# Math 251:H1 - Workshop #5 - 04/23/2009

Each team has to turn in all six problems.
Due date: Last day of class (05/04/2009)

1. Use Stoke's theorem to compute the flux of curl($\mathbf{F}$) through the part of the cone $z^2 = x^2 + y^2$ such that $2 \leq z \leq 4$, where $\mathbf{F} = \langle x^2 + y^2, x + z^2, 0 \rangle$.

2. If a circle $C$ with radius 1 rolls along the outside of a circle $x^2 + y^2 = 16$, a fixed point $P$ on $C$ traces out a curve called an epicycloid, with parametric equations: $x(t) = 5\cos(t) - \cos(5t)$ and $y(t) = 5\sin(t) - \sin(5t)$ (See Figure 1). Compute the area enclosed by the epicycloid.

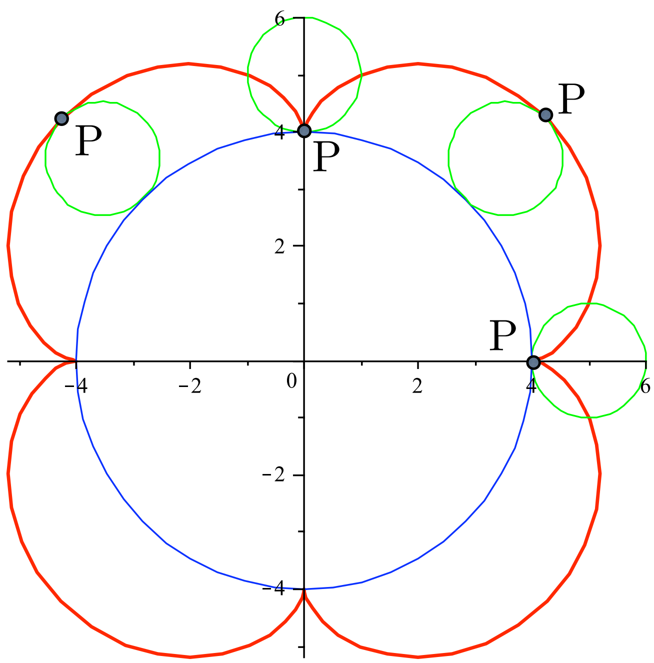


Figure 1: Epicycloid given by $\mathbf{r}(t) = \langle 5\cos(t) - \cos(5t), 5\sin(t) - \sin(5t) \rangle$

3. Find the area of the surface of the *ice cream cone* that lies under the paraboloid $z = 25 - x^2 - y^2$ and above the cone $z = \sqrt{x^2 + y^2}$.

4. Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations. For instance, if $\mathbf{E}$ is an electric field, then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the *electric flux* of $\mathbf{E}$ through the surface $S$. One of the important laws of electrostatic is **Gauss's Law**, which says that the net charge enclosed by a closed surface $S$ is

$$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S},$$

where $\varepsilon_0$ is a constant (called the permittivity of free space) that depends on the units used (in the international system of units (SI), $\varepsilon_0 \approx 8.8542 \times 10^{-12} \frac{C^2}{N \cdot m^2}$).

**Q:** Find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$, it the electric field is $\mathbf{E}(x, y, z) = \langle x, y, 2z \rangle$.

5. Find a parametric representation

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle, \quad (u, v) \in D \subset \mathbb{R}^2,$$

for the torus obtained by rotating about the $z$-axis the circle in the $xz$-plane with center $(b, 0, 0)$ and radius $a < b$ (See Figure 2). Using this representation, find the area of the torus.

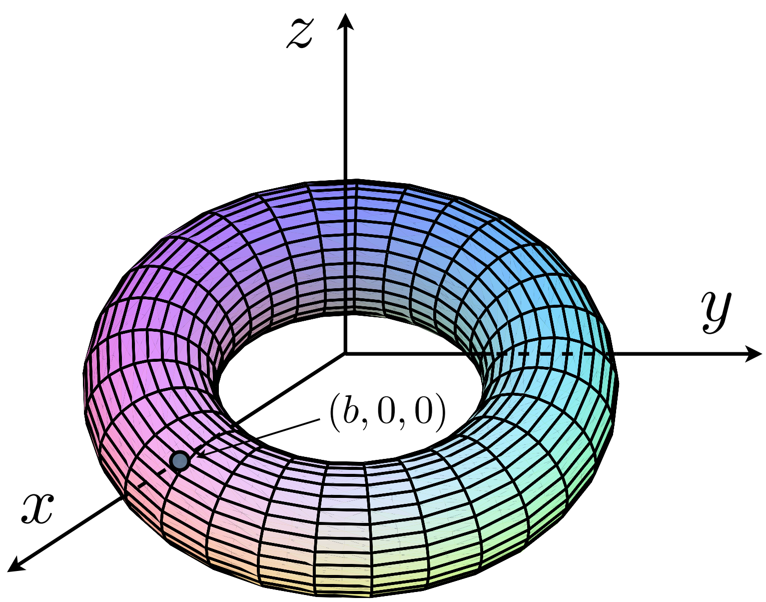


Figure 2: Torus

6. Let $\mathbf{F} = \langle F_1, F_2 \rangle$ be a continuous vector field whose components have continuous first partial derivatives. Let $D$ be a region in $\mathbb{R}^2$ whose boundary is a simple closed curve $C$ ($C$ positively oriented). Let $\mathbf{n}$ be unit outward normal for $C$. Assume that $f$ and $g$ are smooth functions from $\mathbb{R}^2$ to $\mathbb{R}$. Show that

   (a)
   $$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \left( \frac{\partial F_1}{\partial x}(x, y) + \frac{\partial F_2}{\partial y}(x, y) \right) \, dA.$$

   (b) Show that $\nabla \cdot (f\mathbf{F}) = f \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$.

   (c) Prove Green's first identity:
   $$\iint_D f \nabla^2 g \, dA = \int_C f \nabla g \cdot \mathbf{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA$$

   (d) Prove Green's second identity:
   $$\iint_D \left( f \nabla^2 g - g \nabla^2 f \right) \, dA = \int_C \left( f \nabla g - g \nabla f \right) \cdot \mathbf{n} \, ds$$