^2 + \eta_s \|\mathcal{D}(\mathbf{u}_h^{n+1})\|_0^2 \\
 &= \frac{\operatorname{Re}}{k} (\Pi_h^V(\mathbf{u}_h^n \circ \tilde{y}^n), \mathbf{u}_h^{n+1}) - (\mathbf{c}_h^{n+1} : \mathcal{D}(\mathbf{u}_h^{n+1})) \\
 &\leq \frac{\operatorname{Re}}{k} (\mathbf{u}_h^n \circ \tilde{y}^n, \mathbf{u}_h^{n+1}) - \frac{1}{2k} \|\mathbf{c}_h^{n+1}\|_{L^1} + \frac{1}{2k(1+k\alpha)} \|\mathbf{c}_h^n\|_{L^1} + \frac{d\beta|\Omega|}{2(1+k\alpha)}. \quad (4.12)
 \end{aligned}$$

Applying the Cauchy–Schwarz inequality and the standard kick-back argument, we obtain the following relation:

$$\begin{aligned}
 & \frac{\operatorname{Re}}{2k} \|\mathbf{u}_h^{n+1}\|_0^2 + \eta_s \|\mathcal{D}(\mathbf{u}_h^{n+1})\|_0^2 + \frac{1}{2k} \|\mathbf{c}_h^{n+1}\|_{L^1} \\
 & \leq \frac{\operatorname{Re}}{2k} \|\mathbf{u}_h^n\|_0^2 + \frac{1}{2k(1+k\alpha)} \|\mathbf{c}_h^n\|_{L^1} + \frac{d\beta|\Omega|}{2(1+k\alpha)}. \quad (4.13)
 \end{aligned}$$

We are now in the position to show the first estimate (4.3). Multiplying $2k$ to both sides of (4.13) and using Korn’s inequality, we obtain that

$$\begin{aligned}
 \kappa \|\mathbf{u}_h^{n+1}\|_0^2 + \|\mathbf{c}_h^{n+1}\|_{L^1} &\leq \operatorname{Re} \|\mathbf{u}_h^n\|_0^2 + \frac{1}{1+k\alpha} \|\mathbf{c}_h^n\|_{L^1} + \frac{kd\beta|\Omega|}{1+k\alpha} \\
 &\leq \exp(-C_1 k) (\kappa \|\mathbf{u}_h^n\|_0^2 + \|\mathbf{c}_h^n\|_{L^1}) + \frac{kd\beta|\Omega|}{1+k\alpha}, \quad (4.14)
 \end{aligned}$$

where $\kappa = \operatorname{Re} + 2k\eta_s C_\Omega$ and C_Ω is a positive constant depending only on Ω . Here $C_1 > 0$ is chosen to be a constant such that

$$\max\left(\frac{\operatorname{Re}}{\operatorname{Re} + 2k\eta_s C_\Omega}, \frac{1}{1+k\alpha}\right) \leq \exp(-C_1 k), \quad 0 \leq k \leq 1. \quad (4.15)$$

Now, we use the induction argument to obtain:

$$\begin{aligned}
 \kappa \|\mathbf{u}_h^n\|_0^2 + \|\mathbf{c}_h^n\|_{L^1} &\leq \exp(-C_1 t^n) (\kappa \|\mathbf{u}_h^0\|_0^2 + \|\mathbf{c}_h^0\|_{L^1}) + \frac{kd\beta|\Omega|}{1+k\alpha} \sum_{l=0}^{n-1} \exp(-C_1 t^l) \\
 &\leq \exp(-C_1 t^n) (\kappa \|\mathbf{u}_h^0\|_0^2 + \|\mathbf{c}_h^0\|_{L^1}) + C_2^n, \quad (4.16)
 \end{aligned}$$

where

$$C_2^n = kd\beta|\Omega| \frac{1 - \exp(-C_1 t^n)}{1 - \exp(-C_1 k)}. \quad (4.17)$$

It is clear that we can choose generic constants c_1 and c_2 such that

$$\kappa \|\mathbf{u}_h^0\|_0^2 + \|\mathbf{c}_h^0\|_{L^1} \leq c_1 (\operatorname{Re} \|\mathbf{u}_h^0\|_0^2 + \|\mathbf{c}_h^0\|_{L^1}) \quad \text{and} \quad C_2^n \leq c_2. \quad (4.18)$$

We then obtain the desired result (4.3).

