Solution of Exam #1   Math 251:H1   2/26/2009

1. [15 pts] Consider the points $P = (2, 0, 0)$, $Q = (0, 4, 0)$ and $R = (0, 0, 2)$.

(a) [10pts] Find the equation of the plane Π_0 containing P, Q and R.

Solution: Let $\mathbf{v}_1 = \overrightarrow{PQ} = \langle -2, 4, 0\rangle$ and $\mathbf{v}_2 = \overrightarrow{PR} = \langle -2, 0, 2\rangle$. Hence, $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 8, 4, 8\rangle$ is a vector normal to the plane Π_0. Since $P = (2, 0, 0)$ is a point in the plane, the equation is given by

$$\Pi_0 : \quad \mathbf{n} \cdot \langle x - 2, y, z\rangle = 8(x - 2) + 4y + 8z = 0.$$

(b) [5 pts] Find the equation of the plane Π_1 parallel to Π_0 and containing the point $(1, 1, 1)$.

Solution: Since Π_1 is parallel to Π_0, $\mathbf{n}$ is a vector normal to Π_1 as well. Hence, we know that the equation of Π_1 is of the form

$$\Pi_1 : \quad \mathbf{n} \cdot \langle x, y, z\rangle = 8x + 4y + 8z = d,$$

where d is to be determined. Now, since the point $(1, 1, 1)$ is in Π_1, we must have that $8(1)+4(1)+8(1) = 20 = d$. This implies that $d = 20$. Therefore, the equation of the plane Π_1 is

$$\Pi_1 : \quad 8x + 4y + 8z = 20.$$

2. **[10 pts]** Consider a curve $\mathcal{C}$ given by a differentiable vector function $\mathbf{r}(t)$. Assuming that we have the equality $\mathbf{r}'(t) = (5t)\mathbf{T}(t)$, what is the arc length of the section of $\mathcal{C}$ that starts at $\mathbf{r}(0)$ and ends at $\mathbf{r}(2)$?

Solution: Since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$, we get that $\mathbf{r}'(t) = \|\mathbf{r}'(t)\|\mathbf{T}(t) = (5t)\mathbf{T}(t)$, by hypothesis. Hence, we get that $\|\mathbf{r}'(t)\| = 5t$. We use this fact to compute the arc length of the section of $\mathcal{C}$ that starts at $\mathbf{r}(0)$ and ends at $\mathbf{r}(2)$:

$$\int_0^2 \|\mathbf{r}'(u)\| du = \int_0^2 5u \; du = 10.$$

3. [10 pts] Consider the function $f(x,y) = \frac{(x^2+y^2)^3}{2x^2+3y^2}$. Show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Solution: Let $\varepsilon > 0$. Now,

$$|f(x,y) - 0| = \frac{(x^2+y^2)^3}{2x^2+3y^2} \leq \frac{(x^2+y^2)^3}{2x^2+2y^2} = \frac{1}{2}(x^2+y^2)^2.$$

Hence, letting $\delta = \delta(\varepsilon) = (2\varepsilon)^{\frac{1}{4}}$, we have that

$$x^2 + y^2 < \delta^2 \Rightarrow |f(x,y)| \leq \frac{1}{2}(x^2+y^2)^2 < \frac{1}{2}\delta^4 = \frac{1}{2}2\varepsilon = \varepsilon.$$

By definition of the limit, we showed that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

4. **[15 pts]** Consider the function $f(x,y) = y^2x - yx^2 + xy$. Find all critical points of $f(x,y)$ and determine whether they are local minima, local maxima or saddle points.

<u>**Solution:**</u> We want to find points such that $f_x(y,x) = y^2 - 2xy + y = y(y-2x+1) = 0$ and $f_y(x,y,) = 2xy - x^2 + x = x(2y-x+1) = 0$. We have 4 possibilities:

1. $x = 0, y = 0$.
   Then $P_0 = (0,0)$ is a critical point.

2. $x = 0, y \neq 0$.
   In this case, $f_x(0,y) = y(y+1) = 0$ and then $y = -1$. Then $P_1 = (0,-1)$ is a critical point.

