

# Uniform Stability Analysis of Austin, Manteuffel and McCormick Finite Elements and Fast and Robust Iterative Methods for the Stokes-like Equations

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## SUMMARY

In this paper, we discuss uniformly stable discretizations and fast solvers for the steady and semi-discrete time dependent Stokes equations, which will be called the Stokes-like equations. We study two pairs of conforming finite elements that are uniformly accurate with respect to all relevant parameters in the equations, one of which is the Scott-Vogelius [17] element and the other of which is the finite element pair by Austin, Manteuffel and McCormick, [4]. We prove that the finite element pair by Austin, Manteuffel and McCormick is uniformly accurate and we design and analyze the fast and robust solution techniques for the linear algebraic systems arising from the discretizations of the Stokes-like equations as well. Our method is based on the Augmented Lagrangian Uzawa iterative methods, [8] and the multigrid methods designed by Austin, Manteuffel and McCormick, [4] for the linear algebraic equations from the discretizations of the differential operator  $\rho^2 \mathbf{I} - \kappa^2 \Delta - \mu^2 \nabla \text{div}$ . We prove our methods are robust with respect to all relevant parameters that appear in the equations and also the mesh size, thereby, achieve the fast and robust methods for the solutions to the Stokes-like equations. In particular, the convergence analysis for the multigrid methods designed by Austin, Manteuffel and McCormick, has been posed as an open question, [4], which is resolved in this paper. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: *Stokes Equations, Fast Solver, Multigrid, Finite Elements*

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## 1. Introduction

In this paper, we shall discuss the solution to the following equations :

$$\rho^2 \mathbf{u} - \kappa^2 \Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.1)$$

where  $\mathbf{u}, p$  are velocity and pressure and  $\mathbf{f}$  is the body force. We assume that the no-slip boundary condition that  $\mathbf{u} = 0$  is given on  $\partial\Omega$ . Two constants in (1.1),  $\rho^2$  and  $\kappa^2$  are material dependent parameters. Throughout this paper, we shall assume that  $\Omega$  is a bounded polygonal domain in  $\mathbb{R}^2$ . Note that we do **not** assume here that  $\Omega$  is convex. When it comes to discretize the problem (1.1) by the finite elements by Austin, Mantueffel and McCormick, [4], we shall further require that  $\Omega$  is a domain composed of number of rectangles. We shall call the equations (1.1) the Stokes-like equation since these arise from the temporal discretizations of the time dependent Stokes equations. Navier-Stokes equations or PDEs that describe Non-Newtonian flows can also reduce to the Stokes-like equations when they are discretized employing the Eulerian-Lagrangian discretizations in time, (see [14] and references cited therein), which are author's main concern and interest.

For simplicity, we use the notation  $\lesssim$  or  $\gtrsim$ . Namely, we write  $x_1 \lesssim x_2$  or  $x_1 \gtrsim x_2$ , when there exist generic constants  $C_1$  and  $C_2$  (independent of time step size  $h_t$ , mesh size  $h$  or some other important parameters such as physical quantities), such that

$$x_1 \leq C_1 x_2 \quad \text{and} \quad x_1 \geq C_2 x_2. \quad (1.2)$$

We also introduce the standard notation of the Sobolev spaces.  $H^k(\Omega)$  denotes the Sobolev space of scalar functions on  $\Omega$  whose derivatives up to order  $k$  are square integrable, with the norm  $\|\cdot\|_k$ . The notation  $|\cdot|_k$  denotes the semi-norm. We shall also discuss the corresponding spaces restricted to the sub-domain of  $\Omega$ . For any  $\Omega' \subset \Omega$ , we denote  $\|\cdot\|_{k,\Omega'}$  and  $|\cdot|_{k,\Omega'}$  denotes the norm  $\|\cdot\|_k$  and the semi-norm  $|\cdot|_k$  restricted to the domain  $\Omega'$ . The space  $L_0^2(\Omega)$  denotes the subspace of  $L^2(\Omega)$ , the space of square integrable functions for which the integral of the functions are zero. The space  $H_0^k(\Omega)$  denotes the subspace of  $H^k(\Omega)$  whose trace on the boundary is zero. The symbols  $\operatorname{div}$  or **curl** shall denote the usual divergence and curl operators, respectively, and for any vector  $\mathbf{u} \in (H^1(\Omega'))^2$ , we shall consider the semi-norm  $|\mathbf{u}|_{\operatorname{div},\Omega'}$ , which denotes  $\|\operatorname{div} \mathbf{u}\|_{0,\Omega'}$ . For the case  $\Omega' = \Omega$ , we shall simply identify  $\|\cdot\|_{k,\Omega}$ ,  $|\cdot|_{k,\Omega}$  and  $|\cdot|_{\operatorname{div},\Omega}$  with  $\|\cdot\|_k$ ,  $|\cdot|_k$  and  $|\cdot|_{\operatorname{div}}$ , respectively.

Throughout this paper, we shall restrict the variations of the parameters  $\rho^2$  and  $\kappa^2$  so that

$$0 \leq \rho^2 \lesssim 1, \quad \text{and} \quad 0 < \kappa^2 \lesssim 1. \quad (1.3)$$

Since it is the case for quite *general* physical problems of concern. To be more precise, we consider a non-dimensional temporal discrete system of momentum equation for viscoelastic

models, given by  $(\text{Re}/h_t)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \mathbf{f}$ , where  $\text{Re}$  is the Reynolds number, where  $\mu \leq 1$  is the Newtonian viscosity and  $h_t$  is the time step size. For the viscoelastic flows, it is generally true that  $\text{Re}$  is vanishingly small, [16] and therefore, by taking  $\rho^2 = \text{Re}$  and  $\kappa^2 = h_t\mu$ , the assumption (1.3) can be shown to be valid. On the other hand, for the case when  $\text{Re}$  is large, one can rescale the equations so that  $\rho^2 = 1$  and  $\kappa^2 = h_t\mu/\text{Re}$ , which make the condition (1.3) valid.

Our aim and contributions in this paper can be summarized below.

1. To show that the finite element introduced by Austin, Mantueffel and McCormick enjoys the parameters independent optimal approximation properties for the equations (1.1). In fact, the author has to modify their elements introduced in [4] and show that the modified finite element spaces satisfy the uniform stability property.
2. To develop and justify fast and robust iterative methods for the linear algebraic systems arising from the discretization of the equations (1.1). Based on the Augmented Lagrangian Uzawa methods, [8] and the multigrid iterative methods developed by Austin, Manteuffel and McCormick, [4], for the linear algebraic equation that arise from the discretizations of the operator  $\rho^2\mathbf{I} - \kappa^2\Delta - \mu^2\nabla\text{div}$ , we design fast and robust iterative methods for the equation (1.1). In particular, the convergence analysis of the multigrid methods designed for the aforementioned operator equations is missing in the literature and posed as an open question by Austin, Manteuffel and McCormick, [4]. We present a rigorous convergence analysis for the multigrid methods and, thereby, resolve the open question.

To elaborate the second contribution in this paper further, we shall cast the discrete analogue of equations (1.1) into the following operator equations:

$$\begin{pmatrix} A_p & B^* \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h \\ p_h \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}, \tag{1.4}$$

where  $A_p = \rho^2\mathbf{I}_h - \kappa^2\Delta_h$ ,  $B^* = \nabla_h$ ,  $B = -\text{div}$  and  $\mathbf{u}_h, p_h$  are the discrete approximation for the solutions  $\mathbf{u}$  and  $p$  with  $\mathbf{I}_h$ , the identity operator,  $\Delta_h$  is the discrete Laplacian and  $\nabla_h$  is the discrete gradient. To solve the equation (1.4), we shall apply the Augmented Lagrangian Uzawa iterative method, [8], which can be interpreted as the Uzawa method, [7] applied to the following problem :

$$\begin{pmatrix} A_p + \mu^2 B^* B & B \\ B^* & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h \\ p_h \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}, \tag{1.5}$$

where  $\mu^2 \geq 0$  is some parameter which can be chosen in an arbitrary manner. Namely, for a

given initial iterate,  $(\mathbf{u}^0, p^0)$ , we generate iterates  $(\mathbf{u}^\ell, p^\ell)$  by the following relation that

$$(A_p + \mu^2 B^* B) \mathbf{u}^{\ell+1} = \mathbf{f} - B p^\ell \quad (1.6)$$

$$p^{\ell+1} = p^\ell + \mu^2 B \mathbf{u}^{\ell+1}. \quad (1.7)$$

For  $\mu^2 \gg 1$ , the convergence rate  $\sigma$  estimate for the aforementioned methods have been well-known to be such that  $\sigma \lesssim O(\mu^{-2})$ , [12]. However, the price to pay is to invert the operator  $A_p + \mu^2 B^* B$ , [11], which is given by the following discrete operator :

$$\rho^2 \mathbf{I} - \kappa^2 \Delta_h - \mu^2 \nabla_h \operatorname{div} \quad (1.8)$$

It is clear then that if one can handle the operator equation (1.8), we achieve the fast methods for the solution to the Stokes-like equation from the aforementioned arguments. In fact, the multigrid algorithms have been well-known and analyzed for the case  $\kappa = 0$ , [2]. However, for the case when  $\rho, \kappa \neq 0$ , although the multigrid methods have been designed and tested by Austin, Manteuffel and McCormick, [4], the convergence analysis is apparently missing and it is posed as an open question. On the other hand, the multigrid convergence analysis for the equation (1.8) has been worked out recently by author and his collaborators, [11] for the particular case when  $\rho = 0$  and Scott-Vogelius finite elements, [17] have been used for the discretizations. We hereby, extend the multigrid analysis presented in [11] to the case when  $\rho \neq 0$  using somewhat different, but standard techniques.

The rest of the paper is organized as follows. In §2, we study uniformly finite elements for the Stokes-like equations, (1.1). In particular, we review conditions that the finite elements pair should satisfy for which the uniform stability of solutions and error estimates can be obtained. In §3, we review the Augmented Lagrangian Uzawa iterative methods and some issues to make it fast and robust methods and perform the convergence analysis for the multigrid methods that are needed to design robust and efficient iterative solvers for the solution to the Stokes-like equations. In §4, we take the finite elements introduced by Austin, Manteuffel and McCormick, [4] and show that they are uniformly stable finite elements and satisfy all issues that are necessary to develop the fast and robust iterative methods for the Stokes-like equations. We conclude the paper with some remarks in §5.

## 2. Uniformly Accurate Finite Elements

We shall consider the mixed finite element discretizations of the equations (1.1) and study uniformly stable or accurate finite element methods with respect to two parameters  $\rho$  and  $\kappa$ . Our discussion in this section is based mainly on the recent work by Xie, Xue and Xu, [22].

We begin by formulating the weak form of the equation (1.1) as follows: Find  $(\mathbf{u}, p) \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$  such that

$$\begin{cases} a_p(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in (H_0^1(\Omega))^2 \\ b(\mathbf{u}, q) &= 0, \quad \forall q \in L_0^2(\Omega) \end{cases} \quad (2.1)$$

where  $\langle \cdot, \cdot \rangle$  is the dual pairing,

$$a_p(\mathbf{u}, \mathbf{v}) = \rho^2(\mathbf{u}, \mathbf{v})_{0,\Omega} + \kappa^2(\nabla \mathbf{u}, \nabla \mathbf{v})_{0,\Omega} \quad \text{and} \quad b(\mathbf{v}, p) = - \int_{\Omega} \operatorname{div} \mathbf{v} p \, dx.$$

Now, we introduce a norm denoted by  $||| \cdot |||$  on  $(H_0^1(\Omega))^2$  as follows :

$$|||\mathbf{u}|||^2 := \|\mathbf{u}\|_{a_p}^2 + \|\operatorname{div} \mathbf{u}\|_0^2, \quad \forall \mathbf{u} \in (H_0^1(\Omega))^2, \quad (2.2)$$

where  $\|\mathbf{u}\|_{a_p}^2 = a_p(\mathbf{u}, \mathbf{u}) = \rho^2\|\mathbf{u}\|_0^2 + \kappa^2\|\mathbf{u}\|_1^2$ . It is easy to show that the bilinear forms  $a_p(\cdot, \cdot) : (H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2 \mapsto \mathbb{R}$  and  $b(\cdot, \cdot) : (H_0^1(\Omega))^2 \times L_0^2(\Omega) \mapsto \mathbb{R}$  are continuous in the following senses:

$$a_p(\mathbf{u}, \mathbf{v}) \lesssim |||\mathbf{u}||| |||\mathbf{v}|||, \quad \text{and} \quad b(\mathbf{v}, q) \lesssim |||\mathbf{v}||| \|q\|_0, \quad \forall \mathbf{u}, \mathbf{v} \in (H_0^1(\Omega))^2, q \in L_0^2.$$

We now note that the following inequalities hold true. First of all, we have

$$a_p(\mathbf{u}, \mathbf{u}) = |||\mathbf{u}|||^2, \quad \forall \mathbf{u} \in \{\mathbf{v} \in (H_0^1(\Omega))^2 : \operatorname{div} \mathbf{v} = 0\} \quad (2.3)$$

and from the fact that

$$\sup_{\mathbf{v} \in (H_0^1(\Omega))^2} \frac{b(\mathbf{v}, q)}{|||\mathbf{v}|||} \gtrsim \|q\|_0, \quad \forall q \in L_0^2, \quad \text{and} \quad \|\operatorname{div} \mathbf{u}\|_0 \leq \|\mathbf{u}\|_1, \quad \forall \mathbf{u} \in (H_0^1(\Omega))^2, \quad (2.4)$$

we can deduce that since  $\rho^2$  and  $\kappa^2 \lesssim 1$ ,

$$\sup_{\mathbf{v} \in (H_0^1(\Omega))^2} \frac{b(\mathbf{v}, q)}{|||\mathbf{v}|||} \gtrsim \sup_{\mathbf{v} \in (H_0^1(\Omega))^2} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_1} \gtrsim \|q\|_0, \quad \forall q \in L_0^2. \quad (2.5)$$

