Math 251:H1 Spring 2009: Solution of Exam #2

Name :

Problem Number	**Points Possible**	**Points**
1	10	
2	10	
3	10	
4	10	
5	15	
6	15	
7	10	
8	10	
9	10	
10 (Bonus)	10	
	Total	

Explain your computations !!

1. [10 pts] Evaluate the line integral

$$\int_C xe^y dx \ ,$$

where C is the arc of the curve $x = e^y$ from $(1, 0)$ to $(e, 1)$.

Hint: Note the dx in the integral !

<u>**Solution:**</u>

$$\int_C xe^y dx = \int_C xe^y dx + \int_C 0 \ dy = \int_C \mathbf{F} \cdot d\mathbf{s},$$

where $\mathbf{F} = \langle xe^y, 0 \rangle$. Now, C is parametrized by $c(t) = (e^t, t)$, with $0 \leq t \leq 1$. We get that

$$c'(t) = \langle e^t, 1 \rangle.$$

Hence

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(c(t)) \cdot c'(t) dt = \int_0^1 \langle e^t e^t, 0 \rangle \cdot \langle e^t, 1 \rangle dt = \int_0^1 e^{3t} dt = \frac{e^3 - 1}{3}.$$

Then,

$$\int_C xe^y dx = \frac{e^3 - 1}{3}.$$

2. [**10 pts**] Consider the region D in the xy-plane given by

$$D = \{(x, y) \mid \sqrt[3]{y} \le x \le 2,\ 0 \le y \le 8\}.$$

What is the volume of the solid $\mathcal{W}$ bounded below by D and above by the graph $z = e^{x^4}$.

Solution: The volume is given by

$$V = \int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4} dx\ dy = \int_0^2 \int_0^{x^3} e^{x^4} dy\ dx = \int_0^2 x^3 e^{x^4} dx = \frac{e^{16} - 1}{4}.$$

3. **[10 pts]** Consider the vector field $\mathbf{F}(x, y) = \langle e^y, xe^y \rangle$ and consider the curve C consisting of the union of the piecewise smooth curves C_1, C_2, C_3, C_4, C_5, C_6, C_7 and C_8, where

C_1 : Line segment from $(2, 0)$ to $(2, 1)$
C_2 : Line segment from $(2, 1)$ to $(1, 2)$
C_3 : Line segment from $(1, 2)$ to $(0, 2)$
C_4 : Line segment from $(0, 2)$ to $(-1, 1)$
C_5 : Line segment from $(-1, 1)$ to $(-1, 0)$
C_6 : Line segment from $(-1, 0)$ to $(0, -1)$
C_7 : Line segment from $(0, -1)$ to $(1, -1)$
C_8 : Line segment from $(1, -1)$ to $(2, 0)$

Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{s}$.

Solution: Since $\frac{\partial \mathbf{F}_1}{\partial y} = e^y = \frac{\partial \mathbf{F}_2}{\partial x}$ and since the vector field $\mathbf{F}$ is defined everywhere on the plane (which is simply connected), then $\mathbf{F}$ is conservative. Since $\mathbf{F}$ is conservative, there exists a potential function φ such that $\mathbf{F} = \nabla\varphi$. Note that C is an oriented closed curve. By the *Fundamental Theorem for Gradient Vector Fields*, we can conclude that

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \varphi(2, 0) - \varphi(2, 0) = 0.$$

4. [10 pts] Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = 2x + 6y + 10z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

<u>**Solution:**</u> Consider the function $g(x, y, z) = x^2 + y^2 + z^2 - 1$. Hence, we want to optimize f subject to the constraint $g = 0$. Since $\nabla f(x, y, z) = \langle 2, 6, 10\rangle$ and $\nabla g(x, y, z) = \langle 2x, 2y, 2z\rangle$, the Lagrange multipliers equation $\nabla f = \lambda \nabla g$ implies that

$$\lambda = \frac{1}{x} = \frac{3}{y} = \frac{5}{z}.$$

Note that $x \neq 0$, $y \neq 0$ and $z \neq 0$because that would violate $\nabla f = \lambda \nabla g$. Hence, $y = 3x$ and $z = 5x$. Plugging these values in $g = 0$, on gets that

$$g(x, y, z) = g(x, 3x, 5x) = 35x^2 - 1 = 0.$$

Hence, $x = \pm\frac{1}{\sqrt{35}}$. That means that we get two points

$$P_1 = \left(\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}\right), \quad \text{and} \quad P_2 = \left(-\frac{1}{\sqrt{35}}, -\frac{3}{\sqrt{35}}, -\frac{5}{\sqrt{35}}\right).$$

Finally, $f(P_1) = \frac{70}{\sqrt{35}}$ is the maximum and $f(P_2) = -\frac{70}{\sqrt{35}}$ is the minimum value.

