Math 251:H1 - Workshop #1 (Solution)

1. Consider the following planes:

$$x - 2y + z = 1 \qquad \text{and} \qquad 2x + y + z = 1.$$

Believe it or not...they intersect. Let L be their line of intersection.

   (a) What is the angle between these two planes?

   (b) Use normal vectors of the planes to find a vector in the direction of L. Then find a point on L, and give a equation for L.

   (c) Find the equation for the plane perpendicular to L that passes through the point $(1, 2, 3)$.

   (d) Plot all three planes and the line (in Maple) in *different* colors.

Solution: (a) The angle between the two planes is given by the angle between the normal vectors $\mathbf{n}_1 = \langle 1, -2, 1\rangle$ and $\mathbf{n}_2 = \langle 2, 1, 1\rangle$ given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}\right) = \cos^{-1}\left(\frac{1}{6}\right) \approx 80.4059^\circ.$$

(b) Note that $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -3, 1, 5\rangle$ is a direction vector for the line L. A point (x, y, z) in the intersection of the planes is such that $z = 1 - x + 2y$ and $z = 1 - 2x - y$. Hence $1 - x + 2y = 1 - 2x - y$ and then $x = -3y$. We can chose $x = y = 0$ which implies that $z = 1$. We then get that $P = (0, 0, 1)$ belongs to the intersection of the two planes. We then get that the equation of the line is

$$L: \ \mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v} = \langle 0, 0, 1\rangle + t\langle -3, 1, 5\rangle = \langle -3t, t, 1 + 5t\rangle.$$

(c) The plane we are looking for has a normal vector parallel to the direction vector $\mathbf{v} = \langle -3, 1, 5\rangle$ of the line L. Hence letting $\mathbf{n} = \langle -3, 1, 5\rangle$ and $P_0 = (1, 2, 3)$, the equation of the plane is $\mathbf{n} \cdot \langle x - 1, y - 2, z - 3\rangle = -3(x - 1) + (y - 2) + 5(z - 3) = 0$.

2. The Pythogorean Theorem tells us that $\sin^2\theta + \cos^2\theta = 1$. We know that $\sin\theta$ and $\cos\theta$ are related to the cross and dot products. So the above trig. identity should give us some sort of vector identity.

   (a) What is this identity?

   (b) Prove the following identity:

   $$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

   (c) Use the identity from part (b) to prove the <u>Jacobi identity</u>:

   $$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}.$$

<u>**Solution:**</u> (a) We know that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$ and that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$ which implies that

$$(\mathbf{u} \cdot \mathbf{v})^2 + (\|\mathbf{u} \times \mathbf{v}\|)^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(\cos^2\theta + \sin^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2.$$

Hence, the identity is

$$(\mathbf{u} \cdot \mathbf{v})^2 + \|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2. \tag{1}$$

(b) Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$. Then

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \langle u_1, u_2, u_3\rangle \times \langle v_2w_3 - w_2v_3, -v_1w_3 + w_1v_3, v_1w_2 - w_1v_2\rangle \\
&= \langle u_2(v_1w_2 - w_1v_2) - (-v_1w_3 + w_1v_3)u_3, \\
&\quad -u_1(v_1w_2 - w_1v_2) + (v_2w_3 - w_2v_3)u_3, \\
&\quad u_1(-v_1w_3 + w_1v_3) - (v_2w_3 - w_2v_3)u_2\rangle \\
&= \langle (u_2w_2 + u_3w_3)v_1 - (u_2v_2 + u_3v_3)w_1, \\
&\quad (u_1w_1 + u_3w_3)v_2 - (u_1v_1 + u_3v_3)w_2, \\
&\quad (u_1w_1 + u_2w_2)v_3 - (u_1v_1 + u_2v_2)w_3\rangle \\
&= \langle (u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1, \\
&\quad (u_1w_1 + u_2w_2 + u_3w_3)v_2 - (u_1v_1 + u_2v_2 + u_3v_3)w_2, \\
&\quad (u_1w_1 + u_2w_2 + u_3w_3)v_3 - (u_1v_1 + u_2v_2 + u_3v_3)w_3\rangle \\
&= (u_1w_1 + u_2w_2 + u_3w_3)\langle v_1, v_2, v_3\rangle - (u_1v_1 + u_2v_2 + u_3v_3)\langle w_1, w_2, w_3\rangle \\
&= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.
\end{aligned}$$

(c) Using (b), we get that

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} + (\mathbf{v} \cdot \mathbf{u})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \\
&\quad + (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v} \\
&= 0.
\end{aligned}$$

3. Let $\mathbf{r}(t)$ be the parametrization of a smooth curve C. Also, let $s(t)$ be the associated arc-length function. Suppose that $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$, $\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s}$ exist at each point on the curve. We define the <u>binormal vector</u> by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ (Therefore, $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{T} \times \mathbf{N}] = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \times \mathbf{N} + \mathbf{T} \times \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s}$ exists as well.)