3. $x \neq 0, y = 0$.
   In this case $f_y(x,0) = x(-x+1) = 0$ and then $x = 1$. Then $P_2 = (1,0)$ is a critical point.

4. $x \neq 0, y \neq 0$.
   In this case, we obtain that $y - 2x + 1 = 0$ and $2y - x + 1 = 0$. We get that $x = \frac{1}{3}$ and $y = -\frac{1}{3}$. Then $P_3 = \left(\frac{1}{3}, -\frac{1}{3}\right)$ is a critical point.

We now use the second derivative test to conclude about the nature of the 4 critical points. We first compute the partial derivatives

$$f_{xx} = -2y, \qquad f_{xy} = 2y - 2x + 1, \qquad f_{yy} = 2x,$$

and then the discriminant

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 = -4ab - [2b - 2a + 1]^2.$$

1. $D(P_0) = D(0,0) = -1 < 0$. Hence, $P_0 = (0,0)$ is a saddle point.

2. $D(P_1) = D(0,-1) = -1 < 0$. Hence, $P_1 = (0,-1)$ is a saddle point.

3. $D(P_2) = D(1,0) = -1 < 0$. Hence, $P_2 = (1,0)$ is a saddle point.

4. $D(P_3) = D\left(\frac{1}{3}, -\frac{1}{3}\right) = \frac{1}{3} > 0$ and $f_{xx}(P_3) = \frac{2}{3} > 0$. Hence, P_3 is a local minimum.

5. [10 pts] Consider $\mathbf{u}$ and $\mathbf{v}$ two vectors of three components. Show that

$$(\mathbf{u}\cdot\mathbf{v})^2 + \|\mathbf{u}\times\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2 = 0.$$

Solution: Let θ the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Combining the identities

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \quad \text{and} \quad \|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta,$$

with the identity $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$(\mathbf{u}\cdot\mathbf{v})^2 + \|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta + \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(\sin^2\theta + \cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2.$$

6. **[10 pts]** Consider the surface $\mathcal{S}$ generated by $xz^2 = 2y + x^2y^2$. Find the equation of the plane tangent to the surface $\mathcal{S}$ at the point $P = (-1, -1, 1)$.

__**Solution:**__ Define $F(x, y, z) = xz^2 - 2y - x^2y^2$. Then $\mathcal{S}$ is the level surface $F(x, y, z) = 0$. Hence, $\nabla F(-1, -1, 1) = \langle z^2 - 2xy^2, -2 - 2x^2y, 2xz\rangle_{|(x,y,z)=(-1,-1,1)} = \langle 3, 0, -2\rangle$ is a vector normal to $\mathcal{S}$ at $P = (-1, -1, 1)$. This implies that $\mathbf{n} = \langle 3, 0, -2\rangle$ is a normal vector to the plane tangent $\mathcal{S}$ at the point P. Therefore, the equation of the plane is

$$\mathbf{n} \cdot \langle x + 1, y + 1, z - 1\rangle = 3(x + 1) - 2(z - 1) = 0.$$

7. **[10 pts]** Consider the lines of equation

$$\mathcal{L}_1 : \ \mathbf{r}_1(t) = \langle 1, 0, 1\rangle + t\langle 2, 1, 0\rangle \ \text{ and } \ \mathcal{L}_2 : \ \mathbf{r}_2(s) = \langle 0, 2, 1\rangle + s\langle -1, 2, 0\rangle.$$

(a) [5pts] What is the point of intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$?

Solution: If $\mathbf{r}_1(t) = \mathbf{r}_2(s)$, we get that

$$\begin{aligned} 1 + 2t &= -s \\ t &= 2 + 2s. \end{aligned}$$

The unique solution is $s = -1$ and $t = 0$. Hence, the point of intersection corresponds to $\mathbf{r}_1(0) = \mathbf{r}_2(-1) = \langle 1, 0, 1\rangle$ and then the point of intersection is $P = (1, 0, 1)$.

(b) [5pts] What is the angle $0° \leq \theta \leq 90°$ of their intersection?