5. [15 pts] Evaluate

$$\iiint_{\mathcal{W}} z\sqrt{x^2+y^2}\;dV,$$

where $\mathcal{W}$ is the region bounded by the surfaces $x^2+y^2=8-z$ and $x^2+y^2=z$.

<u>Solution:</u> Note first that the intersection of the surfaces $x^2+y^2=8-z$ and $x^2+y^2=z$ occurs when $z=8-x^2-y^2=x^2+y^2$. In other words, the projection of the intersection in the x,y plane is $x^2+y^2=4$. Hence, one we can define $D=\{(x,y)\mid x^2+y^2\leq 4\}$ and write

$$\mathcal{W}=\{(x,y,z)\mid (x,y)\in D\ \text{ and }\ x^2+y^2\leq z\leq 8-(x^2+y^2)\}.$$

We then get that

$$\iiint_{\mathcal{W}} z\sqrt{x^2+y^2}\;dV=\iint_D\left[\int_{x^2+y^2}^{8-(x^2+y^2)} z\sqrt{x^2+y^2}dz\right]dA.$$

Using cylindrical coordinates $x=r\cos\theta$, $y=r\sin\theta$ and $z=z$, one gets that

$$\begin{aligned}\iiint_{\mathcal{W}} z\sqrt{x^2+y^2}\;dV &= \int_0^{2\pi}\int_0^2\left[\int_{r^2}^{8-r^2} zr dz\right]rdrd\theta\\ &= \int_0^{2\pi}\int_0^2\frac{(8-r^2)^2-(r^2)^2}{2}r^2drd\theta\\ &= \pi\int_0^2 64r^2-16r^4dr=\frac{1024\pi}{15}.\end{aligned}$$

6. **[15 pts]** Consider the vector field $\mathbf{F}(x,y,z) = e^y\ \mathbf{i} + xe^y\ \mathbf{j} + (z+1)e^z\ \mathbf{k}$.

(a) Show that $\mathbf{F}$ is conservative.

Solution: Since the vector field is defined on all of $\mathbb{R}^3$ which is simply connected (no holes) and since

$$\frac{\partial}{\partial y}(e^y) = e^y = \frac{\partial}{\partial x}(xe^y), \quad \frac{\partial}{\partial z}(xe^y) = 0 = \frac{\partial}{\partial y}((z+1)e^z), \quad \frac{\partial}{\partial x}((z+1)e^z) = 0 = \frac{\partial}{\partial z}(e^y),$$

we can conclude that $\mathbf{F}$ is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

Solution: Since $\mathbf{F}$ is conservative, we know that there exists a potential function f such that $\mathbf{F} = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle e^y, xe^y, (z+1)e^z \rangle$. Hence, we get that $\frac{\partial f}{\partial x} = e^y$, which implies that $f(x,y,z) = \int e^y dx = xe^y + c(y,z)$. Now, $\frac{\partial f}{\partial y} = xe^y = xe^y + \frac{\partial}{\partial y}c(y,z)$ implies that $\frac{\partial}{\partial y}c(y,z) = 0$. In other words, it means that $c(y,z)$ is independent of the variable y. Then let $c(y,z) = c(z)$. Finally, using that $\frac{\partial f}{\partial z} = (z+1)e^z = \frac{\partial}{\partial z}[xe^y + c(y,z)] = c'(z)$, one can conclude that $c(z) = \int (z+1)e^z dz = ze^z + k$, where k is a constant. Hence, we get that

$$f(x,y,z) = \int e^y dx = xe^y + ze^z + k. \tag{1}$$

(c) Let $\mathbf{F}$ be the vector field given above. Consider the curve C represented by the path $\mathbf{c}(t) = (t, t^2, t^3)$, $0 \le t \le 1$ and evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$.

