Math 115: Precalculus
Special review for the second exam

The following list is not complete, but should help you study for the second exam.

A. Theory

- Transformation of graphs, especially those involving exponential, logarithmic, and trigonometric functions
- Graphs of exponential and logarithmic functions: how the shape of the graph changes as the base varies; intercepts; asymptotes
- Graphs of trigonometric functions: pay attention to important key points (amplitudes, periods, phase shifts)
- Combining and expanding expression involving logarithms
- Solving equations involving exponential and logarithmic functions
- Modeling using exponential and logarithmic functions
- Basic trigonometry of right triangles
- Geometry of the unit circle

B. Sample problems

1. Problems 15-24, 39-40, pp. 336-337. This problem is aimed as recognizing the graph of some function of the form $c\,a^{x-b}+d$. For these problems, you should not need a calculator or make a table of values to identify which is which. For problems 15-24, 39-40, pay attention to the asymptote, the intercepts, and some sample points on the graph.

   Method: The (horizontal) asymptote gives you the value of d. The intercepts and sample points give you information about a, b, and c.

   Let's look at Problem 39, p. 337. The graph is known to be of the form $c\,a^x$. (In particular, b and d are given to be zero, great!)

   On the graph, one sample point is known: (2,12). This says

   $$c\,a^2 = 12.$$

   Since there are two unknowns, we should expect another relation to find them. The graph has no x-intercept, but y-intercept, which is 3. This says

   $$c = c\,a^0 = 3.$$

Okay, it should be easy by now. Substitute $c = 3$ into the previous equation, we get

$$3\,a^2 = 12,$$

so

$$a^2 = 4,$$

so $a = 2$. (The choice $a = -2$ is excluded since as a base, a is necessarily positive.)

We conclude that the given graph is that of $f(x) = 3\,2^x$.

2. Problems 37-46, p. 350. This is in the same spirit as those we consider in B.1, but with logarithms instead. The method is still the same: pay close attention to asymptote, intercepts and sample points.

   For example, let's work on Problem 39 again (p. 350, not p. 337 though). The function is of the form $\log_a x$. It's even simpler than before, we have only one unknown. The intercept doesn't help much, since the x-intercept of $\log_a x$ is always 1 no matter what the base is. But we are safe because of the sample point $(3, 1/2)$. This gives

   $$\frac{1}{2} = \log_a 3$$

   which is equivalent to

   $$a^{1/2} = 3.$$

   It should be clear at this moment that $a = 9$, and so the wished function is $f(x) = \log_9 x$.

3. Problems 25-38, p. 337. These problems involve graph transformations in which the original graph is that of an exponential function. What you need to recognize is the sequence of transformations, not only what kinds of transformations but also their order.

   Consider Problem 38, p. 337. We need to draw the graph of $y = e^{x-3} + 4$, given the graph of $y = e^x$. In this particular example, what we need to do is first shift the graph of $y = e^x$ to the right 3 units, then shift the resulting graph upward 4 units. Your job now is just simply draw the graphs.

4. Problems 49-58, p. 350. These are the counterparts of what we were looking at in B.3.

   Let me pick a random problem, say Problem 56. Okay, we need to draw the graph of $y = 1 + \ln(-x)$ from the graph of $y = \ln x$. What I expect you to do as you read along this sentence is to ask yourself what is the sequence of transformations needed. "Clearly" (isn't it?), you should first apply reflection about the y-axis to the graph of $y = \ln x$, and then raise what you obtained 1 unit up.

5. Problems 13-38, p. 357. What you need to do in these problems is to expand an expression involving logarithms. The key point is to know how logarithms change a product/quotient into a sum/difference.

   Somebody please give me a number, please! Problem 26? Thank you! You are asked to expand $\ln\sqrt[3]{3r^2s}$. Here is how it works:
$$\ln\sqrt[3]{3r^2s} = \ln\sqrt[3]{3\qquad r^2\qquad s} = \ln\sqrt[3]{3\qquad\qquad r^2\qquad\qquad s} = \dots$$
   Hope that this JOKE reenergizes you. The right answer is
$$\begin{aligned}\ln\sqrt[3]{3r^2s} &= \ln(3r^2s)^{1/3} = \frac{1}{3}\ln(3r^2s) = \frac{1}{3}\big[\ln 3 + \ln r^2 + \ln s\big]\\ &= \frac{1}{3}\big[\ln 3 + 2\ln r + \ln s\big] = \frac{1}{3}\ln 3 + \frac{2}{3}\ln r + \frac{1}{3}\ln s.\end{aligned}$$

6. Problems 39-48m p. 357. These are the converse to those we had in B.5. You just need to transform a sum/difference of logarithms into a single logarithm of a product/quotient.