The two inequalities (2.3) and (2.5) show that the equation (2.1) is uniformly stable. These considerations can be transferred to the discrete levels. Let us assume that we are given an appropriate finite element spaces,  $\mathbf{V}_h \subset (H_0^1(\Omega))^2$  for the velocity and  $S_h \subset L_0^2(\Omega)$  for the pressure and consider the discrete weak formulation: Find  $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times S_h$  such that

$$\begin{cases} a_p(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= \langle \mathbf{f}, \mathbf{v}_h \rangle, \quad \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h, q_h) &= 0, \quad \forall q_h \in S_h. \end{cases} \quad (2.6)$$

As demonstrated in Xie, Xue and Xu [22], the main issues on uniform accuracy or stability can be achieved if the finite element pairs  $\mathbf{V}_h \times S_h$  satisfies the following two conditions :

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \gtrsim \|q_h\|_0^2, \quad \forall q_h \in S_h. \quad (2.7)$$

$$\operatorname{div} \mathbf{V}_h \subset S_h, \quad (2.8)$$

where  $\operatorname{div} \mathbf{V}_h = \{\operatorname{div} \mathbf{u}_h : \forall \mathbf{u}_h \in \mathbf{V}_h\}$ . More precisely, under the aforementioned two conditions (2.7) and (2.8), we can easily show that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|} \gtrsim \|q_h\|_0^2, \quad \forall q_h \in S_h. \quad (2.9)$$

$$a(\mathbf{u}_h, \mathbf{u}_h) \gtrsim \|\mathbf{u}_h\|^2, \quad \forall \mathbf{u}_h \in \{\mathbf{v}_h \in \mathbf{V}_h : \operatorname{div} \mathbf{v}_h = 0.\} \quad (2.10)$$

Therefore, for the uniform accuracy of the Stokes-like equations, we have to devise finite element pairs  $\mathbf{V}_h$  and  $S_h$  so that two conditions (2.7) and (2.8) are satisfied. In fact, the Scott-Vogelius finite elements are well-known to satisfy both conditions, so they are uniformly stable pairs. To author's best knowledge, except for the very recent work by S. Zhang, [25] there are no other conforming uniformly stable finite elements to date. However, as stated in the previous section, in fact, the finite elements (more precisely, modified finite elements) introduced by Austin, Manteuffel and McCormick, can be shown to satisfy both conditions 2.3 and 2.5. We study and prove the uniform stability analysis of the Austin, Manteuffel and McCormick finite elements in §4.

### 3. Fast and Robust Iterative methods for the Stokes-like equations

In this section, we shall consider the fast solution techniques for the mixed finite element formulation, (2.6) of the Stokes-like equation (1.1) for which, the uniformly stable conforming finite elements such as Scott-Vogelius elements, [17] and the finite element pairs by Austin, Manteuffel and McCormick, [4] are assumed to be employed. We note that the usual Hood-Taylor finite elements are not uniformly stable, [21] and therefore, it is not of our concern to develop the fast solution methods for the resulting system of equations.

Our solution techniques shall be based on the Augmented Lagrangian Uzawa iterative methods, and therefore, we shall consider the reformulation of the weak formulation (2.1) as follows:

$$\begin{cases} a_p(\mathbf{u}_h, \mathbf{v}_h) + \mu^2 a_s(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= \langle \mathbf{f}, \mathbf{v}_h \rangle, \quad \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h, q_h) &= 0, \quad \forall q_h \in S_h, \end{cases} \quad (3.1)$$

where  $a_s(\mathbf{u}_h, \mathbf{u}_h) = (\operatorname{div} \mathbf{u}_h, \operatorname{div} \mathbf{v}_h)_0$ . We note that under the conditions that  $\mathbf{V}_h$  and  $S_h$  are conforming uniformly stable elements, in particular, from the condition (2.8), the strong divergence free condition, it is easy to see that two formulations, (2.6) and (3.1) are equivalent. In this setting, the Augmented Lagrangian Uzawa method can be made fast and robust if one can invert the following operator equation fast and robust, [12], :

$$a_p(\mathbf{u}_h, \mathbf{v}_h) + \mu^2 a_s(\mathbf{u}_h, \mathbf{v}_h) = \langle \mathbf{g}, \mathbf{v}_h \rangle, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (3.2)$$

where  $\mathbf{g}$  is a function that depends on  $\mathbf{f}$  and  $p^\ell$  in the equation (1.6). Therefore, the design of the fast methods will be made from the fast and robust solution methods for the equation (3.2), which will be made by the multigrid techniques. Our main concern here is then how to construct the fast and robust multigrid methods for the parameter dependent problems like (3.2). This is the main subject of the next subsection.

### 3.1. Multigrid Convergence Analysis for Parameter Dependent Elliptic Equations

Let  $V$  be a real Hilbert space with an inner product  $a(\cdot, \cdot)$  and (energy) norm  $\|\cdot\|_a = a(\cdot, \cdot)^{1/2}$ . The variational formulation of our interest in this section is as follows:

$$\text{Find } u \in V \text{ such that } a(u, v) = f(v), \quad \forall v \in V, \quad (3.3)$$

where the bilinear form  $a(\cdot, \cdot)$  can be decomposed into the following form:

$$a(u, v) = a_p(u, v) + \mu^2 a_s(u, v), \quad \forall u, v \in V, \quad (3.4)$$

where  $a_p(u, v) = \rho^2 a_1(u, v) + \kappa^2 a_{11}(u, v)$  and  $\rho, \kappa$  and  $\mu$  are parameters and the bilinear forms  $a_1(\cdot, \cdot)$ ,  $a_{11}(\cdot, \cdot)$  and  $a_s(\cdot, \cdot)$  are assumed to be

- (i)  $a_1$  and  $a_{11}$  are symmetric and positive definite,
- (ii)  $a_s$  is symmetric and positive semi-definite.

We shall denote  $\|\cdot\|_{a_p}$ ,  $\|\cdot\|_{a_1}$  and  $\|\cdot\|_{a_{11}}$  by the induced norms from the bilinear forms  $a_p$ ,  $a_1$  and  $a_{11}$ , respectively. Note that the bilinear form (3.2) is given by the following choice:

$$a_1(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{u} \mathbf{v} \, dx, \quad a_{11}(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \nabla \mathbf{u} \nabla \mathbf{v} \, dx, \quad \text{and} \quad a_s(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, dx.$$

In the following discussion, for any closed subspace  $W \subset V$ , by  $W^\perp$  we denote the orthogonal complement of  $W$  with respect to the inner product  $a(\cdot, \cdot)$  and by  $\mathcal{N}$  the null space of the bilinear form  $a_s$  defined through:

$$\mathcal{N} = \{v \in V : a_s(v, w) = 0, \quad \forall w \in V\}. \quad (3.5)$$

The construction of the general subspace correction methods is based on the space decomposition that the space  $V$  is decomposed into a number of subspaces  $V_k$ , with  $k = 1, \dots, J$  and the introduction of the local subspace solver in each subspace. In particular, to achieve the parameter independent convergence property of the method, the following formal sets of assumptions should be justified.

**A0** There exist a number of closed subspaces  $V_k$  with  $k = 1, \dots, J$  such that

$$V = \sum_{k=1}^J V_k.$$

**A1** The null space  $\mathcal{N}$  can be represented by sum of elements in subspaces, namely,

$$\mathcal{N} = \sum_{k=1}^J (V_k \cap \mathcal{N}).$$

For each subspace  $V_k$ , we define the orthogonal projection  $P_k : V \mapsto V_k$  with respect to  $a(\cdot, \cdot)$  inner product by

$$a(P_k v, v_k) = a(v, v_k), \quad \forall v \in V, v_k \in V_k. \quad (3.6)$$

Note that the bilinear form  $a(\cdot, \cdot)$  is coercive on  $V_k$  for each  $k = 1, \dots, J$ , therefore, the operator  $P_k$  is well-defined. These shall be used as our local subspace solvers. Now, all components of the subspace correction methods are in place. The method of subspace corrections can be stated as follows, [24, 13] :

**Algorithm 3.1 (MSSC)** *Let  $u^0 \in V$  be given.*

**for**  $\ell = 1, 2, \dots$

$$u_0^{\ell-1} = u^{\ell-1}$$

**for**  $i = 1, \dots, J$

*Let  $e_i \in V_i$  solve*

$$a(e_i, v_i) = f(v_i) - a(u_{i-1}^{\ell-1}, v_i), \quad \forall v_i \in V_i \quad (3.7)$$

$$u_i^{\ell-1} = u_{i-1}^{\ell-1} + e_i$$

**endfor**

$$u^\ell = u_J^{\ell-1}$$

**endfor**

First, we introduce for each  $k = 1, \dots, J$ , the local null space  $\mathcal{N}_k$  which are defined through

$$\mathcal{N}_k = \{u_k \in V_k : a_s(u_k, v_k) = 0, \quad \forall v_k \in V_k\}. \quad (3.8)$$

It is easy to establish the identity:  $\mathcal{N}_k = V_k \cap \mathcal{N}$ , [13]. The orthogonal complement of  $\mathcal{N}_k$  restricted to the subspace  $V_k$  would be denoted by (with some abuse of notation)  $\mathcal{N}_k^\perp$ . More precisely,

$$\mathcal{N}_k^\perp = \{u_k \in V_k : a(u_k, c_k) = 0, \quad \forall c_k \in \mathcal{N}_k\}. \quad (3.9)$$

For  $1 \leq k \leq J$ , the space  $V_k$  can be written as

$$V_k = \mathcal{N}_k \oplus \mathcal{N}_k^\perp. \quad (3.10)$$

Note that the orthogonality  $\oplus$  is with respect to  $a(\cdot, \cdot)$  inner product. Obviously it can be thought to be with respect to  $a_p(\cdot, \cdot)$  inner product, the symmetric positive part of the inner product  $a(\cdot, \cdot)$ . We shall introduce four additional projections with respect to the bilinear forms  $a_s(\cdot, \cdot)$ ,  $a_p(\cdot, \cdot)$ ,  $a_l(\cdot, \cdot)$  and  $a_{ll}(\cdot, \cdot)$  for the discussion that follows. For each  $k = 1, \dots, J$ , we define projections,  $P_{k,s} : V \mapsto \mathcal{N}_k^\perp$ ,  $P_{k,p} : V \mapsto V_k$ ,  $P_{k,l} : V \mapsto V_k$  and  $P_{k,ll} : V \mapsto V_k$  by

$$\begin{aligned} a_s(P_{k,s}v, v_k) &= a_s(v, v_k), \quad \forall v \in V, v_k \in V_k \\ a_p(P_{k,p}v, v_k) &= a_p(v, v_k), \quad \forall v \in V, v_k \in V_k \\ a_l(P_{k,l}v, v_k) &= a_l(v, v_k), \quad \forall v \in V, v_k \in V_k, \text{ and} \\ a_{ll}(P_{k,ll}v, v_k) &= a_{ll}(v, v_k), \quad \forall v \in V, v_k \in V_k. \end{aligned}$$

By the coercivity of  $a_s$  on  $\mathcal{N}_k^\perp$ , which is trivial in the finite dimensional case, we can prove that the operator  $P_{k,s}$  maps  $\mathcal{N}$  into  $\{0\}$ .

We now introduce a simple, but instrumental result.

**Lemma 3.1.** *The following inequality holds true:*

$$\frac{\|P_{k,p}v\|_{a_p}^2}{\|v\|_{a_p}^2} \leq \frac{\|P_{k,l}v\|_{a_l}^2}{\|v\|_{a_l}^2} + \frac{\|P_{k,ll}v\|_{a_{ll}}^2}{\|v\|_{a_{ll}}^2}, \quad \forall v \in \mathbf{V}. \quad (3.11)$$

**Proof** The proof can be done by invoking the Cauchy-Schwarz inequality and the definitions of projections as follows:

$$\begin{aligned} \|P_{k,p}v\|_{a_p}^2 &= a_p(P_{k,p}v, P_{k,p}v) = \rho^2 a_l(v, P_{k,p}v) + \kappa^2 a_{ll}(v, P_{k,p}v) \\ &= \rho^2 a_l(P_{k,l}v, P_{k,p}v) + \kappa^2 a_{ll}(P_{k,ll}v, P_{k,p}v) \\ &\leq \frac{1}{2} (\rho^2 a_l(P_{k,l}v, P_{k,l}v) + \kappa^2 a_{ll}(P_{k,ll}v, P_{k,ll}v)) \\ &\quad + \frac{1}{2} (\rho^2 a_l(P_{k,p}v, P_{k,p}v) + \kappa^2 a_{ll}(P_{k,p}v, P_{k,p}v)). \end{aligned}$$

Furthermore since  $\|v\|_{a_p}^2 = \rho^2\|v\|_{a_1}^2 + \kappa^2\|v\|_{a_{II}}^2$ . This completes the proof.  $\blacksquare$

The following result can be easily established from the assumptions **A0**, **A1** and the orthogonal decomposition (3.10), [11, 12] :

**Theorem 3.1.** *Under the assumptions **A0** and **A1**, the estimate of the energy norm for the error transfer operator  $E_J = (I - P_J) \cdots (I - P_1)$  can be established as follows:*

$$\|E_J\|_a^2 = 1 - \frac{1}{K}, \quad (3.12)$$

where

$$\begin{aligned} K &= \sup_{v \in V} \inf_{\sum_{k=1}^J v_k = v} \frac{\sum_{k=1}^J \left\| P_k \sum_{j \geq k} v_j \right\|_a^2}{(v, v)_a} \\ &\gtrsim \sup_{v \in \mathcal{N}^\perp} \inf_{\sum_{k=1}^J v_k = v} \left( \frac{\sum_{k=1}^J \left\| P_{k,s} \sum_{j \geq k} v_j \right\|_{a_s}^2}{(v, v)_{a_s}} + \frac{\sum_{k=1}^J \left\| P_{k,p} \sum_{j \geq k} v_j \right\|_{a_p}^2}{(v, v)_{a_p}} \right) \\ &\quad + \sup_{v_c \in \mathcal{N}} \inf_{\sum_{k=1}^J v_{k,c} = v_c} \frac{\sum_{k=1}^J \left\| P_{k,p} \sum_{j \geq k} v_{j,c} \right\|_{a_p}^2}{(v_c, v_c)_{a_p}}, \end{aligned} \quad (3.13)$$

where  $v_k$  belongs to  $V_k$  and  $v_{k,c}$ 's are in  $\mathcal{N}_k$  for each  $k = 1, \dots, J$ .