Solution: Since $\mathbf{F} = \nabla f$, where f is given by (1), we can use the *Fundamental Theorem for Gradient Vector Fields* to conclude that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(1)) - f(\mathbf{c}(0)) = f(1,1,1) - f(0,0,0) = e + e + k - (0 + 0 + k) = 2e.$$

7. [10 pts] Evaluate

$$\iint_{\Phi(\mathcal{R})} e^{\frac{y}{x+y}}\ dA,$$

where $\mathcal{R} = [0,1] \times [0,1]$ and $\Phi(u,v) = (u - uv, uv)$.

Solution: First of all,

$$|Jac(\Phi)| = \left| \begin{matrix} \frac{\partial}{\partial u}(u-uv) & \frac{\partial}{\partial v}(u-uv) \\ \frac{\partial}{\partial u}(uv) & \frac{\partial}{\partial v}(uv) \end{matrix} \right| = \left| \begin{matrix} 1-v & -u \\ v & u \end{matrix} \right| = u.$$

Hence, since $\frac{y}{x+y} = \frac{uv}{(u-uv)+(uv)} = v$, one gets that

$$\iint_{\Phi(\mathcal{R})} e^{\frac{y}{x+y}}\ dA = \int_0^1 \int_0^1 e^v u\ dudv = \frac{e-1}{2}.$$

8. **[10 pts]** Let $f(x,y,z) = x + yz$ and let $\mathcal{C}$ be the line segment from $P = (0,0,0)$ to $Q = (6,2,2)$. Evaluate

$$\int_{\mathcal{C}} f(x,y,z)\ ds.$$

Solution: Since $\mathbf{c}(t) = (6t, 2t, 2t)$, $0 \leq t \leq 1$ is a parametrization of $\mathcal{C}$, then

$$\int_{\mathcal{C}} f(x,y,z)\ ds = \int_0^1 f(6t,2t,2t)\|\langle 6,2,2\rangle\| = \sqrt{44}\int_0^1 6t + 4t^2\ dt = \frac{13\sqrt{44}}{3}.$$

9. **[10 pts]** Consider the region $\mathcal{W} = \{(x, y, z) \mid 2 \leq x^2 + y^2 + z^2 \leq 4\}$. Evaluate

$$\iiint_{\mathcal{W}} \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\, dV.$$

Solution: Using the spherical coordinates $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$, one gets that

$$\begin{aligned}
\iiint_{\mathcal{W}} \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\, dV &= \int_0^{2\pi} \int_0^{\pi} \int_{\sqrt{2}}^{2} \frac{1}{\rho^3} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta \\
&= 2\pi[\ln(2) - \ln(\sqrt{2})] \int_0^{\pi} \sin\phi \; d\phi \\
&= 2\pi \ln 2.
\end{aligned}$$

10 (Bonus). **[10 pts]** Show that if $\mathbf{F}$ is a constant vector field in $\mathbb{R}^3$ and $\mathcal{C}$ is any oriented path connecting the two 3-dimensional points P to Q, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot ds = \mathbf{F} \cdot \overrightarrow{PQ}.$$

<u>**Solution:**</u> Consider $\mathbf{F} = \langle A, B, C \rangle$ the constant vector field and let $P = (P_1, P_2, P_3)$, $Q = (Q_1, Q_2, Q_3)$. Assume that we parametrized the curve $\mathcal{C}$ by $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$ $(a \le t \le b)$ such that $c(a) = (c_1(a), c_2(a), c_3(a)) = (P_1, P_2, P_3) = P$ and $c(b) = (c_1(b), c_2(b), c_3(b)) = (Q_1, Q_2, Q_3) = Q$. Then

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot ds &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\ dt \\
&= \int_a^b \langle A, B, C \rangle \cdot \langle c_1'(t), c_2'(t), c_3'(t) \rangle\ dt \\
&= A \int_a^b c_1'(t)\ dt + B \int_a^b c_2'(t)\ dt + C \int_a^b c_3'(t)\ dt \\
&= A[c_1(b) - c_1(a)] + B[c_2(b) - c_2(a)] + C[c_3(b) - c_3(a)] \\
&= A(Q_1 - P_1) + B(Q_2 - P_2) + C(Q_3 - P_3) \\
&= \langle A, B, C \rangle \cdot \langle Q_1 - P_1, Q_2 - P_2, Q_3 - P_3 \rangle \\
&= \mathbf{F} \cdot \overrightarrow{PQ}.
\end{aligned}$$