We now drive the other estimate (4.4). First we multiply $2k$ to both sides of (4.13) and take summation for $l = 1, 2, \dots, n$ for both sides to obtain:

$$\begin{aligned}
 2\eta_s \sum_{l=1}^n k \|\mathcal{D}(\mathbf{u}_h^l)\|_0^2 &\leq c_1 (\operatorname{Re} \|\mathbf{u}_h^0\|_0^2 + \|\mathbf{c}_h^0\|_{L^1}) + \sum_{l=1}^n kd\beta|\Omega|, \\
 &\leq c_1 (\operatorname{Re} \|\mathbf{u}_h^0\|_0^2 + \|\mathbf{c}_h^0\|_{L^1}) + c_2 t^n.
 \end{aligned}$$

This completes the proof. \square

4.2. Uniform boundedness

We now establish that the sequence generated from Algorithm 1 is bounded uniformly. To begin with, we shall recall simple inverse inequalities⁵, which will play an important role to establish our main results.

$$\|\mathbf{v}\|_\infty \lesssim h^{-1}\|\mathbf{v}\|_0 \quad \text{and} \quad \|\nabla\mathbf{v}\|_0 \lesssim h^{-1}\|\mathbf{v}\|_0, \quad \forall \mathbf{v} \in \mathbf{V}_h. \quad (4.19)$$

Lemma 4.2 (Uniform Boundedness). *Suppose that $\mathbf{f} \in (L^2(\Omega))^d$. For sufficiently small k , the sequence generated by Algorithm 1 is uniformly bounded in L^2 -norm for the velocity and L^1 -norm for the stress field, respectively.*

Proof. Using the strong divergence-free finite elements as in §3, we have $\nabla \cdot \mathbf{u}_h^{n,\ell} = 0$, for $\ell = 0, 1, 2, \dots$. It holds that $\det(\nabla \tilde{\mathbf{y}}^n) = 1$. Employing the energy method and the inverse inequality (4.19), we derive that

$$\begin{aligned} \operatorname{Re} \|\mathbf{u}_h^{n,\ell+1}\|_0^2 + k \|\nabla \mathbf{u}_h^{n,\ell+1}\|_0^2 &\leq \frac{\operatorname{Re}}{2} \|\mathbf{u}_h^{n,0}\|_0^2 + \frac{\operatorname{Re}}{2} \|\mathbf{u}_h^{n,\ell+1}\|_0^2 + k \|\mathbf{f}\|_0 \|\mathbf{u}_h^{n,\ell+1}\|_0 \\ &\quad + k \|\mathbf{c}_h^{n,\ell}\|_{L^1} \|\nabla \mathbf{u}_h^{n,\ell+1}\|_\infty. \end{aligned}$$

Therefore, using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \operatorname{Re} \|\mathbf{u}_h^{n,\ell+1}\|_0^2 + k \|\nabla \mathbf{u}_h^{n,\ell+1}\|_0^2 &\leq \frac{\operatorname{Re}}{2} \|\mathbf{u}_h^{n,0}\|_0^2 + \frac{\operatorname{Re}}{2} \|\mathbf{u}_h^{n,\ell+1}\|_0^2 \\ &\quad + k \left(\frac{\nu^{-2}}{2} \|\mathbf{f}\|_0^2 + \frac{\nu^2}{2} \|\mathbf{u}_h^{n,\ell+1}\|_0^2 \right) + k \left(\frac{C_1 h^{-2}}{2} \|\mathbf{c}_h^{n,\ell}\|_{L^1}^2 + \frac{1}{2} \|\nabla \mathbf{u}_h^{n,\ell+1}\|_0^2 \right), \end{aligned}$$

where ν is chosen, such that $|\mathbf{u}_h^{n,\ell+1}|_1^2 \geq \nu^2 \|\mathbf{u}_h^{n,\ell+1}\|_0^2$. We can then get, for all $\ell = 0, 1, 2, \dots$, that

$$\operatorname{Re} \|\mathbf{u}_h^{n,\ell+1}\|_0^2 \leq \operatorname{Re} \|\mathbf{u}_h^{n,0}\|_0^2 + C_1 k h^{-2} \|\mathbf{c}_h^{n,\ell}\|_{L^1}^2 + C_0 k \|\mathbf{f}\|_0^2, \quad (4.20)$$

where C_0 and C_1 are generic constants independent of k and h . Without loss of generality, we can assume that both C_0 and C_1 are greater than 1.