The estimate is still dependent on the parameters since  $a_p$  norm depends on  $\rho$  and  $\kappa$ . To completely eliminate parameter-dependent expressions for the estimates of  $K$ , we use the Lemma 3.1 to obtain

**Theorem 3.2.** *Under the same assumptions **A0** and **A1**, the quantity  $K$  in the Theorem 3.1 can be further estimated as follows:*

$$\begin{aligned} K &\gtrsim \sup_{v \in \mathcal{N}^\perp} \inf_{\sum_{k=1}^J v_k = v} \left( \frac{\sum_{k=1}^J \left\| P_{k,s} \sum_{j \geq k} v_j \right\|_{a_s}^2}{(v, v)_{a_s}} + \frac{\sum_{k=1}^J \left\| P_{k,I} \sum_{j \geq k} v_j \right\|_{a_I}^2}{(v, v)_{a_I}} \right. \\ &\quad \left. + \frac{\sum_{k=1}^J \left\| P_{k,II} \sum_{j \geq k} v_j \right\|_{a_{II}}^2}{(v, v)_{a_{II}}} \right) \\ &\quad + \sup_{v_c \in \mathcal{N}} \inf_{\sum_{k=1}^J v_{k,c} = v_c} \left( \frac{\sum_{k=1}^J \left\| P_{k,I} \sum_{j \geq k} v_{j,c} \right\|_{a_I}^2}{(v_c, v_c)_{a_I}} + \frac{\sum_{k=1}^J \left\| P_{k,II} \sum_{j \geq k} v_{j,c} \right\|_{a_{II}}^2}{(v_c, v_c)_{a_{II}}} \right) \end{aligned} \quad (3.14)$$

where  $v_k$  belongs to  $V_k$  and  $v_{k,c}$ 's are in  $\mathcal{N}_k$  for each  $k = 1, \dots, J$ .

**Proof** From the Theorem 3.1, we obtain that the constant  $K$  can be estimated as given in (3.13). We now apply the Lemma 3.1 to obtain the following inequality that

$$\frac{\sum_{k=1}^J \left\| P_{k,p} \sum_{j \geq k} v_j \right\|_{a_p}^2}{(v, v)_{a_p}} \lesssim \frac{\sum_{k=1}^J \left\| P_{k,l} \sum_{j \geq k} v_j \right\|_{a_l}^2}{(v, v)_{a_l}} + \frac{\sum_{k=1}^J \left\| P_{k,u} \sum_{j \geq k} v_j \right\|_{a_u}^2}{(v, v)_{a_u}}.$$

The inequality holds for  $v_j$  is replaced by  $v_{j,c}$ . This completes the proof.

### 3.2. Main Properties of (Multilevel) Finite Element Spaces and Design of the Robust Multigrid Methods

In this section, we shall summarize properties that can be found at both finite elements introduced by Scott-Vogelius, [17] and Austin, Manteuffel and McCormick, [4] and construct the Multilevel Finite Element Spaces based upon which the design and analysis of the robust multigrid method is performed. Throughout this section, we assume that we have a nested sequence of triangulations (or rectangulations)  $\mathcal{T}_k = \{\tau_k^i\}$ ,  $1 \leq k \leq L$  of  $\Omega$  with characteristic mesh size  $h_k$  proportional to  $\gamma^{2k}$  with  $\gamma \in (0, 1)$ . The nested sequence of triangles will be assumed to be formed in a way that the refined triangle is obtained by connecting the midpoints of the coarse triangles for simplicity. Let  $\mathcal{T}_h = \mathcal{T}_L$  and  $\mathbf{V}_k$  and  $S_k$  denote the spaces corresponding  $\mathbf{V}_h$  and  $S_h$  defined on the triangulations  $\mathcal{T}_k$ . In the rest of this section, to simplify notation, we shall omit the subscript  $h$  when referring to a fixed finest triangulation, namely,  $\mathbf{V} = \mathbf{V}_h = \mathbf{V}_L$ ,  $S = S_h = S_L$ , and etc. For the discussions that follow, we describe the space decompositions and subspace corrections, we let  $\{x_k^\ell\}_\ell$  be vertices of the triangulation  $\mathcal{T}_k$ . We denote  $\mathcal{T}_k^\ell$  by the set of triangles in  $\mathcal{T}_k$  meeting at the vertex  $x_k^\ell$  and set

$$\Omega_k^\ell = \bigcup_{\tau \in \mathcal{T}_k^\ell} \tau.$$

Namely,  $\Omega_k^\ell$  is the star or the patch of the vertex  $x_k^\ell$ . These form an overlapping covering of  $\Omega$  with the finite overlap property, namely, we have that

$$\sum_{\ell=1}^{N_k} |\Omega_k^\ell| \lesssim |\Omega_k|, \tag{3.15}$$

where  $|\Omega_k|$  and  $|\Omega_k^\ell|$  are the area of  $\Omega_k$  and  $\Omega_k^\ell$ , respectively. We build the finite element functions on  $\Omega_k^\ell$ . For  $1 \leq k \leq L$ ,

$$\mathbf{V}_k^\ell = \{\mathbf{v}_k \in \mathbf{V}_k : \text{supp}(\mathbf{v}_k) \subseteq \overline{\Omega_k^\ell}\}, \quad \forall 1 \leq \ell \leq N_k$$

and also the corresponding finite element spaces are defined, which will be denoted by  $W_k^\ell$  and  $S_k^\ell$ . It is then clear to see that

$$\mathbf{V} = \sum_{k=1}^L \mathbf{V}_k = \sum_{k=1}^L \sum_{\ell=1}^{N_k} \mathbf{V}_k^\ell, \quad (3.16)$$

where  $N_k$  is the number of vertices for the triangulation  $\mathcal{T}_k$ . We introduce additional projection operators, the  $L^2$  projection,  $Q_k : L^2(\Omega) \mapsto \mathbf{V}_k$ ,  $H^1$  projections  $P_k^w : H^1(\Omega) \mapsto W_k$  and  $P_k^v : H^1(\Omega) \mapsto \mathbf{V}_k$  for each  $k = 1, \dots, J$  defined as follows:

$$\begin{aligned} (Q_k \mathbf{u}, \mathbf{v}_k) &= (\mathbf{u}, \mathbf{v}_k), \quad \forall \mathbf{u} \in \mathbf{V}, \mathbf{v}_k \in \mathbf{V}_k \\ (P_k^w \phi, \phi_k)_1 &= (\phi, \phi_k)_1, \quad \forall \phi \in W, \phi_k \in W_k. \\ (P_k^v \mathbf{u}, \mathbf{u}_k)_1 &= (\mathbf{u}, \mathbf{u}_k)_1, \quad \forall \mathbf{u} \in (H^1(\Omega))^2, \mathbf{u}_k \in \mathbf{V}_k. \end{aligned}$$

The corresponding projections onto  $W_k^\ell$  and  $\mathbf{V}_k^\ell$  will be denoted by  $Q_k^\ell, P_k^{w,\ell}$  and  $P_k^{v,\ell}$ . The multigrid algorithm shall be constructed in particular, in terms of the algorithm (MSSC) with the local exact solver,  $P_k^{v,\ell}$  in each subspace  $\mathbf{V}_k^\ell$ .

We are in a position to state properties of the finite element spaces based on which the convergence of the multigrid methods can be obtained in an optimal fashion for the problem (3.4). We would like to remark that these are sufficient conditions and may not be necessary in a sense that although some of properties are not satisfied, the uniform estimate of  $K$  can possibly, be made. The interesting facts are that the uniform stability conditions for the finite element spaces for the Stokes-like equations are, in particular the crucial components of such properties.

B1. The spaces are nested, namely,

$$\mathbf{V}_1 \subset \dots \subset \mathbf{V}_k \subset \dots \subset \mathbf{V}_L \quad \text{and} \quad \mathcal{N}_1 \subset \dots \subset \mathcal{N}_k \subset \dots \subset \mathcal{N}_L.$$

This condition is true for both Scott-Vogelius finite elements, [11] and the finite element spaces by Austin, Manteuffel and McCormick, [4].

B2. The space  $\mathbf{V}$  is further decomposed so that the local problems are sufficiently small with **A1** being still valid, namely,

$$\mathbf{V} = \sum_{k=1}^L \mathbf{V}_k = \sum_{k=1}^L \sum_{\ell=1}^{N_k} \mathbf{V}_k^\ell, \quad \text{and} \quad \mathcal{N} = \sum_{k=1}^L \sum_{\ell=1}^{N_k} \mathbf{V}_k^\ell \cap \mathcal{N}. \quad (3.17)$$

This condition can be easily validated under the setting that  $\Omega_k^\ell$  is the patch of the vertex  $x_k^\ell$  and the finite element spaces  $\mathbf{V}_k^\ell$  is constructed based on this domain.

B3. For each  $1 \leq k \leq L$ , there exists a finite element space  $W_k$  that is contained in  $H_0^2(\Omega)$  and the following sequence is exact :

$$W_k \xrightarrow{\mathbf{curl}} \mathbf{V}_k \xrightarrow{\text{div}} S_k \longrightarrow^0 0.$$

Furthermore, the spaces are nested, namely,  $W_1 \subset \dots \subset W_k \subset \dots \subset W_L$ .

The exact sequence property can be shown by proving  $\mathbf{curl}W_k = \mathcal{N}_k$  and  $\text{div}\mathbf{V}_k = S_k$ . In fact, it is well-known for the Scott-Vogelius finite elements, [17]. However, the fact  $\text{div}\mathbf{V}_k = S_k$  seems to be nontrivial and a key factor for showing the stability for the finite element spaces by Austin, Manteuffel and McCormick, [4] and it is shown in §4.

B4. There exists an interpolation operators  $\Pi_h^v : (H_0^1(\Omega))^2 \mapsto \mathbf{V}$  such that

$$\Pi_h^v \mathbf{v} = \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{V}, \quad \text{and} \quad \|\mathbf{v} - \Pi_h^v \mathbf{v}\|_0 \lesssim h|\mathbf{v}|_1, \quad \forall \mathbf{v} \in (H_0^1(\Omega))^2.$$

B5. There exists an interpolation operators  $\Pi_h^w : H_0^2(\Omega) \mapsto W$  such that

$$\Pi_h^w q = q, \quad \forall q \in W_h, \quad \text{and} \quad \|q - \Pi_h^w q\|_0 \lesssim h^2|q|_2, \quad \forall q \in H_0^2(\Omega).$$

For the Scott-Vogelius finite elements, the operator  $\Pi_h^v$  can be constructed directly from the Scott-Zhang interpolation operator, [18] since the finite element spaces are based on the Lagrange finite elements with nodal values that consist of the point evaluations of functions. Furthermore, the operator  $\Pi_h^w$  is just from the Girault-Scott interpolation operator devised in [10]. On the other hand, for the case of the Austin, Manteuffel, McCormick finite elements, the operator  $\Pi_h^v$  should be constructed by using the idea by Girault and Scott, since the nodal values contain the point evaluations of the partial derivatives and so should the operator  $\Pi_h^w$ . These will be discussed in details in the Appendix, 6.

B6. For each  $1 \leq k \leq L$ , the finite element pair  $\mathbf{V}_k$  and  $S_k$  satisfies the following inf-sup condition that

$$\sup_{\mathbf{v} \in \mathbf{V}_k} \frac{(\text{div}\mathbf{v}, s)_0}{\|\mathbf{v}\|_1} \gtrsim \|s\|_0, \quad \forall s \in S_k.$$

The condition B6 is well-known for the Scott-Vogelius finite elements under mild conditions on the mesh, [17], however, it has not been known for the finite elements by Austin, Manteuffel and McCormick,[4, 3]. One of our main contributions is to show this is the case for the finite elements by Austin, Manteuffel and McCormick. The proof is provided in §4.

We shall now list several consequences that can be obtained from the aforementioned assumptions B1 – B6, which shall be crucially used for the multigrid convergence analysis

in §3.3. In fact, many of the results below are obtained in the recent work, [11], therefore, we shall not provide the proofs of results, rather we shall focus on the discussions about results that are different and new considering the previous work, [11].

Due to B3, for  $1 \leq k \leq L$ , we define an adjoint operator  $\mathbf{G}_k : S_k \mapsto \mathbf{V}_k$  of div as follows: for  $s \in S_k$ ,

$$(\mathbf{G}_k s, \mathbf{v})_1 = -(s, \operatorname{div} \mathbf{v})_0, \quad \forall \mathbf{v} \in \mathbf{V}_k. \quad (3.18)$$

By the well-known Closed Range Theorem, we obtain the discrete Helmholtz decomposition :

$$\mathbf{V}_k = \mathcal{N}(\operatorname{div}) \oplus \mathcal{R}(\mathbf{G}_k) = \mathbf{curl} W_k \oplus \mathbf{G}_k(S_k), \quad (3.19)$$

where  $\mathcal{N}(\operatorname{div}) = \mathcal{N}_k$  denotes the null space of the divergence operator and  $\mathcal{R}(\mathbf{G}_k)$  is the range of the dual of the divergence operator. Note that the orthogonality is with respect to  $(H^1)^2$  semi- inner product and  $\mathcal{N}_k$  can be expressed as follows :

$$\begin{aligned} \mathcal{N}_k &= \{ \mathbf{u} \in \mathbf{V}_k : (\operatorname{div} \mathbf{u}, s) = 0, \quad \forall s \in \operatorname{div} \mathbf{V}_k \} \\ &= \{ \mathbf{u} \in \mathbf{V}_k : (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}_k \}. \end{aligned}$$

The symbol  $\mathcal{N}_k^\perp$  denotes the orthogonal complement of  $\mathcal{N}$  with respect to  $(H^1)^2$  semi-inner product. Furthermore, as discussed in §2, the inf-sup condition B6, [9] implies that

$$\|\operatorname{div} \mathbf{v}\|_0 \gtrsim \|\mathbf{v}\|_1 \gtrsim \|\mathbf{v}\|_{a_p}, \quad \forall \mathbf{v} \in \mathcal{N}_k^\perp, \quad (3.20)$$

Note that when  $\kappa^2$  approaches zero, the norm  $\|\cdot\|$  approaches to  $\|\cdot\|_0$ , therefore, the orthogonality becomes the usual  $L^2$ -Helmholtz decomposition, [9].