   Let me work out the HARDEST one, Problem 40.
$$\log 12 + \frac{1}{2}\log 7 - \log 2 = \log 12 + \log 7^{1/2} - \log 2 = \log\frac{12\sqrt{7}}{2}.$$

7. Problem 1-18, p. 366; Problems 35-41, p. 367. These are the simplest equations involving exponential and/or logarithmic functions. I think I have done enough of this in our recitations.

8. Problem 19-22, p. 366. We are concern with equations involving exponential functions of different bases. There are typically two methods. One is to stay with exponential functions, and the other is to apply logarithm to both sides. The former is what I did in our classes, the latter, to what I knew, is how Prof. Klein did in some of his lectures. The two methods are equivalent; the difference is that some of you might like the first one while others like the second.

   For example, let's solve
$$3^{2x-3} = 2^{3x-2}.$$
   Method 1: Divide both sides by 2^{3x-2}, and then proceed using properties of exponential functions.
$$\begin{aligned}\frac{3^{2x-3}}{2^{3x-2}} &= 1\\ \frac{3^{2x}\,3^{-3}}{2^{3x}\,2^{-2}} &= 1\\ \frac{3^{2x}}{2^{3x}}\,\frac{4}{27} &= 1\end{aligned}$$

$$
\begin{aligned}
\frac{(3^2)^x}{(2^3)^x}\,\frac{4}{27} &= 1 \\
\frac{9^x}{8^x}\,\frac{4}{27} &= 1 \\
\left(\frac{9}{8}\right)^x \frac{4}{27} &= 1 \\
\left(\frac{9}{8}\right)^x &= \frac{27}{4} \\
x &= \log_{\left(\frac{9}{8}\right)}\left(\frac{27}{4}\right).
\end{aligned}
$$

Method 2: Apply ln (or other logarithm of your favorite base) to both sides.

$$
\begin{aligned}
(2x-3)\ln 3 &= (3x-2)\ln 2 \\
(2\,\ln 3)x - 3\,\ln 3 &= (3\,\ln 2)x - 2\,\ln 2 \\
(2\,\ln 3 - 3\,\ln 2)x &= 3\,\ln 3 - 2\,\ln 2 \\
x &= \frac{3\,\ln 3 - 2\,\ln 2}{2\,\ln 3 - 3\,\ln 2}.
\end{aligned}
$$

Some of you might have observed that the answers according to the two methods don't seem to match up. A good practice is to show that the two are actually the same, using what we have gained from B.5 and B.6.

9. Problems 43-50, p. 367. While B.8 is the second simplest type of equations involving exponential functions, this is the second simplest type of equations involving logarithmic functions. The only difference is that this time we have the same base. (Well, if you have different bases, you can convert them to some common base using the Change of Base formula.) In contrast to B.8, this time we don't have two different methods. Living with logarithm seems to be effective enough that one doesn't need to think of another method.

   For instance, let's consider Problem 44. We need to solve

$$2\,\log x = \log 2 + \log(3x-4).$$

   What we need to do is to switch all of them to one side, then combine them into a single logarithm. Here we go:

$$
\begin{aligned}
2\log x - \log 2 - \log(3x-4) &= 0 \\
\log x^2 - \log 2 - \log(3x-4) &= 0 \\
\log \frac{x^2}{2(3x-4)} &= 0.
\end{aligned}
$$

Applying in $10^{\square}$ to both side, we get

$$\begin{aligned} \frac{x^2}{2(3x-4)} &= 1 \\ x^2 &= 2(3x-4) \\ x^2 &= 6x-8 \\ x^2-6x+8 &= 0. \end{aligned}$$

Solving this quadratic equation, we get $x = 2$ and $x = 4$.

The final step is to check that these figures make sense in the original equation. This is because we have ignored the meaningfulness of the expressions in the original equation. In this case, both of the above values are valid.

10. Problems 73-80, p. 340; Problems 67-74 p. 367. We now enter the realm of modeling using exponential and logarithmic functions. The first model is that for compound interest. When analyzing the problem, try to pick out as many pieces of information from the following list as you can:

    - What is the method of compounding interest? Is is compounded continuously or not?
    - If interest is not compounded continuously, how often is it compounded? Weekly? Monthly? Quaterly? Annually? Or else?
    - What is the principal (the amount of money originally invested)?
    - What is the interest rate?
    - How long has the principal been invested?
    - What is the final sum?

    Typically, all but one of the above information are given and you need to find the remain. The central formulas for this type of problem are as follows.

    Continuous compounding method:

    $$\text{Final sum} = \text{Principal} \times e^{\text{Interest rate} \times \text{Time}}.$$

    Discrete compounding method:

    $$\text{Final sum} = \text{Principal} \times \left(1 + \frac{\text{Interest rate}}{n}\right)^{n \times \text{Time}},$$

    where n is the number of times interest is compounding in a year. In any case, time is measure in years.