The equation (3) in Step 1 for updating the conformation tensor reveals the following inequality

$$\|\mathbf{c}_h^{n,\ell+1}\|_{L^1} \leq \|\mathbf{F}_h^{n,\ell+1}\|_\infty^2 \|\mathbf{c}_h^{n,0}\|_{L^1} + d|\Omega|\beta k. \quad (4.21)$$

Combining the last two inequalities, (4.20) and (4.21), we obtain

$$\begin{aligned} \operatorname{Re} \|\mathbf{u}_h^{n,\ell+1}\|_0^2 &\leq \operatorname{Re} \|\mathbf{u}_h^{n,0}\|_0^2 + 2C_1 k h^{-2} \|\mathbf{F}_h^{n,\ell}\|_\infty^4 \|\mathbf{c}_h^{n,0}\|_{L^1}^2 \\ &\quad + C_0 k \|\mathbf{f}\|_0^2 + 2C_1 d^2 |\Omega|^2 \beta^2 k^2. \end{aligned} \quad (4.22)$$

Now we define

$$\bar{C} := (\operatorname{Re} \|\mathbf{u}_h^{n,0}\|_0^2 + 4C_1 \|\mathbf{c}_h^{n,0}\|_{L^1}^2 + C_0 \|\mathbf{f}\|_0^2 + 2C_1 d^2 |\Omega|^2 \beta^2)^{\frac{1}{2}}.$$

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And, we will show that, if k is small enough, \bar{C} is a uniform upper bound for $\|\mathbf{u}_h^{n,\ell}\|_0$ and $\|\mathbf{c}_h^{n,\ell}\|_{L^1}$. This is apparently true for $\ell = 0$. Now suppose that this is also true for ℓ , we can now prove that it is true for $\ell + 1$ with a fixed time step size k .

Using the inequality (3.19) and the inverse inequality, we have

$$\|\Pi_h^S(\nabla \mathbf{u}_h^{n,\ell})\|_\infty \leq h^{-1} \|\nabla \mathbf{u}_h^{n,\ell}\|_0 \leq C_2 h^{-2} \|\mathbf{u}_h^{n,\ell}\|_0 \leq \bar{C} C_2 h^{-2}.$$

Hence, we can choose $2\bar{C}C_2k \leq h^2$, which implies that $\boldsymbol{\delta} - k\Pi_h^S(\nabla \mathbf{u}_h^{n,\ell})$ is invertible and that $\mathbf{F}_h^{n,\ell}$ is well-defined. Furthermore, we also have that

$$\|\mathbf{F}_h^{n,\ell}\|_\infty \leq \frac{1}{1 - kh^{-1} \|\nabla \mathbf{u}_h^{n,\ell}\|_0} \leq \frac{1}{1 - \bar{C}C_2kh^{-2}} \leq 2.$$

Plugging the inequality above to (4.22), we obtain

$$\text{Re} \|\mathbf{u}_h^{n,\ell+1}\|_0^2 \leq \text{Re} \|\mathbf{u}_h^{n,0}\|_0^2 + 32C_1kh^{-2} \|\mathbf{c}_h^{n,0}\|_{L^1}^2 + C_0k \|\mathbf{f}\|_0^2 + 2C_1d^2|\Omega|^2\beta^2k^2 \leq \bar{C}^2,$$

if $k \leq \min(1, h^2/8)$. The equation (4.21) then immediately implies that $\|\mathbf{c}_h^{n,\ell+1}\|_{L^1}$ is bounded uniformly, which completes the proof. \square

We would like to note that from the proof, we can see that there is an upper bound for the time step size:

$$k \leq \min\left(1, \frac{h^2}{8}, \frac{h^2}{2\bar{C}C_2}\right),$$

which depends on neither nonlinear iteration step ℓ nor time level n . Furthermore, this still holds for the infinite Weissenberg number case where $\alpha = \beta = 0$.

4.3. Existence and uniqueness

Lemma 4.3 (Positivity of the Discrete Conformation Tensor). *If the initial condition \mathbf{c}_h^0 is symmetric positive-definite and the time step size k is small enough, then the discrete conformation tensor \mathbf{c}_h^n is always symmetric positive-definite for all $n = 1, 2, 3, \dots$*

Proof. It is trivial to the symmetry is kept for all n . We only need to check the positivity here. From the proof of Lemma 4.2, we have seen that, if k is small enough, $\mathbf{F}_h^{n,\ell}$ is well-defined. Since Π_h^S is a positivity-preserving interpolation and $\beta > 0$, we can obtain that $\mathbf{c}_h^{n,\ell}$ is positive-definite by induction. As this is true for all ℓ and n , it completes the proof. \square

We are in a position to show the existence of the solution from the compactness argument. To begin with, it is helpful to notice that for any invertible matrices \mathbf{A} and \mathbf{B} , we have that

$$\mathbf{A}^{-1} - \mathbf{B}^{-1} = \mathbf{A}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{B}^{-1}. \quad (4.23)$$

We will show that the sequence converges to a unique limit and conclude our main result in this section.

