Math 115: Precalculus
Special review for the not-so-scary final exam: Trigonometry

(1) Problems 3-22, p. 416: Drill but important exercises.

(2) Problems 15-40, p. 429, 25-28, pp. 451-452: These were covered in the review for the second exam. Again, if you graph the function, make sure to label all x-intercepts, maximum, minimum points, amplitude, period and phase shift. Without labelling, the graph is almost meaningless.

(3) Problems 41-48, p. 429, 29,30, p. 452: These are the converse to the problems in (2). What you have to do is to regconizing the equation of a sine/cosine function from its graph. The main point is to find the amplitude, period and phase shift.

The amplitude should be easy to find out. The period is exactly the distance between two consecutive maximum points. (Minimum points are okay as well.) For phase shift, there's a difference if you want to use sine or cosine. For sine, take any number on the center axis where the graph is going up. For cosine, take any number at which the graph is at its highest level.

Consider for example problem 46, p. 429. Obviously, the amplitude is $\frac{1}{10}$. The period is the distance between two consecutive peaks, which is π. If we use sine, a phase shift is $-\frac{\pi}{2}$. If we use cosine, a phase shift is $-\frac{\pi}{4}$. The answer is

$$\frac{1}{10}\sin(2x+\pi) = \frac{1}{10}\cos(2x+\frac{\pi}{2}).$$

(Try to see this using addition formulas!)

(4) Problems 31-41, pp. 452-453. You can try to work on these to gain some more experience. I strongly recommend them.

(5) Problems 1-25, p. 474. Another sets of drill exercises. What you need to do is to convert between radians and degrees.

(6) Problems 49-58, p. 475. These are problems concerning the length of a circular arc. The formula is

$$s = r\,\theta.$$

Here, s is the length of the given arc, r is the radius, and θ is the central angle measured in radians. If θ is given in degrees, make sure that you convert it into radians first.

Let's work out problem 58. In this problem, s was given to be 4 ft, the central angle θ is 135^o, which is $\frac{3\pi}{4}$ radians. To find the radius, we write

$$4 = r\,\frac{3\pi}{4}.$$

So $r = \frac{16}{3\pi}$ ft.

(7) Problems 59-65, p. 475. Area of circular sector is given by the formula

$$A = \frac{1}{2}\,r^2\,\theta.$$

Again, the central angle is measured in radians.

(8) Problems 1-14, 29-36, pp. 484-485. These are about trigonometry of right triangles. You should be able to work them out easily.

(9) Problems 39-42, 45-60, pp. 485-487. These are more complicated versions of (8). They are quite straightforward, more or less comprehensive reading. The key factor to solve these problem is to bypass unimportant information, and pick the right thing, funny enough.

Let me do, for example, problem 55. The horizontal line splits the water tower into two parts, the top and the bottom. Focus on the lower right triangle, we have

$$\tan 25^o = \frac{\text{height of the bottom part}}{\text{distance from the building to the tower}},$$

so the height of the bottom part is $325 \times \tan 25^o$ feets. Similarly, the height of the top part is $325 \times \tan 39^o$ feets. Adding up these figures, we get the height of the tower.

(10) Problems 43-50, p. 496. Again, drill exercises.

(11) Problems 1-18, p. 513. These are just direct applications of the law of cosines.

For example, let's do problem 16. a, c and $\angle C$ are given. We first find b. By the law of cosines

$$2500 = c^2 = a^2 + b^2 - 2ab\,\cos\angle C = 4225 + b^2 - 130b\,\cos 52^o,$$

so

$$b^2 - 130b\ \cos 52^o + 1725 = 0.$$

This equation has no solution as its discriminant is negative. So there are no triangles having $a = 65$, $c = 50$ and $\angle C = 52^o$.

(12) Problems 37-49, pp. 514-515. These are more delicate applications of the Law of Cosines, with more words to distract our attention.

Consider problem 44. Let C denote the place where the pilot realized that he was on a wrong direction. What we need to find is the distance BC and some angle related to $\angle C$. Let's see what were given to us: the distance AB, an angle related to $\angle A$ and some information about AC. The angle $\angle A$ is seen to be $90^o - 50^o = 40^o$. AC is the distance travelled by the airplane in 30 minutes at speed 200 mi/h, so $AC = 200 \times \frac{1}{2} = 100$ miles.

