Math 251:H1 - Workshop #1

1. Consider the following planes:

$$x - 2y + z = 1 \qquad \text{and} \qquad 2x + y + z = 1.$$

Believe it or not...they intersect. Let L be their line of intersection.

(a) What is the angle between these two planes?

(b) Use normal vectors of the planes to find a vector in the direction of L. Then find a point on L, and give a equation for L.

(c) Find the equation for the plane perpendicular to L that passes through the point $(1, 2, 3)$.

(d) Plot all three planes and the line (in Maple) in *different* colors.

2. The Pythogorean Theorem tells us that $\sin^2\theta + \cos^2\theta = 1$. We know that $\sin\theta$ and $\cos\theta$ are related to the cross and dot products. So the above trig. identity should give us some sort of vector identity.

(a) What is this identity?

(b) Prove the following identity:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

(c) Use the identity from part (b) to prove the Jacobi identity:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}.$$

3. Let $\mathbf{r}(t)$ be the parametrization of a smooth curve C. Also, let $s(t)$ be the associated arc-length function. Suppose that $\frac{d\mathbf{T}}{ds}$, $\frac{d\mathbf{N}}{ds}$ exist at each point on the curve. We define the binormal vector by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ (Therefore, $\frac{d\mathbf{B}}{ds} = \frac{d}{ds}[\mathbf{T} \times \mathbf{N}] = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds}$ exists as well.)

(a) Show that $\frac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{B}$.

(b) Show that $\frac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{T}$ (*Hint:* Use the fact that $\mathbf{B}$ is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$, and differentiate $\mathbf{B} \cdot \mathbf{T}$ with respect to s.)

(c) Show that $\mathbf{T} = \mathbf{N} \times \mathbf{B}$ and that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$. (*Hint:* You may use results from other problems in this workshop!)

(d) Use the previous parts to show that $\frac{d\mathbf{B}}{ds}$ is a multiple of $\mathbf{N}$.

(e) Therefore, $\frac{d\mathbf{B}}{ds} = -\tau(s)\mathbf{N}$ for some function $\tau(s)$. τ is called the *torsion* of the curve C. Note that if the curvature $\kappa(s)$ of C is identically zero, then C is a straight line. What can be said about the curve if $\tau(s) = 0$ for all s?

(f) Differentiate $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ with respect to s. Show that $\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$.

Putting all of this together we have the following fundamental space curve formulas, called the *Frenet-Serret* formulas:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

4. You are the captain of a spaceship that has been given the mission of flying from Deep Space τ located at coordinates $(1, 1, 0)$ to Research Station κ at coordinates $(3, 2, 2)$. You must be there in exactly one hour! Both stations require that you arrive and depart with zero velocity. You must also arrive at Research Station κ with zero acceleration. After considering calculating various forces acting on your craft, you have determined that you are able to fire your engines so that the acceleration of your spacecraft is given by the formula (t is measured in number of hours after departure): $\mathbf{a}(t) = t^2\mathbf{b} + t\mathbf{c} + \mathbf{d}$ where $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are constant vectors. Determine the vectors $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ so that you can complete your mission.

5. Prove the following formula for torsion:

$$\tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2},$$

   where $\mathbf{r}(t)$ is a regular parametrization.

6. Let C be the curve parametrized by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t^2\mathbf{k}$

   (a) Find $\mathbf{T}(\pi)$, $\mathbf{N}(\pi)$, and $\mathbf{B}(\pi)$.

   (b) Compute the length of C between $t = 0$ and $t = \pi$.

   (c) Show that $\kappa(t) = \sqrt{\frac{5+4t^2}{(4t^2+1)^3}}$. Even though the first two components of this curve describe uniform circular motion, $\lim_{t\to\infty} \kappa(t) = 0$. Explain briefly why this can happen.

   (d) Compute $\tau(t)$.