Solution: The angle between $\mathcal{L}_1$ and $\mathcal{L}_2$ is the angle between the direction vectors $\mathbf{v}_1 = \langle 2, 1, 0\rangle$ and $\mathbf{v}_2 = \langle -1, 2, 0\rangle$. Since $\langle 2, 1, 0\rangle \cdot \langle -1, 2, 0\rangle = 0$, it means that the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\theta = 90°$. Hence the angle between $\mathcal{L}_1$ and $\mathcal{L}_2$ is $\theta = 90°$

8. [**10pts**] Suppose that at the point $(1, -1)$, the directional derivative of the function $z = f(x, y)$ in the direction $-2\mathbf{i} + \mathbf{j}$ is equal to 1 and the directional derivative of $z = f(x, y)$ in the direction $\mathbf{i} - \mathbf{j}$ is equal to -2. Determine the gradient of f at the point $(1, -1)$.

Solution: Denote $\nabla f(1, -1) = \langle a, b\rangle$. Define $\mathbf{u}_1 = \langle -2, 1\rangle$ and $\mathbf{v}_1 = \langle 1, -1\rangle$. Since $\mathbf{u}_1$ and $\mathbf{v}_2$ are not unit, we have to scale them in order to compute the directional derivatives. Remember that directional derivatives are defined only for unit vectors! Hence, define

$$\mathbf{u} = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{5}}\langle -2, 1\rangle \quad \text{and} \quad \mathbf{v} = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}\langle 1, -1\rangle.$$

Then we know that

$$\begin{aligned} D_{\mathbf{u}}(1, -1) &= \nabla f(1, -1) \cdot \mathbf{u} = \langle a, b\rangle \cdot \frac{1}{\sqrt{5}}\langle -2, 1\rangle = \frac{1}{\sqrt{5}}(-2a + b) = 1 \\ D_{\mathbf{v}}(1, -1) &= \nabla f(1, -1) \cdot \mathbf{v} = \langle a, b\rangle \cdot \frac{1}{\sqrt{2}}\langle 1, -1\rangle = \frac{1}{\sqrt{2}}(a - b) = -2. \end{aligned}$$

Solving for a and b, we get that $a = 2\sqrt{2} - \sqrt{5}$ and $b = 4\sqrt{2} - \sqrt{5}$. Hence, we finally conclude that

$$\nabla f(1, -1) = \langle 2\sqrt{2} - \sqrt{5}, 4\sqrt{2} - \sqrt{5}\rangle.$$

9. [10 pts] Consider the curve $\mathcal{C}$ traced by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. Find the equation of the plane perpendicular to the curve $\mathcal{C}$ at the point $P = (2, 4, 8)$.

Solution: The point $P = (2, 4, 8)$ corresponds to $\mathbf{r}(2) = \langle 2, 2^2, 2^3 \rangle = \langle 2, 4, 8 \rangle$. Hence, let us compute the tangent vector $\mathbf{r}'(2)$, which will give a normal vector for the plane perpendicular to the $\mathcal{C}$ at P. We have that $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ and then that $\mathbf{r}'(2) = \langle 1, 4, 12 \rangle$. Define $\mathbf{n} = \langle 1, 4, 12 \rangle$. Therefore, the equation of the plane is

$$\mathbf{n} \cdot \langle x - 2, y - 4, z - 8 \rangle = (x - 2) + 4(y - 4) + 12(z - 8) = 0.$$

10. [Bonus Question: 10 pts] Recall that $v(t) = \|\mathbf{r}'(t)\|$. Show that

$$\mathbf{N}(t) = \frac{\mathbf{r}''(t) - v'(t)\mathbf{T}(t)}{\|\mathbf{r}''(t) - v'(t)\mathbf{T}(t)\|}$$

Solution: First observe that $\mathbf{r}'(t) = v(t)\mathbf{T}(t)$. Hence $\mathbf{r}''(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$. This implies that $v(t)\mathbf{T}'(t) = \mathbf{r}''(t) - v'(t)\mathbf{T}(t)$. However,

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{v(t)}{v(t)}\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{v(t)\mathbf{T}'(t)}{\|v(t)\mathbf{T}'(t)\|} = \frac{\mathbf{r}''(t) - v'(t)\mathbf{T}(t)}{\|\mathbf{r}''(t) - v'(t)\mathbf{T}(t)\|}.$$