The Lemmas 3.2-3.4 can be crucial in the multigrid convergence analysis. The proofs can be found at [11].

**Lemma 3.2.** *Under the assumption B5, the operator  $P_k^w$  is stable with respect to  $|\cdot|_{1+\alpha}$  semi-norm for  $\alpha \in [0, 1]$ , namely,*

$$|P_k^w q|_{1+\alpha} \lesssim |q|_{1+\alpha}, \quad \forall q \in W.$$

**Lemma 3.3.** *Under the assumption B5, for any  $q \in W_h$ , the following holds true:*

$$\sum_{k=1}^J |(P_k^w - P_{k-1}^w)q|_2^2 \lesssim |q|_2^2. \quad (3.21)$$

For the aforementioned results, one has to use the regularity estimate for the solution  $u$  to the biharmonic equations posed in polygonal domains, [6]. Namely, there exists  $\alpha \in (0, 1]$  such that

$$|u|_{2+\alpha} \lesssim \|f\|_{-2+\alpha}. \quad (3.22)$$

The results below are related to the strengthened Cauchy-Schwarz inequality, [1]. The proof can be found at [23].

**Lemma 3.4.** *Assume that  $1 \leq i \leq j \leq L$ . The following holds true that*

$$(\operatorname{div} \mathbf{u}_j, \operatorname{div} \mathbf{v}_i) \lesssim \gamma^{j-i} h_j^{-1} \|\mathbf{u}_j\|_0 \|\operatorname{div} \mathbf{v}_i\|_0, \quad \forall \mathbf{u}_j \in \mathbf{V}_j, \mathbf{v}_i \in \mathbf{V}_i. \quad (3.23)$$

We would like to remark that the identical result is valid when  $\operatorname{div}$  is replaced by  $\nabla$ , namely, for each  $1 \leq i \leq j \leq L$ , we have that for all  $\mathbf{u}_j \in \mathbf{V}_j, \mathbf{v}_i \in \mathbf{V}_i$ ,

$$(\nabla \mathbf{u}_j, \nabla \mathbf{v}_i) \lesssim \gamma^{j-i} h_j^{-1} \|\mathbf{u}_j\|_0 \|\nabla \mathbf{v}_i\|_0. \quad (3.24)$$

We shall now establish various decomposition results, which will be used in the multigrid analysis that follows. The one of the main ingredients for the convergence rate estimate shall be a partition of unity,  $\{\theta_\ell\}_{\ell=1}^{N_k}$ , defined on  $\Omega$ , which satisfies  $\sum_{\ell=1}^{N_k} \theta_\ell = 1$  and, for  $\ell = 1, \dots, N_k$ , (see K. T. Smith, [19]).

$$\operatorname{supp}(\theta_\ell) \subset \Omega_k^\ell, \quad (3.25)$$

and

$$\|D^\alpha \theta_\ell\|_\infty \lesssim h_k^{-|\alpha|}, \quad \forall \ell = 1, \dots, N_k, \quad |\alpha| = 1 \text{ and } 2. \quad (3.26)$$

**Lemma 3.5.** *Under the assumptions B4, for each  $k = 1, \dots, L$  and a given  $\mathbf{v} \in \mathbf{V}_k$ , there exists a decomposition*

$$\mathbf{v} = \sum_{\ell=1}^{N_k} \mathbf{v}^\ell$$

such that  $\mathbf{v}^\ell \in \mathbf{V}_k^\ell$  for each  $\ell = 1, \dots, N_k$  and

$$\sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\mathbf{v}^j\|_{0, \Omega_k^\ell}^2 \lesssim \|\mathbf{v}\|_0^2, \quad (3.27)$$

where  $\Omega_k^\ell$  is the patch of the vertex  $x_\ell$  and  $N_k(\ell) = \{j = \ell, \dots, N_k : \Omega_k^j \cap \Omega_k^\ell \neq \emptyset\}$ .

**Proof** Using the partition of unity  $\{\theta_\ell\}_{\ell=1, \dots, N_k}$ , for a given function  $\mathbf{v} \in \mathbf{V}_k$ , we can consider the following decomposition of  $\mathbf{v}$  :

$$\mathbf{v} = \sum_{j=1}^{N_k} \theta_j \mathbf{v}, \quad \operatorname{supp}(\theta_j \mathbf{v}) \subset \Omega_k^j \cap \partial\Omega, \quad \forall j = 1, \dots, N_k.$$

From the assumption B4, namely,  $\Pi_h^v$  is a projection, the following identity holds true :

$$\mathbf{v} = \sum_{j=1}^{N_k} \Pi_k^v(\theta_j \mathbf{v}) = \sum_{j=1}^{N_k} \mathbf{v}^j,$$

where  $\mathbf{v}^j = \Pi_k^v(\theta_j \mathbf{v})$  for  $j = 1, \dots, N_k$ . We now observe that the approximation property of the operator  $\Pi_k^v$  leads to the following estimate:

$$\begin{aligned} \|\Pi_k^v \theta_j \mathbf{v}\|_{0, \Omega_k^\ell} &\leq \|\Pi_k^v \theta_j \mathbf{v} - \theta_j \mathbf{v}\|_{0, \Omega_k^\ell} + \|\theta_j \mathbf{v}\|_{0, \Omega_k^\ell} \lesssim h_k \|\nabla(\theta_j \mathbf{v})\|_{0, \Omega_k^\ell} + \|\theta_j \mathbf{v}\|_{0, \Omega_k^\ell} \\ &\lesssim h_k \|\nabla \theta_j\|_{\mathbf{L}^\infty} \|\mathbf{v}\|_{0, \Omega_k^\ell} + h_k \|\theta_j\|_{\mathbf{L}^\infty} \|\mathbf{v}\|_{1, \Omega_k^\ell} + \|\theta_j\|_{L^\infty} \|\mathbf{v}\|_{0, \Omega_k^\ell} \lesssim \|\mathbf{v}\|_{0, \Omega_k^\ell}. \end{aligned}$$

The last inequality is due to the inverse inequality. The proof can now be completed by observing

$$\sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\mathbf{v}^j\|_{0, \Omega_k^\ell}^2 = \sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\Pi_k^v \theta_j \mathbf{v}\|_{0, \Omega_k^\ell}^2 \lesssim \sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\mathbf{v}\|_{0, \Omega_k^\ell}^2 \lesssim \|\mathbf{v}\|_{0, \Omega}^2.$$

The last inequality is due to the finite overlap property of the overlapping covering  $\{\Omega_k^\ell\}$ . This completes the proof.  $\blacksquare$

**Lemma 3.6.** *Under the assumption B5, for each  $k = 1, \dots, L$ , and any  $\phi \in W_k$ , there exists a decomposition  $\{\phi^j\}_{j=1}^{N_k} \in W_k$  such that  $\phi^j \in W_k^j$  and*

$$\sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\phi^j\|_{0, \Omega_k^\ell}^2 \lesssim \|\phi\|_0^2,$$

where  $N_k(\ell) = \{j = \ell, \dots, N_k : \Omega_k^j \cap \Omega_k^\ell \neq \emptyset\}$ .

**Proof** For a given  $\phi$ , using the partition of unity, we can consider the decomposition of  $\phi$  as follows:

$$\phi = \sum_{j=1}^{N_k} \theta_j \phi, \quad \text{and} \quad \phi = \sum_{j=1}^{N_k} \Pi_k^w(\theta_j \phi^k), \quad \text{due to B5,}$$

where  $\Pi_k^w(\theta_j \phi)$  has a support contained in  $\Omega_k^j$ . We shall set  $\phi^j = \Pi_k^w(\theta_j \phi)$  for  $j = 1, \dots, N_k$ . Thanks to the approximation property of the operator  $\Pi_k^w$ , we obtain that for  $j = 1, \dots, N_k$ , we have

$$\begin{aligned} \|\Pi_k^w \theta_j \phi\|_{0, \Omega_k^\ell} &\leq \|\Pi_k^w \theta_j \phi - \theta_j \phi\|_{0, \Omega_k^\ell} + \|\theta_j \phi\|_{0, \Omega_k^\ell} \lesssim h_k^2 \|\theta_j \phi\|_{2, \Omega_k^\ell} + \|\theta_j \phi\|_{0, \Omega_k^\ell} \\ &\lesssim h_k^2 \|\theta_j\|_{2, \mathbf{L}^\infty} \|\phi\|_{0, \Omega_k^\ell} + h_k^2 \|\theta_j\|_{1, \mathbf{L}^\infty} \|\phi\|_{1, \Omega_k^\ell} + h_k^2 \|\theta_j\|_{\mathbf{L}^\infty} \|\phi\|_{2, \Omega_k^\ell} \\ &\quad + \|\theta_j\|_{L^\infty} \|\phi\|_{0, \Omega_k^\ell} \lesssim \|\phi\|_{0, \Omega_k^\ell}, \end{aligned}$$

where the last inequality is due to the inverse inequality. Therefore, we obtain that

$$\sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\phi^j\|_{0,\Omega_k^\ell}^2 = \sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\Pi_k^w(\theta_{k,j}\phi)\|_{0,\Omega_k^\ell}^2 \lesssim \sum_{\ell=1}^{N_k} \|\phi\|_{0,\Omega_k^\ell}^2 \lesssim \|\phi\|_{0,\Omega}^2,$$

where the last inequality is due to the finite overlap property of the covering  $\{\Omega_k^\ell\}$  for  $\Omega$ . This completes the proof.  $\blacksquare$

**Lemma 3.7.** *For each  $k = 1, \dots, L$  and let  $\mathbf{v}_c \in \mathcal{N}_k$  be given by  $\mathbf{v}_c = \mathbf{curl} q$  with  $q = (P_k^w - P_{k-1}^w)\phi$  for some  $\phi \in W_k$ , then there exists a decomposition*

$$\mathbf{v}_c = \sum_{\ell=1}^{N_k} \mathbf{v}_c^\ell$$

such that  $\mathbf{v}_c^\ell \in \mathcal{N}_k^\ell$  for each  $\ell = 1, \dots, N_k$  and

$$\sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\mathbf{v}_c^j\|_{0,\Omega_k^\ell}^2 \lesssim \|\mathbf{v}_c\|_0^2, \tag{3.28}$$

where  $\Omega_k^\ell$  is the patch of the vertex  $x^\ell$  and  $N_k(\ell) = \{j = \ell, \dots, N_k : \Omega_k^j \cap \Omega_k^\ell \neq \emptyset\}$ .

### 3.3. Multigrid Methods and Convergence Estimates

As mentioned in the previous section, the design of the multigrid method is presented in terms of the Algorithm 3.1, which is based on the space decomposition as described in (3.16) and the local exact solvers denoted by  $P_k^{v,\ell}$  for each  $\mathbf{V}_k^\ell$  with  $1 \leq k \leq L$  and  $1 \leq \ell \leq N_k$  for each  $k$ . This can be interpreted as the multigrid "v" cycle with the block Gauss-Seidel smoothing. Under this setting, the norm of the product operator  $E_L$  can be written as follows:

$$E = E_L = \Pi_{k=1}^L \Pi_{\ell=1}^{N_k} (I - P_k^\ell). \tag{3.29}$$

Following [13], to express the estimate of the norm of  $E$ ,  $\|E\|$ , we shall introduce several notation that for given  $\{\mathbf{v}_k^\ell \in \mathbf{V}_k^\ell\}$ ,

$$\sum_{(\ell,j) \geq (k,i)} \mathbf{v}_\ell^j = \sum_{j=i}^{N_k} \mathbf{v}_k^j + \sum_{\ell=k+1}^L \sum_{j=1}^{N_\ell} \mathbf{v}_\ell^j. \tag{3.30}$$

The following can be easily obtained by applying the abstract results given in the previous section and the several technical lemmas in Section 3.2.

**Theorem 3.3.** *Under the assumptions B1 – B6, the following estimate holds true:*

$$\|E_L\|^2 = 1 - \frac{1}{K} < \delta < 1 \quad (3.31)$$

where  $K$  can be estimated as follows :

$$K \lesssim \sup_{\mathbf{v} \in \mathcal{N}^\perp} \inf_{\sum_{k=1}^L \sum_{i=1}^{N_k} \mathbf{v}_k^i = \mathbf{v}} (I + II) + \sup_{\mathbf{v}_c \in \mathcal{N}} \inf_{\sum_{k=1}^L \sum_{i=1}^{N_k} \mathbf{v}_{k,c}^i = \mathbf{v}_c} III, \quad (3.32)$$

where  $\mathbf{v}_k^i \in \mathbf{V}_k^i$ , and  $\mathbf{v}_{k,c}^i \in \mathcal{N}_k^i$  for each  $k$  and  $i$ .

$$\begin{aligned} I &= \frac{\sum_{k=1}^L \sum_{i=1}^{N_k} \left| P_{k,s}^i \sum_{(\ell,j) \geq (k,i)} \mathbf{v}_\ell^j \right|_{\text{div}}^2}{(\mathbf{v}, \mathbf{v})_{\text{div}}} \\ II &\lesssim \frac{\sum_{k=1}^L \sum_{i=1}^{N_k} \|Q_k^i \sum_{(\ell,j) \geq (k,i)} \mathbf{v}_\ell^j\|_0^2}{\|\mathbf{v}\|_0^2} + \frac{\sum_{k=1}^L \sum_{i=1}^{N_k} |P_k^{v,i} \sum_{(\ell,j) \geq (k,i)} \mathbf{v}_\ell^j|_1^2}{|\mathbf{v}|_1^2} \\ III &\lesssim \frac{\sum_{k=1}^L \sum_{i=1}^{N_k} \|Q_k^i \sum_{(\ell,j) \geq (k,i)} \mathbf{v}_{\ell,c}^j\|_0^2}{\|\mathbf{v}_c\|_0^2} + \frac{\sum_{k=1}^L \sum_{i=1}^{N_k} |P_k^{v,i} \sum_{(\ell,j) \geq (k,i)} \mathbf{v}_{\ell,c}^j|_1^2}{|\mathbf{v}_c|_1^2}. \end{aligned}$$

and  $\delta$  is a constant independent of the mesh size  $h$  and the parameters,  $\rho^2$ ,  $\kappa^2$  and  $\mu^2$ .