   (a) Show that $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}$ is perpendicular to $\mathbf{B}$.

   (b) Show that $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}$ is perpendicular to $\mathbf{T}$ (*Hint:* Use the fact that $\mathbf{B}$ is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$, and differentiate $\mathbf{B} \cdot \mathbf{T}$ with respect to s.)

   (c) Show that $\mathbf{T} = \mathbf{N} \times \mathbf{B}$ and that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$. (*Hint:* You may use results from other problems in this workshop!)

   (d) Use the previous parts to show that $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}$ is a multiple of $\mathbf{N}$.

   (e) Therefore, $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau(s)\mathbf{N}$ for some function $\tau(s)$. τ is called the *torsion* of the curve C. Note that if the curvature $\kappa(s)$ of C is identically zero, then C is a straight line. What can be said about the curve if $\tau(s) = 0$ for all s?

   (f) Differentiate $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ with respect to s. Show that $\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -\kappa\mathbf{T} + \tau\mathbf{B}$.

   Putting all of this together we have the following fundamental space curve formulas, called the *Frenet-Serret* formulas:

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N}, \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -\kappa\mathbf{T} + \tau\mathbf{B}, \qquad \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = -\tau\mathbf{N}$$

<u>Solution:</u> (a) Since $\|B(s)\| = 1$ for all s, then $B(s) \cdot B(s) = \|B(s)\|^2 = 1$ for all s. Differentiating on both sides with respect to the arc length parameter s, we get that

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{B}(s) \cdot \mathbf{B}(s)) = \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \cdot \mathbf{B} + \mathbf{B} \cdot \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = 2\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \cdot \mathbf{B} = \frac{\mathrm{d}}{\mathrm{d}s}(1) = 0.$$

We can conclude that $\mathbf{B} \cdot \frac{d\mathbf{B}}{ds} = 0$. In other words, we have that $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}$ is perpendicular to $\mathbf{B}$.

(b) Since $\mathbf{B}$ is perpendicular to $\mathbf{T}$, we have that $\mathbf{B} \cdot \mathbf{T} = 0$. Hence,

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}(0) = \frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{B} \cdot \mathbf{T}) = \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \cdot \mathbf{T} + \mathbf{B} \cdot \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}. \tag{2}$$

We know that $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$ is parallel to $\mathbf{N}$, by definition of the normal vector $\mathbf{N}$. This implies that the vector $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$ is perpendicular to $\mathbf{B}$ and then that $\mathbf{B} \cdot \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = 0$. Plugging the last equality in (2), we get that $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \cdot \mathbf{T} = 0$, and the conclusion follows.

(c) By definition, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Hence, $\mathbf{N} \times \mathbf{B} = \mathbf{N} \times (\mathbf{T} \times \mathbf{N}) \overset{2(b)}{=} (\mathbf{N} \cdot \mathbf{N})\mathbf{T} - (\mathbf{N} \cdot \mathbf{T})\mathbf{N} = \|\mathbf{N}\|^2\mathbf{T} - (0)\mathbf{N} = \mathbf{T}$, since $\mathbf{N}$ is a unit vector and the vectors $\mathbf{T}$ and $\mathbf{N}$ are perpendicular. That shows that $\mathbf{N} \times \mathbf{B} = \mathbf{T}$. Now, $\mathbf{B} \times \mathbf{T} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{T} = -\mathbf{T} \times (\mathbf{T} \times \mathbf{N}) \overset{2(b)}{=} -[(\mathbf{T} \cdot \mathbf{N})\mathbf{T} - (\mathbf{T} \cdot \mathbf{T})\mathbf{N}] = -(0)\mathbf{T} + \|T\|^2\mathbf{N} = \mathbf{N}$, since $\mathbf{T}$ is a unit vector and the vectors $\mathbf{T}$ and $\mathbf{N}$ are perpendicular. That shows that $\mathbf{B} \times \mathbf{T} = \mathbf{N}$.