Theorem 4.1 (Convergence of Algorithm 1). *The nonlinear iteration in Algorithm 1 converges if k small enough.*

Proof. To ease our presentation, we define

$$\mathcal{E}_{\mathbf{u}}^{\ell+1} := \mathbf{u}_h^{n,\ell+1} - \mathbf{u}_h^{n,\ell} \quad \text{and} \quad \mathcal{E}_{\mathbf{c}}^{\ell+1} := \mathbf{c}_h^{n,\ell+1} - \mathbf{c}_h^{n,\ell}.$$

By subtracting the momentum equation for $\mathbf{u}_h^{n,\ell+1}$ from the equation for $\mathbf{u}_h^{n,\ell}$ in Algorithm 1 and taking integration by parts, we obtain that

$$\operatorname{Re} \|\mathcal{E}_{\mathbf{u}}^{\ell+1}\|_0^2 + k \|\nabla \mathcal{E}_{\mathbf{u}}^{\ell+1}\|_0^2 \leq k \|\nabla \mathcal{E}_{\mathbf{u}}^{\ell+1}\|_{\infty} \|\mathcal{E}_{\mathbf{c}}^{\ell}\|_{L^1}.$$

Therefore, by the inverse inequality (4.19) and the Cauchy–Schwarz inequality, we conclude that

$$\operatorname{Re} \|\mathcal{E}_{\mathbf{u}}^{\ell+1}\|_0^2 \leq C_3 k h^{-2} \|\mathcal{E}_{\mathbf{c}}^{\ell}\|_{L^1}^2, \quad \forall \ell = 0, 1, 2, \dots \quad (4.24)$$

By subtracting the constitutive equations for $\mathbf{c}_h^{n,\ell+1}$ and $\mathbf{c}_h^{n,\ell}$, we obtain the following inequality:

$$\|\mathcal{E}_{\mathbf{c}}^{\ell+1}\|_{L^1} \leq \|\mathbf{F}_h^{n,\ell+1} \Pi_h^S(\mathbf{c}_h^{n,0} \circ \tilde{y}^n)(\mathbf{F}_h^{n,\ell+1})^T - \mathbf{F}_h^{n,\ell} \Pi_h^S(\mathbf{c}_h^{n,0} \circ \tilde{y}^n)(\mathbf{F}_h^{n,\ell})^T\|_{L^1}.$$

In the proof of Lemma 4.2, we have seen that $\mathbf{F}_h^{n,\ell}$ is well-defined and also that the $\|\mathbf{F}_h^{n,\ell}\|_0 \leq 2$ for $\ell = 0, 1, 2, \dots$. Therefore, we have

$$\begin{aligned} \|\mathcal{E}_{\mathbf{c}}^{\ell+1}\|_{L^1} &\leq \|\mathbf{F}_h^{n,\ell+1} \Pi_h^S(\mathbf{c}_h^{n,0} \circ \tilde{y}^n)(\mathbf{F}_h^{n,\ell+1})^T - \mathbf{F}_h^{n,\ell} \Pi_h^S(\mathbf{c}_h^{n,0} \circ \tilde{y}^n)(\mathbf{F}_h^{n,\ell+1})^T\|_{L^1} \\ &\quad + \|\mathbf{F}_h^{n,\ell} \Pi_h^S(\mathbf{c}_h^{n,0} \circ \tilde{y}^n)(\mathbf{F}_h^{n,\ell+1})^T - \mathbf{F}_h^{n,\ell} \Pi_h^S(\mathbf{c}_h^{n,0} \circ \tilde{y}^n)(\mathbf{F}_h^{n,\ell})^T\|_{L^1}. \end{aligned}$$