**Proof** The estimate (3.32) of  $K$  can be obtained by the direct consequence of the abstract results, the Theorem 3.2. Our main task is to show that  $K = O(1)$ . We shall now estimate I, II and III one by one rigorously. To estimate I and II, for any given  $\mathbf{v} \in \mathcal{N}^\perp$ , we consider the following decomposition:

$$\mathbf{v} = \sum_{k=1}^L (\mathbf{v}_k - \mathbf{v}_{k-1}) = \sum_{k=1}^L \sum_{j=1}^{N_k} \mathbf{v}_k^j, \quad (3.33)$$

where  $\mathbf{v}_k = Q_k \mathbf{v}$  with  $Q_0 = 0$  and  $\mathbf{v}_k^j \in \mathbf{V}_k^j$  for each  $j = 1, \dots, N_k$  and

$$\sum_{\ell=1}^{N_k} \sum_{j \in N_k(\ell)} \|\mathbf{v}_k^j\|_{0, \Omega_k^\ell}^2 \lesssim \|\mathbf{v}_k - \mathbf{v}_{k-1}\|_0^2, \quad (3.34)$$

where  $\Omega_k^\ell$  is the star (or patch) of the vertex  $x_\ell$  and  $N_k(\ell) = \{j \in \{1, \dots, N_k\} : \Omega_k^j \cap \Omega_k^\ell \neq \emptyset\}$ . Under this setting, the estimate of I can be made following the argument given in [11]. Using the BPX type estimate, [5], we obtain that

$$\sum_{k=1}^L \left( \sum_{i=1}^{N_k} |P_{k,s}^i \sum_{(\ell,j) \geq (k,i)} \mathbf{v}_\ell^j|_{\text{div}}^2 \right) \lesssim \sum_{k=1}^L |(Q_k - Q_{k-1})\mathbf{v}_k|_1^2 \lesssim \|\mathbf{v}\|_1^2 \lesssim |\mathbf{v}|_{\text{div}}^2.$$

The last inequality is due to the fact that  $\mathbf{v} \in \mathcal{N}^\perp$  and the inf-sup condition B6. The estimate for II that will result in

$$\sum_{k=1}^L \sum_{i=1}^{N_k} \left\| Q_k^i \sum_{(\ell,j) \geq (k,j)} \mathbf{v}_\ell^j \right\|_0^2 \lesssim \|\mathbf{v}\|_0^2 \quad \text{and} \quad \sum_{k=1}^L \sum_{i=1}^{N_k} \left| P_k^{i,v} \sum_{(\ell,j) \geq (k,j)} \mathbf{v}_\ell^j \right|_1^2 \lesssim |\mathbf{v}|_1^2.$$

will be skipped as this will be similar to the estimate for III. We shall prove the following estimates :

$$\sum_{k=1}^L \sum_{i=1}^{N_k} \left\| Q_k^i \sum_{(\ell,j) \geq (k,j)} \mathbf{v}_{\ell,c}^j \right\|_0^2 \lesssim \|\mathbf{v}_c\|_0^2, \quad \text{and} \quad \sum_{k=1}^L \sum_{i=1}^{N_k} \left| P_k^{i,v} \sum_{(\ell,j) \geq (k,j)} \mathbf{v}_{\ell,c}^j \right|_1^2 \lesssim |\mathbf{v}_c|_1^2. \quad (3.35)$$

In particular, we shall focus on the first estimate (3.35) since the second estimate can be made under the decompositions (3.36) below following the same argument in [11]. For any  $\mathbf{v}_c \in \mathcal{N}$ , let  $q \in W$  be such that  $\mathbf{v}_c = \mathbf{curl} q$ . We consider the following decomposition that

$$\mathbf{v}_c = \sum_{k=1}^L (\mathbf{v}_{k,c} - \mathbf{v}_{k-1,c}), \quad \text{with } \mathbf{v}_{k,c} = \mathbf{curl} P_k^w q. \quad (3.36)$$

First of all, due to the fact that  $\mathbf{v}_{k,c} - \mathbf{v}_{k-1,c} = \mathbf{curl}(P_k^w - P_{k-1}^w)q$  and the Lemma 3.7, we can consider the following further decomposition :

$$\mathbf{v}_{k,c} - \mathbf{v}_{k-1,c} = \sum_{i=1}^{N_k} \mathbf{v}_{k,c}^\ell, \quad (3.37)$$

such that  $\mathbf{v}_{k,c}^\ell \in \mathcal{N}_k^\ell$  for each  $\ell = 1, \dots, N_k$  and the following estimate holds true that

$$\sum_{i=1}^{N_k} \sum_{j \in N_k(i)} \|\mathbf{v}_{k,c}^j\|_{0,\Omega_k^i}^2 \lesssim \|\mathbf{v}_{k,c} - \mathbf{v}_{k-1,c}\|_0^2. \quad (3.38)$$

We note that the following relation holds true :

$$\sum_{k=1}^L \sum_{i=1}^{N_k} \left\| Q_k^i \sum_{(\ell,j) \geq (k,i)} \mathbf{v}_{\ell,c}^j \right\|_0^2 \lesssim \sum_{k=1}^L \left\| Q_k^i \sum_{j=i}^{N_k} \mathbf{v}_{k,c}^j \right\|_0^2 + \sum_{i=1}^{N_k} \|Q_k^i Q_k(\mathbf{v}_c - \mathbf{v}_{k,c})\|_0^2. \quad (3.39)$$

From (3.38), we can estimate the first term in the right hand side of the inequality (3.39) as follows:

$$\sum_{k=1}^L \sum_{i=1}^{N_k} \left\| Q_k^i \sum_{j=i}^{N_k} \mathbf{v}_{k,c}^j \right\|_0^2 \lesssim \sum_{k=1}^L \|\mathbf{v}_{k,c} - \mathbf{v}_{k-1,c}\|_0^2 = \sum_{k=1}^L |(P_k^w - P_{k-1}^w)q|_1^2 \lesssim |q|_1^2 = \|\mathbf{v}_c\|_0^2.$$

For the second inequality, using the finite overlap property, we have

$$\sum_{i=1}^{N_k} \|Q_k^i Q_k(\mathbf{v}_c - \mathbf{v}_{k,c})\|_0^2 \lesssim \|Q_k(\mathbf{v}_c - \mathbf{v}_{k,c})\|_0^2.$$

Furthermore, from the Strengthened Cauchy-Schwarz inequality, we obtain that, (see also [11] for details),

$$\|Q_k(\mathbf{v}_c - \mathbf{v}_{k,c})\|_0^2 \lesssim \sum_{j=k+1}^L \gamma^{j-k} \|Q_k(\mathbf{v}_c - \mathbf{v}_{k,c})\|_0 \|\mathbf{v}_{j,c} - \mathbf{v}_{j-1,c}\|_0$$

Therefore, we obtain that

$$\sum_{k=1}^L \|Q_k(\mathbf{v}_c - \mathbf{v}_{k,c})\|_0^2 \lesssim \sum_{k=1}^L \|\mathbf{v}_{k,c} - \mathbf{v}_{k-1,c}\|_0^2 \lesssim \|\mathbf{v}_c\|_0^2.$$

This completes the proof.  $\blacksquare$

#### 4. Finite Element Spaces by Austin, Manteuffel and McCormick and Stability Analysis

In this section, we shall study the finite elements introduced by Austin, Manteuffel and McCormick, [3, 4] and their properties. Our aim in this section is to show that the finite elements satisfy all the conditions B1–B6, which includes the proof that the Austin, Manteuffel and McCormick finite elements satisfy the uniform stability. As mentioned, these properties have been proven for the Scott-Vogelius element cases in the literatures [17] and [10].

##### 4.1. Finite Elements by Austin, Manteuffel and McCormick

Let  $\Omega$  be a rectangular domain in  $\mathbb{R}^2$  and assume that it is triangulated by rectangles  $\tau_{ij}$  of the form  $\tau_{ij} = I_{h_{x_i}} \times I_{h_{y_j}}$  with  $I_{h_{x_i}} = [x_{i-1}, x_i]$  and  $I_{h_{y_j}} = [y_{j-1}, y_j]$ . The triangulation of the domain  $\Omega$  will be denoted by  $\mathcal{T}_h$ , where  $h$  denotes the maximum mesh size in the following sense :

$$h = \max \left\{ \max_{i=1, \dots, n} |x_i - x_{i-1}|, \max_{j=1, \dots, m} |y_j - y_{j-1}| \right\},$$

where  $n$  and  $m$  are the number of partitions of the domain  $\Omega$ , in  $x$ - and  $y$ -directions respectively. A schematic domain  $\Omega$  triangulated by two rectangles are presented in Figure 4.1. To begin with, following the notation given in [4], we introduce two function spaces as follows:

$$\mathcal{C}^{(m,n)}(\Omega) = \{f(x, y) \in C(\Omega) : \partial_x^m f \in C(\Omega) \text{ and } \partial_y^n f \in C(\Omega)\}$$

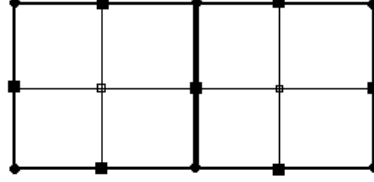


Figure 4.1. Sketch of Domain  $\Omega$  Triangulated by Two Rectangles. Dots : vertices; Squares : midpoints of edges; Blank Squares : Center nodes

and

$$\mathcal{C}^k(\Omega) = \{f(x, y) \in C(\Omega) : \partial_x^{k_1} \partial_y^{k_2} f \in C(\Omega) \text{ for } k_1 + k_2 \leq k\}.$$

We denote the set of all polynomials up to degree  $\ell$  on an interval  $I \subset \mathbb{R}$  by  $\mathcal{P}^\ell(I)$  and denote a rectangular element of  $\mathcal{T}_h$  by  $\tau_{ij} = I_{h_x} \times I_{h_y}$  with  $I_{h_x} = [x_{i-1}, x_i]$  and  $I_{h_y} = [y_{j-1}, y_j]$  and define

$$M_1^h = \{v_h \in \mathcal{C}^{(1,0)} \text{ and } v_h|_\tau \in \mathcal{P}^3(I_{h_x}) \times \mathcal{P}^2(I_{h_y}), \quad \forall \tau \in \mathcal{T}_h\}$$

and

$$M_2^h = \{v_h \in \mathcal{C}^{(0,1)} \text{ and } v_h|_\tau \in \mathcal{P}^2(I_{h_x}) \times \mathcal{P}^3(I_{h_y}), \quad \forall \tau \in \mathcal{T}_h\}.$$

The finite element space,  $\mathbf{V}_h$  for the velocity field will then be defined by

$$\mathbf{V}_h = \{\mathbf{u}_h = (u, v)^t : u \in M_1^h \text{ and } v \in M_2^h\} \cap (H_0^1)^2. \tag{4.1}$$

To introduce the basis functions, let  $\{a_v^\ell\}_{\ell=1, \dots}$  be the vertices and  $\{a_m^\ell\}_{\ell=1, \dots}$  be the midpoints of edges for the triangulation  $\mathcal{T}_h$  respectively. In particular, we may need to distinguish midpoints  $\{a_m^\ell\}_{\ell=1, \dots}$  into two types. The first type will be denoted by  $\{a_{m,x}^\ell\}_\ell$ , which are contained in edges parallel to  $x$ -axis and the second will be denoted by  $\{a_{m,y}^\ell\}_{\ell=1, \dots}$ , which are contained in edges parallel to  $y$ -axis. The basis functions are constructed from the tensor product between cubic Hermite basis functions and quadratic Lagrangian basis functions with nodal values consisting of values at two end points and the edge moment, i.e., on  $\{q(a), \int_a^b q dx, q(b), q \in \mathcal{P}^2([a, b])\}$ . In order to describe the basis functions for  $M_\ell^h$  with  $\ell = 1, 2$ , we recall that for a unit interval  $I = [0, 1]$  with nodes given on  $t_1 = 0, t_2 = 1/2$  and  $t_3 = 1$ , the Hermite cubic basis functions  $c_i$  with  $i = 1, 2, 3, 4$  and the quadratic Lagrange basis functions  $q_j$  with  $j = 1, 2, 3$  are given, respectively, as follows

$$\begin{aligned} c_1(t) &= 2t^3 - 3t^2 + 1; & q_1(t) &= 3t^2 - 4t + 1 \\ c_2(t) &= t^3 - 2t^2 + t; & q_2(t) &= -6t^2 + 6t \\ c_3(t) &= -2t^3 + 3t^2; & q_3(t) &= 3t^2 - 2t \\ c_4(t) &= t^3 - t^2. \end{aligned}$$

For each  $M_\ell^h$ , there are two kinds of nodal basis functions. The first kind is for the value at nodes and the second kind is for the  $x$ -derivative for  $M_1^h$  and the  $y$ -derivative for  $M_2^h$ , namely, for the domain  $I \times I$  with  $I = [0, 1]$ , we consider the following combinations :

$$\widehat{\phi}_\ell(\widehat{x}, \widehat{y}) = c_{2i-1}(\widehat{x})q_j(\widehat{y}), \quad \text{and} \quad \widehat{\alpha}_\ell(\widehat{x}, \widehat{y}) = c_{2i}(\widehat{x})q_j(\widehat{y}),$$

and

$$\widehat{\psi}_\ell(\widehat{x}, \widehat{y}) = c_{2i-1}(\widehat{y})q_j(\widehat{x}), \quad \text{and} \quad \widehat{\beta}_\ell(\widehat{x}, \widehat{y}) = c_{2i}(\widehat{y})q_j(\widehat{x}),$$

where  $(\widehat{x}, \widehat{y}) \in I \times I$  is the reference coordinate,  $\ell = 3(i-1) + j$  with  $i = 1, 2$ , and  $j = 1, 2, 3$ . Now, for any  $\tau = I_x \times I_y$ , we consider the affine mapping  $F : I \times I \mapsto I_x \times I_y$ , we then construct the basis functions on the computational domain by the usual composition that

$$\phi_\ell = \widehat{\phi}_\ell \circ F^{-1}, \quad \alpha_\ell = \widehat{\alpha}_\ell \circ F^{-1}, \quad \psi_\ell = \widehat{\psi}_\ell \circ F^{-1}, \quad \text{and} \quad \beta_\ell = \widehat{\beta}_\ell \circ F^{-1}.$$