(d) We now have that $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \times \mathbf{N} \overset{3(c)}{=} \frac{\mathrm{d}\mathbf{B}}{ds} \times (\mathbf{B} \times \mathbf{T}) \overset{2(b)}{=} \left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \cdot \mathbf{T}\right)\mathbf{B} - \left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \cdot \mathbf{B}\right)\mathbf{T} = (0)\mathbf{B} - (0)\mathbf{T} = \langle 0, 0, 0\rangle$, which follows from part 3(a) and 3(b). Since we proved that $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \times \mathbf{N} = \langle 0, 0, 0\rangle$, we then have that $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}$ and $\mathbf{N}$ are parallel. In other words, $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}$ is a multiple of $\mathbf{N}$.

(e) A curve that has zero torsion is such that $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} = 0$. Hence the binormal vector $B(t)$ is a constant unit vector for all t. That means that the curve lies on a fixed plane. In other words, the curve is a *planar* curve.

(f) Note first that by definition of $\mathbf{N}$ and the definition of the curvature κ, we have that $\mathbf{N} = \frac{\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}}{\left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\|} = \frac{\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}}{\kappa}$. Hence, we get that $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N}$. Now, $\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{B} \times \mathbf{T}) = \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} \times \mathbf{T} + \mathbf{B} \times \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = (-\tau\mathbf{N}) \times \mathbf{T} + \mathbf{B} \times (\kappa\mathbf{N}) = \tau(\mathbf{T} \times \mathbf{N}) - \kappa(\mathbf{N} \times \mathbf{B}) = \tau\mathbf{B} - \kappa\mathbf{T}$. That finally shows that $\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -\kappa\mathbf{T} + \tau\mathbf{B}$.

4. You are the captain of a spaceship that has been given the mission of flying from Deep Space τ located at coordinates $(1, 1, 0)$ to Research Station κ at coordinates $(3, 2, 2)$. You must be there in exactly one hour! Both stations require that you arrive and depart with zero velocity. You must also arrive at Research Station κ with zero acceleration. After considering calculating various forces acting on your craft, you have determined that you are able to fire your engines so that the acceleration of your spacecraft is given by the formula (t is measured in number of hours after departure): $\mathbf{a}(t) = t^2\mathbf{b} + t\mathbf{c} + \mathbf{d}$ where $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are constant vectors. Determine the vectors $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ so that you can complete your mission.

<u>**Solution:**</u> Denote the velocity vector by $\mathbf{v}(t)$ and the position vector by $\mathbf{r}(t)$. Hence, $\mathbf{v}(t) = \int \mathbf{a}(t)dt = \frac{t^3}{3}\mathbf{b} + \frac{t^2}{2}\mathbf{c} + t\mathbf{d} + \mathbf{e}$, where $\mathbf{e}$ is a constant vector. Now, since we know that we must leave τ with zero velocity, that means that $\mathbf{v}(0) = \mathbf{e} = \langle 0, 0, 0\rangle$. That implies that $\mathbf{v}(t) = \frac{t^3}{3}\mathbf{b} + \frac{t^2}{2}\mathbf{c} + t\mathbf{d}$. Knowing that we must arrive at κ with zero velocity, we have that

$$\mathbf{v}(1) = \frac{1}{3}\mathbf{b} + \frac{1}{2}\mathbf{c} + \mathbf{d} = \langle 0, 0, 0\rangle. \tag{3}$$

Now, $\mathbf{r}(t) = \int \mathbf{v}(t)dt = \frac{t^4}{12}\mathbf{b} + \frac{t^3}{6}\mathbf{c} + \frac{t^2}{2}\mathbf{d} + \mathbf{f}$. Since we depart from τ at the point $(1, 1, 0)$, that implies that $\mathbf{r}(0) = \mathbf{f} = \langle 1, 1, 0\rangle$ and since we arrive at κ at time $t = 1$ at the point $(3, 2, 2)$, we get that

$$\mathbf{r}(1) = \frac{1}{12}\mathbf{b} + \frac{1}{6}\mathbf{c} + \frac{1}{2}\mathbf{d} + \langle 1, 1, 0\rangle = \langle 3, 2, 2\rangle. \tag{4}$$

Now, we know that we must arrive at κ with zero acceleration. This implies that

$$\mathbf{a}(1) = \mathbf{b} + \mathbf{c} + \mathbf{d} = \langle 0, 0, 0\rangle. \tag{5}$$

Combining equations (3), (4) and (5), we can solve for the vectors $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ to get that

$$\mathbf{b} = \langle 72, 36, 72\rangle, \quad \mathbf{c} = \langle -96, -48, -96\rangle, \quad \mathbf{d} = \langle 24, 12, 24\rangle.$$

5. Prove the following formula for torsion:

$$\tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t))\cdot\mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2},$$

where $\mathbf{r}(t)$ is a regular parametrization.