We are in good shape to provoke the law of cosines:

$$\begin{aligned} BC &= \sqrt{AB^2 + AC^2 - 2AB.AC\ \cos\angle A} \\ &= \sqrt{90000 + 10000 - 60000 \cos 40^o} \\ &= 232.46 \text{ miles.} \end{aligned}$$

This gives (a).

For (b), we use the law of cosines again:

$$\cos\angle C = \frac{BC^2 + AC^2 - AB^2}{2BC.AC} = \frac{54037.33 + 10000 - 90000}{2 \times 232.46 \times 100} = -.56$$

Since $\angle C$ is obstute, we must have $\angle C = \cos^{-1}(-.56) = 123.95^o$. The answer to (b) is $123.95^o - 90^o = 33.95^o$.

(13) Problems 1-88, pp. 523-524. Doing these is the most effective way to learn the trigonometric identities. Try to do as many as you can. Of course, you have to save time for other issues as well.

For me, the easiest way to tackle these problems is to rewrite the expressions in terms of sines and cosines, and then manipulate. If you think you like this, the magical artifacts you would need are:

$$\sin^2 x + \cos^2 x = 1$$

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x},$$

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}.$$

Other formulas are more or less consequences of them.

For example, let me work out problem 50 for you. I need to show that $(\tan y + \cot y) \sin y \cos y = 1$. As said above, I first turn $(\tan y + \cot y) \sin y \cos y$ into something that involves only sines and cosines:

$$(\tan y + \cot y) \sin y \cos y = \left(\frac{\sin y}{\cos y} + \frac{\cos y}{\sin y} \right) \sin y \cos y.$$

Now, it's just manipulating rational expression:

$$\begin{aligned} \ldots &= \left(\frac{\sin^2 y}{\sin y \cos y} + \frac{\cos^2 y}{\sin y \cos y} \right) \sin y \cos y \\ &= \sin^2 y + \cos^2 y = 1. \end{aligned}$$

(14) Problems 13-18, p. 539, 23-40, p. 540, 15-40, p. 548. They are exercises for all the YOU-NAME-IT formulas.

(15) Problems 1-48, pp. 568-569. Equations of trigonometric functions. They are among the simplest. Basicly, what you need to do is to factor the expression, then set each factor to be zero. When you factor, it'd be helpful if you see some sort of polynomials of trigonometric functions occuring in your equation.

For example, let's look at problem 18. You need to solve $\tan x \sin x + \sin x = 0$. There is no polynomial here, but a common factor $\sin x$. Factor it out, we get

$$\sin x(\tan x + 1) = 0.$$

Now we solve two sub-equations: $\sin x = 0$ and $\tan x + 1 = 0$.

$\sin x = 0$ is easy. In $[0, 2\pi)$ (one period of sine), $\sin x$ vanishes at 0 and at π. Since sine has period 2π, it vanishes at any point of the form 2π or $\pi + 2\pi$. The solutions are $k\pi$ where k is some integer.

The second equation is equivalent to $\tan x = -1$. We do the same thing as in the previous case. Inside $[0, \pi)$, $\tan x = -1$ only if $x = \frac{3\pi}{4}$. So all solutions to $\tan x = -1$ is $\frac{3\pi}{4} + k\pi$. Here we used the fact that period of tan is π.

In conclusion, all solutions of the original equation are $k\pi$ or $\frac{3\pi}{4} + k\pi$, where k is some integer.

Let's do some other example, say problem 20. This time, I will only solve the equation in $[0, 2\pi)$, you will need to find all the remained solutions. The equation we are considering is $2\sin^2 x - \sin x - 1 = 0$. The left hand side is a polynomial of $\sin x$. If we put $\Box = \sin x$, we have $\Box^2 - \Box - 1 = 0$. This polynomial factors into $(2\Box + 1)(\Box - 1)$. So $\Box$ has to be 1 or $-1/2$.

Going back to $\sin x$, we now solve two sub-equations: $\sin x = 1$ and $\sin x = -1/2$. For the first, a solution in $[0, 2\pi)$ is $\pi/2$. For the second, it's $\frac{11\pi}{6}$. You can now continue to find all other solutions.