We shall now summarize the set of all interpolation nodal variables for the triangulation  $\mathcal{T}_h$  for both  $M_1^h$  and  $M_2^h$  as follows: For  $\mathbf{u} = (u, v)^t \in (H^l(\Omega))^2$  with  $l > 2$ ,

$$D_i^u u(a_v^i) = u(a_v^i), \quad \text{and} \quad D_i^v v(a_v^i) = v(a_v^i), \quad (4.2)$$

$$D_i^u u(a_v^i) = \partial_x u(a_v^i), \quad \text{and} \quad D_i^v v(a_v^i) = \partial_y v(a_v^i), \quad (4.3)$$

$$D_i^u u(a_{m,y}^i) = \int_{e_y^i} D_i^u u \, d\nu, \quad \text{and} \quad D_i^v v(a_{m,x}^i) = \int_{e_x^i} D_i^v v \, d\nu. \quad (4.4)$$

where  $e_y^i$  is the edge containing the midpoint  $a_{m,y}^i$  and  $e_x^i$  is the edge containing the midpoint  $a_{m,x}^i$ , respectively. For the convenience of the presentation that follows, we shall denote all relevant nodes by  $a_i$ 's and use the symbol  $D_i$  for both  $D_i^u$  and  $D_i^v$  in case that the different use of the operator  $D_i$  does not make any confusions. We also denote the basis functions for  $M_1^h$  by  $\{\Phi_i\}$  and  $\{\overline{\Phi}_i\}$ , where  $\{\Phi_i\}$  are the basis functions corresponding to the differential operator  $D_i$  with  $|D_i| = 0$  and  $\{\overline{\Phi}_i\}$  is for  $|D_i| = 1$ . The corresponding basis functions for  $M_2^h$  shall be denoted by  $\{\Psi_i\}$  and  $\{\overline{\Psi}_i\}$ . We shall also specify the local bases for both  $M_1^h$  and  $M_2^h$ . For a given  $\tau \in \mathcal{T}_h$ , which consists of four edges,  $\partial\tau_1, \partial\tau_2, \partial\tau_3$  and  $\partial\tau_4$ , where the numbering has been made in counterclockwise, starting from the left edge and denote vertices by  $\{p_i\}_{i=1, \dots, 4}$  from left *top* corner to right *top* corner numbered in counterclockwise and  $p_{12}, p_{23}, p_{34}$  and  $p_{41}$  are midpoints for each edge,  $\partial\tau_1, \partial\tau_2, \partial\tau_3$  and  $\partial\tau_4$ . We shall denote  $\{\Lambda_i\}$  and  $\{\Gamma_i\}$  by the basis for the nodal point evaluations at vertices and  $\{\overline{\Lambda}_i\}$  and  $\{\overline{\Gamma}_i\}$  by the basis for the evaluations of partials for the space  $M_1^h$  and  $M_2^h$ , respectively. Now, we shall denote  $\Lambda_{12}, \Lambda_{34}$  and  $\overline{\Lambda}_{12}, \overline{\Lambda}_{34}$  by the edge moments and edge moments of the partials on  $\partial\tau_1$  and  $\partial\tau_3$  for  $M_1^h$ , respectively

and  $\Gamma_{23}$ ,  $\Gamma_{41}$  and  $\bar{\Lambda}_{23}$ ,  $\bar{\Lambda}_{41}$  by the edge moments and edge moments of the partials on  $\partial\tau_2$  and  $\partial\tau_4$  for  $M_2^h$ , respectively. These definitions lead to write, for all  $u \in M_1^h$ ,  $v \in M_2^h$ ,

$$u = \sum_{|D_i|=0} D_i(u)\Phi_i + \sum_{|D_i|=1} D_i(u)\bar{\Phi}_i, \quad \text{and} \quad v = \sum_{|D_i|=0} D_i(v)\Psi_i + \sum_{|D_i|=1} D_i(v)\bar{\Psi}_i, \quad (4.5)$$

We shall be needing the estimate of the norm of each basis functions for the space  $\mathbf{V}_h$ , which can result from the following norm equivalence relation that

**Lemma 4.1.** *Let  $h \approx 1$  and  $u \in M_1^h$ , we have the following equivalent norm relation that*

$$\|u\|_{0,\tau} \approx \sqrt{\sum_{i=1}^4 u(a_v^i)^2 + (\partial_x u(a_v^i))^2} + \left| \int_{\partial\tau_1} u \, d\nu \right| + \left| \int_{\partial\tau_3} u \, d\nu \right| + \left| \int_{\partial\tau_1} \partial_x u \, d\nu \right| + \left| \int_{\partial\tau_3} \partial_x u \, d\nu \right|, \quad (4.6)$$

where  $\partial\tau_1$  is the left edge and  $\partial\tau_3$  is the right edge of the rectangle  $\tau$ , respectively.

**Lemma 4.2.** *The following inequalities hold true:*

$$\begin{aligned} \|\phi_\ell\|_{0,\Omega} &\lesssim h, & \|\phi_\ell\|_{1,\Omega} &\lesssim 1, & \|\alpha_\ell\|_{0,\Omega} &\lesssim h^2, & \|\alpha_\ell\|_{1,\Omega} &\lesssim h \\ \|\psi_j\|_{0,\Omega} &\lesssim h, & \|\psi_\ell\|_{1,\Omega} &\lesssim 1, & \|\beta_\ell\|_{0,\Omega} &\lesssim h^2, & \|\beta_\ell\|_{1,\Omega} &\lesssim h, \quad \forall \ell. \end{aligned}$$

**Proof** It is enough to show the inequality restricted to  $\tau \in \mathcal{T}_h$ . We denote the reference rectangle of  $\tau$  by  $\hat{\tau}$ . We shall only consider the estimate in  $\|\cdot\|_{0,\Omega}$  for the space  $M_1^h$ , since the corresponding results for  $\|\cdot\|_{1,\Omega}$  and for  $M_2^h$ , can be easily deduced. We consider the basis function,  $\phi_i$  with  $|D_i| = 0$  and observe that

$$\begin{aligned} \|\phi_i\|_{0,\tau}^2 &= \int_{\tau} \phi_i^2 \, dx = \mathcal{J} \int_{\hat{\tau}} \hat{\phi}_i^2 \, d\hat{x} \\ &\lesssim \mathcal{J} \left( \sum_{i=1}^4 \hat{\phi}_i(\hat{a}_v^i)^2 + \partial_{\hat{x}} \hat{\phi}_i(\hat{a}_v^i)^2 + \left| \int_{\partial\hat{\tau}_1} \hat{\phi}_i \, d\hat{s} \right| + \left| \int_{\partial\hat{\tau}_3} \hat{\phi}_i \, d\hat{s} \right| \right) \end{aligned} \quad (4.7)$$

$$+ \left| \int_{\partial\hat{\tau}_1} \partial_{\hat{x}} \hat{\phi}_i \, d\hat{s} \right| + \left| \int_{\partial\hat{\tau}_3} \partial_{\hat{x}} \hat{\phi}_i \, d\hat{s} \right| \quad (4.8)$$

$$\lesssim \mathcal{J} \lesssim h^2,$$

where  $\hat{a}_v^i$  is the vertex for reference rectangle  $\hat{\tau}$ . Since  $\mathcal{J} \approx h^2$  is the determinant of the Jacobian that maps  $\hat{\tau}$  to  $\tau$ . For  $\alpha_i$ , with  $|D_i| = 1$ , by the Poincaré's inequality, we have that

$$\|\alpha_i\|_{0,\tau}^2 \lesssim h^2 \int_{\tau} \left( \frac{\partial \alpha_i}{\partial x} \right)^2 \, dx \lesssim h^4 \int_{\hat{\tau}} \left( \frac{\partial \hat{\alpha}_i}{\partial \hat{x}} \right)^2 \, d\hat{x} \lesssim h^4.$$

This completes the proof.  $\blacksquare$

The similar estimate has been derived for the basis for the space of Morgan and Scott, C1 conforming finite element spaces, [15] by Girault and Scott, [10] (see the Theorem 4.1, page 1051). We shall now define the pressure finite element space denoted by  $S_h$ . The space  $S_h$  is defined as the bi-Lagrangian finite element space as follows:

$$S_h = \{s_h \in C^0(\Omega) : s_h|_\tau \in \mathcal{P}^2(I_x) \times \mathcal{P}^2(I_y), \quad \forall \tau \in \mathcal{T}_h \text{ with } \int_\Omega s_h dx = 0\}.$$

The set of all interpolation nodal variables for the triangulation  $\mathcal{T}_h$  consist of nodal values at vertices and all edge midpoints together with moments in each rectangle, namely, for  $s \in H^l(\Omega)$  with  $l > 1$ , the set of all interpolation nodal variables are

$$s(a_i), \quad \forall a_i \text{ vertices}, \quad \int_{\partial\tau} s d\nu, \quad \text{four edge moments and,} \quad \int_\tau s dx \quad \forall \tau \in \mathcal{T}_h. \quad (4.9)$$

We shall introduce additional finite element spaces, denoted by  $W_h$ , whose **curl** becomes the null space of the divergence operator in  $\mathbf{V}_h$ . The space  $W_h$  is the bi-Hermite cubic space defined by

$$\begin{aligned} W_h &= \{w_h \in C^1(\Omega) : w_h|_\tau \in \mathcal{P}^3(I_x) \times \mathcal{P}^3(I_y), \quad \forall \tau \in \mathcal{T}_h\} \cap H_0^2(\Omega) \\ &= \left\{ w_h \in C^1(\Omega) : w_h|_\tau \in \mathcal{P}^3(I_x) \times \mathcal{P}^3(I_y), \text{ and } w_h = 0, \frac{\partial w_h}{\partial \mathbf{n}} = 0, \text{ on } \partial\Omega \right\}, \end{aligned}$$

where  $\partial/\partial \mathbf{n}$  is the outer normal derivative. For the space  $W_h$ , the set of all interpolation nodal variables for the triangulation  $\mathcal{T}_h$  will be given by, for  $q \in H^l(\Omega)$  with  $l > 3$ ,

$$q(a_i), \quad \partial_x q(a_i), \quad \partial_y q(a_i), \quad \text{and} \quad \partial_{xy} q(a_i), \quad (4.10)$$

where  $a_i$ 's are vertices for the triangulation  $\mathcal{T}_h$ . Note that these degrees of freedom have not been specified in Austin, Manteuffel and McCormick, [4] and the degrees of freedom given in (4.10) uniquely determine functions in  $W_h$  so that they belong to  $C^1$ . These specifications shall play a crucial role in constructing  $\Pi_h^w$  so that it can be used for the multigrid analysis as discussed in §3.2. Let  $N$  be total degrees of freedom for the space  $W_h$ . These definitions lead us to write

$$w = \sum_{i=1}^N D_i^w(w) \zeta_i, \quad \forall w \in W_h. \quad (4.11)$$

For the space  $W_h$ , we can easily show that the following relation holds true, (see the Lemma 3.2 in [4])

$$\mathbf{curl} W_h = \mathcal{N}_h. \quad (4.12)$$

We are now in a position to establish the exact sequence property, B3 between finite element spaces,  $\mathbf{V}_h, W_h$  and  $S_h$ , namely, the following sequence is exact :

$$W_h \xrightarrow{\text{curl}} \mathbf{V}_h \xrightarrow{\text{div}} S_h \longrightarrow^0 0. \tag{4.13}$$

Note that the exact sequence property has been shown to be true for the Scott-Vogelius elements, see [17] or [11] for details and the exact sequence property for the Austin, Manteuffel and McCormick’s elements can be shown if we can show that  $\text{div}\mathbf{V}_h = S_h$ , which turns out to be non-trivial. On the other hand, it is true and it is the key factor to establish the inf-sup condition of the pair  $\mathbf{V}_h$  and  $S_h$  as shall be demonstrated in the following section §4.2. We remark that some relevant results have been stated without proof in Austin, Manteuffel and McCormick, [4]. However, their definition for  $S_h$  does not seem to be appropriate since a boundary condition for a function in  $S_h$  is assigned and for such a definition, (see the remark that follows the Definition 3.3 in [4], where authors assign zero boundary condition for functions in  $S_h$ ) and it is not so clear if the statement  $\text{div}\mathbf{V}_h = S_h$  is true in such a situation. We begin by stating and proving a simple but instrumental lemma

**Lemma 4.3.** *For any  $q \in S_h$ , there exists  $\mathbf{v} \in \mathbf{V}_h$  such that*

$$\int_{\tau} \text{div}\mathbf{v} \, dx = \int_{\tau} q \, dx, \quad \forall \tau \in \mathcal{T}_h \tag{4.14}$$

*and the nodal variables for  $\mathbf{v}$  are all zero except for the edge moments.*

**Proof** For a fixed  $\tau \in \mathcal{T}_h$ , we set  $\partial\tau = \sum_{i=1}^4 \partial\tau_i$ , where  $\partial\tau_i$ ’s are edges of  $\tau$  with  $\partial\tau_1$ , left,  $\partial\tau_2$ , bottom,  $\partial\tau_3$ , right, and  $\partial\tau_4$ , top and observe that showing the equality (4.14) would be equivalent to choose  $\mathbf{v}$  so that  $\mathbf{v} = (u, v)^t$  are chosen so that

$$\begin{aligned} \int_{\tau} q \, d\mathbf{x} &= - \int_{\partial\tau} \mathbf{v} \cdot \mathbf{n} \, d\nu \\ &= - \left( - \int_{\partial\tau_1} u \, d\nu - \int_{\partial\tau_2} v \, d\nu + \int_{\partial\tau_3} u \, d\nu + \int_{\partial\tau_4} v \, d\nu \right). \end{aligned} \tag{4.15}$$

Now, we shall use some combinatoric arguments to show that by an appropriate choices of  $\mathbf{v}$ , one can satisfy the aforementioned equation (4.15) in each element in  $\mathcal{T}_h$ . Let  $n_e$  be the total number of elements. We then see that the conditions that need to be satisfied will be  $n_e - 1$ , which are  $\int_{\tau_k} q \, d\mathbf{x}$  with  $k = 1, \dots, n_e$  with a constraint  $\int_{\Omega} q \, d\mathbf{x} = 0$ . Starting from the assumption that  $\Omega = \tau$ , we argue as follows. The general case can be easily deduced by the same argument. If  $\Omega = \tau$ , we have no degree of freedom for  $\mathbf{v}$  to take as a value on  $\partial\tau$  other than zero due to the boundary condition, however, with the homogeneous boundary

condition, the condition that  $\int_{\tau} q \, d\mathbf{x} = 0$  is naturally satisfied. Now, we refine  $\tau$  by bisecting the rectangle. The way to refine  $\tau$  so that the internal degree of freedoms is generated smallest is to refine  $\tau$  in only one direction for instance, in which way, the internal vertex edges can not be generated. Note that the number of degrees of freedom is the same with the interval vertex edges and we should consider the worst possible case where we have the smallest degrees of freedom. In this case, the internal nodes are only edges that intersects two elements. A simple counting indicates that the number of internal edges are equal to  $n_e - 1$ . This concludes that the constraints  $\int_{\tau_k} q \, d\mathbf{x}$  can be realized by the choice of the nodal values that correspond only to the edge moments. By defining  $\mathbf{v} \in \mathbf{V}_h$  by a vector for which the nodal values corresponding to the edge moments are chosen to satisfy the constraints  $\int_{\tau_k} \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\tau_k} q \, d\mathbf{x}$ , for all  $\tau_k \in \mathcal{T}_h$  with other nodal values being zero. This completes the proof.