Using an argument similar to that in the proof of Lemma 4.2, we obtain

$$\begin{aligned} \|\mathcal{E}_{\mathbf{c}}^{\ell+1}\|_{L^1} &\leq 2(\|\mathbf{F}_h^{n,\ell+1}\|_{\infty} + \|\mathbf{F}_h^{n,\ell}\|_{\infty}) \|\mathbf{c}_h^{n,0}\|_{L^1} \|\mathbf{F}_h^{n,\ell+1} - \mathbf{F}_h^{n,\ell}\|_{\infty} \\ &\leq 8 \|\mathbf{c}_h^{n,0}\|_{L^1} \|\mathbf{F}_h^{n,\ell+1} - \mathbf{F}_h^{n,\ell}\|_{\infty} \leq 32 k h^{-1} \|\mathbf{c}_h^{n,0}\|_{L^1} \|\nabla \mathbf{u}_h^{n,\ell+1} - \nabla \mathbf{u}_h^{n,\ell}\|_0, \end{aligned}$$

where the last inequality is from the inverse inequality and the following fact:

$$\begin{aligned} \mathbf{F}_h^{n,\ell+1} - \mathbf{F}_h^{n,\ell} &= \mathbf{F}_h^{n,\ell+1} \left((\mathbf{F}_h^{n,\ell})^{-1} - (\mathbf{F}_h^{n,\ell+1})^{-1} \right) \mathbf{F}_h^{n,\ell} \\ &= \mathbf{F}_h^{n,\ell+1} \left(\boldsymbol{\delta} - k \Pi_h(\nabla \mathbf{u}_h^{n,\ell}) - \boldsymbol{\delta} + k \Pi_h(\nabla \mathbf{u}_h^{n,\ell+1}) \right) \mathbf{F}_h^{n,\ell} \\ &= k \mathbf{F}_h^{n,\ell+1} \Pi_h(\nabla \mathbf{u}_h^{n,\ell+1} - \nabla \mathbf{u}_h^{n,\ell}) \mathbf{F}_h^{n,\ell}. \end{aligned}$$

By invoking the inverse inequality again, we conclude that

$$\operatorname{Re} \|\mathcal{E}_{\mathbf{u}}^{\ell+1}\|_0^2 \leq C_3 k h^{-2} \|\mathcal{E}_{\mathbf{c}}^{\ell}\|_{L^1}^2 \leq C_4 k^3 h^{-4} \|\mathbf{c}_h^{n,0}\|_{L^1}^2 \|\mathcal{E}_{\mathbf{u}}^{\ell}\|_0^2. \quad (4.25)$$

For sufficiently small k , more specifically $C_4 \bar{C}^2 k^3 h^{-4} \leq 1/2$, the equation (4.25) implies that the sequences $\{\|\mathcal{E}_{\mathbf{u}}^{\ell}\|_0\}$ and $\{\|\mathcal{E}_{\mathbf{c}}^{\ell}\|_{L^1}\}$ are Cauchy sequences. Hence, $\mathbf{u}_h^{n,\ell}$ converges in the L^2 sense and $\mathbf{c}_h^{n,\ell}$ converges in the L^1 sense. \square

Theorem 4.2 (Global existence of the discrete solution). *For any initial guess \mathbf{u}_h^0 and \mathbf{c}_h^0 , there is a positive constant κ_0 , such that the discrete systems (2.17) have a unique solution for all $n \geq 0$ as long as $k \leq \kappa_0 h^2$.*

Proof. This theorem follows directly from Theorem 4.1 by noting that, in its proof, the time step size k and all other constants that appear in the proof of Theorem 4.1 are independent of the time level t^n . \square

4.4. Computational complexity

We shall now discuss the computational complexity of Algorithm 1. First of all, if k is small enough, then the nonlinear equation for the flow map in Step 1, Algorithm 1 is solvable. Specifically, Step 1 can be solved in a fixed number of iterations. Secondly, Theorem 4.1 guarantees that the fixed-point iteration for solving the coupled system in Step 2 can also be terminated in a finite number of iterations to any give tolerance.

Finally, Stokes-type systems can be solved by optimal multilevel methods independent of h , k , Re , and η_s . For example, we can apply the preconditioned minimum residual (MinRes) method with the diagonal block preconditioners^{31,4}, in which we need to solve $d + 1$ Poisson-type linear systems at each iteration. The MinRes with this preconditioner is uniform and user-friendly; furthermore, it can be easily parallelized; see Xu³⁷. An alternative effective solver for the Stokes-type systems can be designed combining the Augmented Lagrange methods^{13,14} and the multigrid methods based on block Gauss-Seidel smoothers as discussed in Lee et al.^{21,22}. We, therefore, arrive at the following optimality result:

Corollary 4.1 (Computation Complexity). *If the time step size k is small enough, Algorithm 1 converges uniformly with respect to Re and Wi and the computational complexity is $O(N \log N)$, where N is the total spatial degrees of freedom.*

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