    Let's look at Problem 71, p. 367. First we study what is given and what is not.

- Compounding method: continuously
- Principal: \$1000
- Interest rate: 8.5% = 0.085
- How long has the principal been invested: not known, need to find
- What is the final sum: double the principle, so is $2 \times 1000 =$ \$2000.

The equation is then

$$2000 = 1000 \times e^{0.085 \times t},$$

and we need to solve for t. It should be clear to you how to proceed now.

$$\begin{aligned} 2 &= e^{0.085t} \\ 0.085t &= \ln 2 \\ t &= \frac{\ln 2}{0.085} \\ t &\approx 8.15(\text{yrs}) \end{aligned}$$

Hence the time needed is 8.15 years.

Let's next try something with discrete compounding, say Problem 69, p. 367. Here is a summary of the problem:

- Compounding method: discretely, quarterly. So the number of times interest is compounded in a year is 4.
- Principal: \$5000
- Interest rate: 7.5% = 0.075
- Time of investment: not known, need to find
- Final sum: \$8000.

The equation is

$$8000 = 5000 \left(1 + \frac{0.075}{4}\right)^{4t}.$$

I believe that you know how to proceed from here.

11. Problems 1-13, pp. 379-380. These are problems about population growth. These are in spirit the same as those we consider in B.10 with continuous compounding method. I suppose that you can recognize the similarity.
12. Problems 14-22, pp. 380-381. In contrast to those problems we studied in B.10 and B.11, we now deal with some quantity that decays exponentially. Even so, the key factors you want to keep track of is similar:
    - What is the initial mass?

- What is the half-life/rate of decay?
- How long has it been decaying?
- What is the final mass?

This time, we have two equation. One relates the half-life and the rate of decay, the other relates initial/final mass, rate of decay, and time.

$$\text{Rate of decay} = \frac{\ln 2}{\text{Half-life}},$$
$$\text{Final mass} = \text{Initial mass} \times e^{-\text{Rate of decay} \times \text{Time}}.$$

As always, it's good to see an example. Consider Problem 17, p. 380, for example. We summarize the problem as follows:

- Initial mass: 50 mg
- Half-life: 28 years
- Time: unknown, need to find
- Final mass: 32 mg

Let me write down the equations for the problem.

$$\text{Rate of decay} = \frac{\ln 2}{\text{Half-life}} = \frac{\ln 2}{28} = 0.025,$$
$$32 = \text{Final mass} = \text{Initial mass} \times e^{-\text{Rate of decay} \times \text{Time}} = 50 \times e^{-0.025t}.$$

Great, we have a single equation for time:

$$32 = 50 \times e^{-0.025t}.$$

Solving this equation we get $t = 17.85$ years.

13. Problems 23-26, p. 381. These are about Newton's law of cooling. Important information consists of

- Initial temperature
- Surrounding temperature
- Initial temperature difference
- Cooling constant
- Time
- Final temperature

The equations are

$$\text{Initial temp. diff.} = \text{Initial temp.} - \text{Sur. temp.},$$
$$\text{Final temp.} = \text{Sur. temp.} + \text{Initial temp. diff.} \times e^{-\text{Cooling constant} \times \text{Time}}.$$

14. Problems 27-41. These are modeling problems involving logarithmic functions. If you followed B.10, B.11, B.12, and B.13 well, I think you can start analyzing these problems on your own.

15. Section 5.1 and 6.2: These are fairly simple, I guess you can work them out on your own.

16. Problems 15-40, p. 429. We now switch to sine and cosine function. In there problems, you are asked to find amplitude, period and phase shift of a 'shifted' **sine/cosine** function. No matter what function is given to you, try to turn it into

$$a\ \sin k(x-b)$$

or

$$a\ \cos k(x-b)$$

where k is a positive number. The absolute value of a is the amplitude, p is the phase shift, and the period is related to k by

$$\text{Period} = \frac{2\pi}{k}.$$

If you happen to know some physics, k is 2π times the frequency. Enough of abstract words, let's get our hand dirty. Consider Problem 33, p. 429. You are asked to find the amplitude, period, phase shift and then sketch the graph of

$$y = 5\ \cos\left(3x - \frac{\pi}{4}\right).$$

This is not of the form we want, so we change it into

$$y = 5\ \cos\left(3x - \frac{\pi}{4}\right) = 5\ \cos 3\left(x - \frac{\pi}{12}\right).$$

Okay, we're ready to go. The amplitude is 5, the period is $\frac{2\pi}{3}$, the phase shift is $\frac{\pi}{12}$. Since I'm too lazy to draw a graph in latex, you'll need to do it on your own. Keep in mind that in your graph, you need to show clearly the amplitude, period, phase shift, x-intercepts, and where the function reaches its maximum/minimum values.

See? Math is really this simple. Are you ready yet? Good luck!