**Proposition 4.1.** *The divergence operator maps  $\mathbf{V}_h$  onto  $S_h$ , i.e.,  $\operatorname{div} \mathbf{V}_h = S_h$ .*

**Proof** For any  $\mathbf{v} \in \mathbf{V}_h$ , it is clear to see that  $\operatorname{div} \mathbf{v} \in S_h$ . Therefore, we obtain that  $\operatorname{div} \mathbf{V}_h \subset S_h$ . To show the reverse inclusion, for any  $q \in S_h$ , we shall need to construct  $\mathbf{v} \in \mathbf{V}_h$  so that  $\operatorname{div} \mathbf{v} = q$ . By  $\operatorname{div} \mathbf{v} = q$ , we mean that

$$\operatorname{div} \mathbf{v} = q, \quad \text{for all degrees of freedoms for the space } S_h. \quad (4.16)$$

In particular, we shall have to show that  $\operatorname{div} \mathbf{v} = q$  at vertices,

$$\int_{\partial\tau_i} \operatorname{div} \mathbf{v} \, d\nu = \int_{\partial\tau_i} q \, d\nu, \quad \forall i = 1, \dots, 4 \quad (4.17)$$

and

$$\int_{\tau} \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_{\tau} q \, d\mathbf{x}, \quad \forall \tau \in \mathcal{T}_h. \quad (4.18)$$

To construct  $\mathbf{v}$  that satisfies conditions (4.16), (4.17) and (4.18), we shall construct three functions,  $\mathbf{u}_1$ ,  $\mathbf{u}_2$  and  $\mathbf{u}_3$  as follows. The function  $\mathbf{u}_1$  will be constructed so that

$$\int_{\tau} \operatorname{div} \mathbf{u}_1 \, dx = \int_{\tau} q \, dx \quad \forall \tau \in \mathcal{T}_h, \quad (4.19)$$

with nonzero edge moments. More precisely, let  $\mathbf{u}_1 = (u_1, v_1)$ , then  $u_1|_{\tau}$  can be represented by  $\Lambda_{12}$  and  $\Lambda_{34}$  and  $v_1|_{\tau}$  can be represented by  $\Gamma_{23}$  and  $\Gamma_{41}$ . The function  $\mathbf{u}_2$  shall be constructed so that  $\operatorname{div} \mathbf{u}_2 = q$  at all vertices. In particular, let  $\mathbf{u}_2 = (u_2, v_2)$ , then  $u_2|_{\tau}$  and  $v_2|_{\tau}$  can be represented by  $\{\bar{\Lambda}_i\}$  and  $\{\bar{\Gamma}_i\}$  respectively. More precisely, we define  $\mathbf{u}_2 = (u_2, v_2)^t$  by the

followings :

$$\begin{aligned} u_2|_\tau &= \frac{1}{2} (q(p_1)\bar{\Lambda}_1 + q(p_2)\bar{\Lambda}_2 + q(p_3)\bar{\Lambda}_3 + q(p_4)\bar{\Lambda}_4) \\ v_2|_\tau &= \frac{1}{2} (q(p_1)\bar{\Gamma}_1 + q(p_2)\bar{\Gamma}_2 + q(p_3)\bar{\Gamma}_3 + q(p_4)\bar{\Gamma}_4) \end{aligned}$$

We, now define  $\mathbf{w} = \mathbf{u}_1 + \mathbf{u}_2$  and notice that the only difference between  $\text{div}\mathbf{w}$  and  $q$  is for the edge moments. To match these quantities, we shall construct additional function,  $\mathbf{u}_3$ . We set  $\mathbf{w} = (w_1, w_2)^t$  and define  $\mathbf{u}_3 = (u_3, v_3)^t \in \mathbf{V}_h$  as follows :

$$u_3|_\tau = \left( \int_{\partial\tau_1} -\partial_y w_2 + q \, d\nu \right) \bar{\Lambda}_{12} + \left( \int_{\partial\tau_3} -\partial_y w_2 + q \, d\nu \right) \bar{\Lambda}_{34}$$

and

$$v_3|_\tau = \left( \int_{\partial\tau_2} -\partial_x w_1 + q \, d\nu \right) \bar{\Gamma}_{23} + \left( \int_{\partial\tau_4} -\partial_x w_1 + q \, d\nu \right) \bar{\Gamma}_{41},$$

where both  $\partial_y w_2$  and  $\partial_x w_1$  are well-defined since the derivative is taken along the edge directions. We shall show that  $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$  is the desired function. First of all,  $\mathbf{v}$  belongs to  $\mathbf{V}_h$  and satisfies the moment on  $\tau$  for all  $\tau \in \mathcal{T}_h$  since both functions  $\mathbf{u}_2$  and  $\mathbf{u}_3$  does not affect the evaluations and have zero edge moments. Since  $\mathbf{u}_1$  and  $\mathbf{u}_3$  does not affect the partials at vertices,  $\text{div}\mathbf{v} = q$  at all vertices,  $\{p_i\}$ . Finally, we need to show that the edge moments are identical for both functions,  $\text{div}\mathbf{v}$  and  $q$ . To show this we only prove the case for  $\partial\tau_1$  since other cases are similar.

$$\begin{aligned} \int_{\partial\tau_1} \text{div}\mathbf{v} \, d\nu &= \int_{\partial\tau_1} \text{div}\mathbf{w} + \text{div}\mathbf{u}_3 \, d\nu = \int_{\partial\tau_1} \partial_x w_1 + \partial_y w_2 + \partial_x u_3 + \partial_y v_3 \, d\nu \\ &= \int_{\partial\tau_1} \partial_y w_2 + \partial_x u_3 \, d\nu, \quad \text{since } \int_{\partial\tau_1} \partial_x w_1 \, d\nu = 0, \text{ and } \int_{\partial\tau_1} \partial_y v_3 \, d\nu = 0 \\ &= \int_{\partial\tau_1} \partial_y w_2 \, d\nu + \left( \int_{\partial\tau_1} -\partial_y w_2 + q \, d\nu \right) = \int_{\partial\tau_1} q \, d\nu. \end{aligned}$$

This completes the proof. ■

We shall now show that the conditions B4 and B5 hold true. In fact, for the Scott-Vogelius elements, the existence of the interpolation operator  $\Pi_h^v$  can be proven by invoking the Scott-Zhang interpolation operator and the operator  $\Pi_h^w$  has been introduced in the work by Girault and Scott, [10]. On the other hand, both the interpolation operators  $\Pi_h^v$  and  $\Pi_h^w$  for the Austin, Manteuffel and McCormick finite elements can not be constructed from the Scott-Zhang interpolation technique since the space  $\mathbf{V}_h$  takes the derivative nodal values and the

space  $\mathbf{V}_h$  takes the mixed derivative nodal values. More precisely, for the definition of the nodal values for  $\mathbf{V}_h$ , the functions should belong to at least  $H^l(\Omega)$  with  $l > 2$ . Both interpolation operators  $\Pi_h^v$  and  $\Pi_h^w$  can be constructed following the idea from Girault and Scott, [10]. We shall state the main results related to these issues with proofs given in the Appendix for completeness.

**Theorem 4.1.** *There exists an interpolation operator  $\Pi_h^v : (H_0^1(\Omega))^2 \mapsto \mathbf{V}_h$  such that  $\Pi_h^v \mathbf{u} = \mathbf{u}, \forall \mathbf{u} \in \mathbf{V}_h$ ,*

$$\|\Pi_h^v \mathbf{u}\|_{1,\Omega} \lesssim \|\mathbf{u}\|_{1,\Omega}, \quad \forall \mathbf{u} \in (H_0^1(\Omega))^2, \quad (4.20)$$

and

$$|\mathbf{u} - \Pi_h^v \mathbf{u}|_{m,\Omega} \lesssim h^{1-m} |\mathbf{u}|_{1,\Omega} \quad \text{for } m = 0, 1. \quad (4.21)$$

Furthermore it holds true that

$$\int_{\tau} \operatorname{div} (\Pi_h^v \mathbf{u} - \mathbf{u}) \, d\mathbf{x} = 0, \quad \forall \tau \in \mathcal{T}_h. \quad (4.22)$$

**Proof** See Appendix.

The corresponding result for the space  $W_h$  are given as follows:

**Theorem 4.2.** *There exists an interpolation operator  $\Pi_h^w : H_0^2 \mapsto W_h$  such that  $\Pi_h^w q = q, \forall q \in W_h$  and for  $m = 0, 1$ ,*

$$\|q - \Pi_h^w q\|_m \lesssim h^{2-m} |q|_2, \quad \forall q \in H_0^2(\Omega).$$

**Proof** See Appendix.

We note that the Theorem 4.1 plays a crucial role in establishing the fact that the pair of finite elements  $\mathbf{V}_h$  and  $S_h$  satisfies the inf-sup conditions, which is shown to be crucial in establishing the uniform convergence of the multigrid methods in the previous section §3.3. This is the subject of the next subsection.

#### 4.2. On the inf-sup conditions for the Austin, Manteuffel and McCormick finite elements

It is well-known that the inf-sup conditions for the Scott-Vogelius elements can be established under mild conditions on the triangulations of the domain  $\Omega$ , [17]. However, the stability results for the finite element pairs  $\mathbf{V}_h$  and  $S_h$  by Austin, Manteuffel and McCormick, [4] is not known in the literatures. The aim in this section is to demonstrate the stability result, B6 for the pair of finite element spaces  $\mathbf{V}_h$  and  $S_h$ . The technical tool we shall take is so-called Macroelement techniques, introduced by Stenberg, [20]. Let  $\mathcal{M}_h$  be a macroelement

that consists of one element in  $\mathcal{T}_h$ . Therefore, we have that  $\mathcal{M}_h = \mathcal{T}_h$  and macroelement is the same with an element  $\tau \in \mathcal{T}_h$ . Now, since  $\tau$  is *equivalent* to a reference triangle  $\hat{\tau}$  in the sense of the Stenberg, [20]. A class of equivalent macroelements,  $\mathcal{E}_{\hat{\tau}}$  is equal to  $\mathcal{T}_h$ . For each element  $\tau \in \mathcal{T}_h$ , we then introduce the following usual notation:

$$\begin{aligned} \mathbf{V}_{0,\tau} &= \mathbf{V}_h \cap (H_0^1(\tau))^2 \\ S_\tau &= \{q|_\tau : q \in S_h\}, \quad S_{0,\tau} = S_\tau \cap L_0^2(\tau), \\ N_\tau &= \left\{ q \in S_\tau : (\operatorname{div} \mathbf{v}, q)_{0,\tau} = \int_\tau \operatorname{div} \mathbf{v} q \, dx = 0, \quad \forall \mathbf{v} \in \mathbf{V}_{0,\tau} \right\}. \end{aligned}$$

We shall first prove the following result that is an easy consequence of the Lemma 4.1.