Solution: From the conclusions of problem 3, we get that

$$\mathbf{T}' = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} = \kappa v \mathbf{N}, \qquad \mathbf{N}' = \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}t} = -\kappa v \mathbf{T} + \tau v \mathbf{B}, \qquad \mathbf{B}' = \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = -\tau v \mathbf{N},$$

where $v = v(t) = \|\mathbf{r}'(t)\|$. Note that since $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{v(t)}$, we get that $\mathbf{r}'(t) = v(t)\mathbf{T}(t)$. Hence, we obtain that $\mathbf{r}''(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$ and that $\mathbf{r}'''(t) = v''(t)\mathbf{T}(t) + 2v'(t)\mathbf{T}'(t) + v(t)\mathbf{T}''(t)$. Therefore, $\mathbf{r}' \times \mathbf{r}'' = v\mathbf{T} \times (v'\mathbf{T} + v\mathbf{T}') = vv'(\mathbf{T} \times \mathbf{T}) + v^2(\mathbf{T} \times \mathbf{T}') = v^2(\mathbf{T} \times \kappa v \mathbf{N}) = v^3\kappa\mathbf{B}$. Note that $\mathbf{T}' = \kappa v \mathbf{N}$ implies that $\mathbf{T}'' = \kappa' v \mathbf{N} + \kappa v' \mathbf{N} + \kappa v \mathbf{N}'$. Hence,

$$\begin{aligned}
(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' &= v^3\kappa\mathbf{B} \cdot (v''\mathbf{T} + 2v'\mathbf{T}' + v\mathbf{T}'') \\
&= v^3\kappa v''(\overbrace{\mathbf{B} \cdot \mathbf{T}}^{=0}) + 2v^3\kappa v'(\mathbf{B} \cdot \mathbf{T}') + v^4\kappa(\mathbf{B} \cdot \mathbf{T}'') \\
&= 2v^3\kappa v'(\mathbf{B} \cdot \kappa v \mathbf{N}) + v^4\kappa\mathbf{B} \cdot (\kappa' v \mathbf{N} + \kappa v' \mathbf{N} + \kappa v \mathbf{N}') \\
&= 2v^4\kappa^2 v'(\overbrace{\mathbf{B} \cdot \mathbf{N}}^{=0}) + v^4\kappa \left[\kappa' v(\overbrace{\mathbf{B} \cdot \mathbf{N}}^{=0}) + \kappa v'(\overbrace{\mathbf{B} \cdot \mathbf{N}}^{=0}) + \kappa v(\mathbf{B} \cdot \mathbf{N}')\right] \\
&= v^5\kappa^2(\mathbf{B} \cdot \mathbf{N}') \\
&= v^5\kappa^2\left[\mathbf{B} \cdot (-\kappa v \mathbf{T} + \tau v \mathbf{B})\right] \\
&= -v^6\kappa^3(\overbrace{\mathbf{B} \cdot \mathbf{T}}^{=0}) + \tau v^6\kappa^2(\overbrace{\mathbf{B} \cdot \mathbf{B}}^{=1}).
\end{aligned}$$

Hence, we get that $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = \tau v^6\kappa^2$. Since $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \kappa(t)v^3(t)$, we can conclude that

$$\tau(t) = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{v^6\kappa^2} = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}.$$

6. Let C be the curve parametrized by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t^2\mathbf{k}$

   (a) Find $\mathbf{T}(\pi)$, $\mathbf{N}(\pi)$, and $\mathbf{B}(\pi)$.

   (b) Compute the length of C between $t = 0$ and $t = \pi$.

   (c) Show that $\kappa(t) = \sqrt{\frac{5+4t^2}{(4t^2+1)^3}}$. Even though the first two components of this curve describe uniform circular motion, $\lim_{t\to\infty} \kappa(t) = 0$. Explain briefly why this can happen.

   (d) Compute $\tau(t)$.

   Solution: This is very technical. I don't provide a solution for this problem. I'm lazy!