**Lemma 4.4.** *For every  $\tau \in \mathcal{T}_h$ , the space  $N_\tau$  is one dimensional, consisting of those functions which are constant on  $\tau$  and the following inf-sup condition holds true :*

$$\sup_{\mathbf{v} \in \mathbf{V}_{0,\tau}} \frac{(\operatorname{div} \mathbf{v}, p)_{0,\tau}}{|\mathbf{v}|_{1,\tau}} \gtrsim \|p\|_{0,\tau}, \quad \forall p \in S_{0,\tau}, \quad \forall \tau \in \mathcal{T}_h. \tag{4.23}$$

**Proof** For each  $\tau$ , the divergence operator  $: \mathbf{V}_{0,\tau} \mapsto S_\tau/\mathbb{R}$  is onto by the Lemma 4.1. Let  $p \in S_\tau$  be given and suppose that

$$\int_\tau \operatorname{div} \mathbf{v} p \, d\mathbf{x} = 0, \quad \forall \mathbf{v} \in \mathbf{V}_{0,\tau}.$$

Set  $p_0 = \int_\tau p \, d\mathbf{x} \in \mathbb{R}$  and we choose  $\mathbf{v} \in \mathbf{V}_{0,\tau}$  such that  $\operatorname{div} \mathbf{v} = p - p_0$ . For such a  $\mathbf{v}$ , we obtain that

$$0 = \int_\tau \operatorname{div} \mathbf{v} p \, d\mathbf{x} = \int_\tau \operatorname{div} \mathbf{v} (p - p_0) \, d\mathbf{x} = \int_\tau (p - p_0)^2 \, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{V}_{0,\tau}.$$

Therefore,  $p = p_0$  on  $\tau$ . This implies that the dimension of  $N_\tau$  is one, consisting of only constant functions. The local inf-sup condition (4.23) is the direct consequence of the Lemma 3.1, in the page 13 from R. Stenberg, [20] in the setting that the class of equivalent macroelements is  $\mathcal{T}_h$ . We omit the proof as it is identical with the proof there. This completes the proof. ■

To conclude the stability, the inf-sup condition for the pair  $\mathbf{V}_h$  and  $S_h$ , it is enough to prove the inf-sup condition between  $\mathbf{V}_h$  and the piecewise constant pressure space, denoted by  $\Sigma_h$ , which in turn has already been established in the Theorem 4.1. Note that the Theorem 4.1 constructs the Fortin operator,  $\Pi_h^v$  that can directly validate the inf-sup condition for the pair of spaces  $\mathbf{V}_h$  and  $\Sigma_h$ .

**Theorem 4.3.** *Let  $\mathcal{T}_h$  be a macroelement partition of the elements that consists of a single element. If  $N_\tau$  is of dimension one and the pair  $\mathbf{V}_h$  and  $\Sigma_h$  is stable, then the following inf-sup condition holds true:*

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{v}_h, s_h)}{\|\mathbf{v}_h\|_1} \gtrsim \|s_h\|_0, \quad \forall s_h \in S_h. \quad (4.24)$$

## 5. Conclusion

In this paper, we proved the uniform accuracy of the (modified) Austin, Manteuffel and McCormick's finite elements for the Stokes-like equations with respect to the relevant parameters. The fast and robust solution techniques for the linear algebraic systems arising from the accurate discretization of the Stokes-like equations are discussed and analyzed mathematically as well. The fast and robust solution techniques for the Stokes-like equations are constructed based on the Augmented Lagrangian Uzawa iterative methods and the multigrid methods for the linear algebraic systems of the differential operator  $\rho^2 \mathbf{I} - \kappa^2 \Delta - \mu^2 \nabla \operatorname{div}$ . In particular, posed by Austin, Manteuffel and McCormick in [4] as an open question is the uniform convergence analysis for the multigrid methods for the linear algebraic systems of the differential operator  $\rho^2 \mathbf{I} - \kappa^2 \Delta - \mu^2 \nabla \operatorname{div}$  respect to the parameters  $\rho, \kappa$  and  $\mu$ . This open question is resolved in this paper.

## 6. Appendix

In the foregoing analysis, the operator  $\Pi_h^v$  and  $\Pi_h^w$  have been shown to play crucial roles for both the multigrid analysis and the inf-sup condition. In this section, we shall construct these operators following the idea from Girault-Scott, [10].

In order to define a representation that is equivalent to (4.2) - (4.4) and (4.10) for functions with less regularities, we represent point evaluations or point evaluations of the partials by appropriate integrals. We note that all nodes  $a_i$  with  $|D_i| = 0$  and  $|D_i| = 1$ , belong to the edge of a rectangle  $\tau$  and, therefore, we shall pick  $\kappa_i$ , the edge that containing  $a_i$ , subject only to the restrictions that

- i.  $\kappa_i \in \partial\Omega$ , if  $a_i \in \partial\Omega$ .
- ii. If  $a_i$  is the corner vertex where the point evaluations of  $x$ -derivative, the edge  $\kappa_i$  is chosen parallel to  $x$ -axis
- iii. If  $a_i$  is the corner vertex where the point evaluations of  $y$ -derivative, the edge  $\kappa_i$  is parallel to  $y$ -axis.

- iv. For the edge moment, (e.g.,  $\int_{e_y^i} \partial_x u ds$ ), one has to take into account the partial derivative in the direction perpendicular to the edge (e.g.,  $e_y^i$ ). In this case,  $\kappa_i$  should be the edge in which direction the partial is taken.

The restrictions (i. - iii.) are made for the purpose of preserving homogeneous Dirichlet or Neumann boundary conditions. In this setting, we define  $\mathcal{P}_d^r$  by the space of polynomials of degree  $r$  or less in  $d$ -variables and denote the basis by  $\{\xi_i^\kappa\}$ . Note that  $\dim \kappa_i = 1$ , the space  $\mathcal{P}_d^r$  consists of one variable functions and it is either quadratic Lagrange polynomials of degree two or cubic Hermite polynomials. For both cases, we denote the dual basis of the basis  $\{\xi_i^\kappa\}$  by  $\{\chi_i^\kappa\} \in \mathcal{P}_d^r$  that is defined with respect to the following relation:

$$\int_{\kappa} \chi_i^\kappa \xi_j^\kappa b_\kappa d\mu(\kappa) = \delta_{ij}, \quad \forall \xi_j \in \mathcal{P}_d^r, \tag{6.1}$$

where  $\kappa_i$  is identified with  $\kappa$  for simplicity and  $d\mu(\kappa)$  denotes the Lebesgue measure on  $\kappa$ . The function  $b_\kappa$  is the image on  $\kappa$  of the polynomial  $\widehat{b}_{\widehat{\kappa}}$ , which will be chosen to be one dimensional bubble function  $\widehat{b}_{\widehat{\kappa}} = \widehat{x}(1 - \widehat{x})$  defined on  $\widehat{\kappa} = [0, 1]$  for the definition of  $\Pi_h^v$ . We note that by passing to the reference  $\widehat{\kappa}$  of  $\kappa$ , the formula (6.1) becomes

$$\int_{\widehat{\kappa}} \widehat{\chi}_i^{\widehat{\kappa}} \widehat{\xi}_j^{\widehat{\kappa}} \widehat{b}_{\widehat{\kappa}} \mathcal{J} d\widehat{\mu} = \int_{\kappa} \chi_i^\kappa \xi_j^\kappa b_\kappa d\mu(\kappa) \tag{6.2}$$

where  $\mathcal{J}$  is the Jacobian of the transformation that maps  $\widehat{\kappa}$  to  $\kappa$ . By the uniqueness of the dual basis, we have that

$$\mathcal{J} \widehat{\chi}_i^{\widehat{\kappa}} = \widehat{\chi}_i^{\widehat{\kappa}},$$

where  $\widehat{\chi}_i^{\widehat{\kappa}}$  is the corresponding dual basis on  $\widehat{\kappa}$  with respect to the bubble function  $\widehat{b}_{\widehat{\kappa}}$ . Therefore, there exists a constant  $\widehat{C}$  that depends only on  $r, d, \widehat{\kappa}$  and  $\widehat{b}_{\widehat{\kappa}}$  such that

$$\|\widehat{\chi}_i^{\widehat{\kappa}}\|_{\infty, \widehat{\kappa}} = \|\mathcal{J} \widehat{\chi}_i^{\widehat{\kappa}}\|_{\infty, \kappa} \lesssim \widehat{C}.$$

Now due to the fact that

$$\mathcal{J} \approx h, \quad \text{and} \quad \|b_\kappa\|_{\infty, \kappa} = \|\widehat{b}_{\widehat{\kappa}}\|_{\infty, \widehat{\kappa}} \lesssim \widehat{C},$$

we obtain the inequality that

$$\|b_\kappa \chi_i^\kappa\|_{\infty, \kappa} = \|\widehat{b}_{\widehat{\kappa}} \widehat{\chi}_i^{\widehat{\kappa}}\|_{\infty, \widehat{\kappa}} \lesssim h^{-1}. \tag{6.3}$$

We note that the nodal values related to the partial derivatives does not make sense in the trace sense, for functions  $f \in H^1(\Omega)$ , therefore, the integration by parts will be performed for

such cases to obtain

$$\int_{\kappa} D_i f \chi_i^{\kappa} b_{\kappa} d\mu(\kappa_i) = \int_{\kappa} \tilde{D}_i f \tilde{\chi} d\mu(\kappa_i), \text{ where } |\tilde{D}_i| = 0, \text{ and } \tilde{\chi} = -\frac{\partial}{\partial e} (\chi_i^{\kappa} b_{\kappa}) \quad (6.4)$$

with  $e$ , a unit direction which would be the direction of  $\kappa$ . It is easy to see that  $\|\tilde{\chi}\|_{\infty, \kappa} \lesssim h^{-2}$ .

We define  $\Pi_h^v = (\Pi_{1,h}, \Pi_{2,h})^t : (H_0^1(\Omega))^2 \mapsto \mathbf{V}_h$  as follows: For any  $\mathbf{u} = (u, v) \in (H_0^1(\Omega))^2$ , we shall define

$$\Pi_{1,h} u = \sum_{|D_i|=0} \left( \int_{\kappa_i} u \chi_i^{\kappa_i} b_{\kappa_i} d\mu(\kappa_i) \right) \Phi_i + \sum_{|D_i|=1} \left( \int_{\kappa_i} u \tilde{\chi}^{\kappa_i} d\mu(\kappa_i) \right) \bar{\Phi}_i$$

and

$$\Pi_{2,h} v = \sum_{|D_i|=0} \left( \int_{\kappa_i} v \chi_i^{\kappa_i} b_{\kappa_i} d\mu(\kappa_i) \right) \Psi_i + \sum_{|D_i|=1} \left( \int_{\kappa_i} v \tilde{\chi}^{\kappa_i} d\mu(\kappa_i) \right) \bar{\Psi}_i,$$

We note that the operator  $\Pi_h^v : (H_0^1(\Omega))^2 \mapsto \mathbf{V}_h$  is well-defined and is a projection  $\Pi_h^v \mathbf{u} = \mathbf{u}$ ,  $\forall \mathbf{u} \in \mathbf{V}_h$ , from the relation (6.2). In particular, from the edge moments, it holds true that for  $\mathbf{u} \in (H_0^1(\Omega))^2$  and  $\tau \in \mathcal{T}_h$ ,

$$\begin{aligned} \int_{\tau} \operatorname{div}(\Pi_h^v \mathbf{u} - \mathbf{u}) dx &= - \int_{\partial\tau} (\Pi_h^v \mathbf{u} - \mathbf{u}) \cdot \mathbf{n} dx \\ &= \int_{\partial\tau_1} \Pi_{1,h} u - u d\nu + \int_{\partial\tau_2} \Pi_{2,h} v - v d\nu \\ &\quad - \int_{\partial\tau_3} \Pi_{1,h} u - u d\nu - \int_{\partial\tau_4} \Pi_{2,h} v - v d\nu = 0. \end{aligned}$$

Furthermore, we obtain the following stability estimate that for  $m = 0, 1$  and  $\tau \in \mathcal{T}_h$ ,

$$\begin{aligned} \|\Pi_{1,h} u\|_{m,\tau} &\lesssim \sum_{|D_i|=0} \left| \int_{\kappa_i} u \chi_i^{\kappa_i} b_{\kappa_i} d\mu(\kappa_i) \right| \|\Phi_i\|_{m,\tau} + \sum_{|D_i|=1} \left| \int_{\kappa_i} u \tilde{\chi}^{\kappa_i} d\mu(\kappa_i) \right| \|\bar{\Phi}_i\|_{m,\tau} \\ &\lesssim \sum_{|D_i|=0} \left| \int_{\kappa_i} u \chi_i^{\kappa_i} b_{\kappa_i} d\mu(\kappa_i) \right| h^{-m+1} + \sum_{|D_i|=1} \left| \int_{\kappa_i} u \tilde{\chi}^{\kappa_i} d\mu(\kappa_i) \right| h^{-m+2} \\ &\lesssim h^{-m} \sum_i \int_{\kappa_i} |u| d\mu(\kappa_i) \\ &\lesssim h^{-m} (\|u\|_{0,\tau} + h|u|_{1,\tau}) = h^{-m} \|u\|_{0,\tau} + h^{-m+1} |u|_{1,\tau}. \end{aligned}$$

This has been crucially used to establish the inf-sup condition for the pair  $\mathbf{V}_h$  and  $S_h$ . We now turn to our attention to the operator  $\Pi_h^w$ . For any  $w \in H_0^2(\Omega)$ , we can define  $\Pi_h^w w$  by the

following:

$$\Pi_h^w w = \sum_{i=1}^N \left( \int_{\kappa_i} D_i w \chi_i^{\kappa_i} d\mu(\kappa_i) \right) \zeta_i, \tag{6.5}$$

where we do need the integration by parts only for the degree of freedom that involves the mixed partial derivatives,  $\partial_x \partial_y$ . Obviously, only with performing the integration by parts for such degrees of freedom, it is easy to see that the operator  $\Pi_h^w : H_0^2(\Omega) \mapsto W_h$  is well-defined. Under this setting, using the estimates for the norms of the basis functions presented in the Lemma 4.2, we can establish the following Theorems from the same arguments from Girault-Scott, [10] and Scott-Zhang, [18].

**Proposition 6.1.** *The operator  $\Pi_h^v : (H_0^1(\Omega))^2 \mapsto \mathbf{V}_h$  is stable and also possesses the approximation property as follows:*

$$\|\Pi_h^v \mathbf{u}\|_{1,\Omega} \lesssim \|\mathbf{u}\|_{1,\Omega} \tag{6.6}$$

and for  $m = 0, 1$ ,

$$|\mathbf{u} - \Pi_h^v \mathbf{u}|_{m,\Omega} \lesssim h^{1-m} |\mathbf{u}|_{1,\Omega} \quad \forall \mathbf{u} \in (H_0^1(\Omega))^2. \tag{6.7}$$

**Proposition 6.2.** *The interpolation operator  $\Pi_h^w : H_0^2 \mapsto W_h$  is such that  $\Pi_h^w q = q$ ,  $\forall q \in W_h$  and satisfies for  $m = 0, 1$ ,*

$$\|q - \Pi_h^w q\|_m \lesssim h^{2-m} |q|_2, \quad \forall q \in H_0^2(\Omega). \tag{6.8}$$

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