(1) $g(x) = \sqrt[5]{x-1}$

$g'(x) = \frac{1}{5}(x-1)^{-\frac{4}{5}}$

$L(x) = g'(a)(x-a) + g(a)$

$a = 33$

$g(a) = 2$

$g'(a) = \frac{1}{80}$

$$\boxed{L(x) = \frac{1}{80}(x-33) + 2}$$

$\sqrt[5]{33} = g(34) \approx L(34) = \frac{1}{80}(34-33) + 2$

$$\boxed{\sqrt[5]{33} \approx 2\frac{1}{80}}$$

$g''(x) = -\frac{4}{25}(x-1)^{-\frac{9}{5}}$

$g''(33) = - \cdots < 0$

$\rightarrow$ near 33, g is concave down

$\rightarrow$ $\boxed{\text{overestimate}}$

② $x^3 - 2xy + 2y^2 = 6,\ y \geq 1$

(a) $3x^2 - 2xy' - 2y + 4yy' = 0$

$3x^2 - 2y + (-2x+4y)y' = 0$

$(-2x+4y)y' = -3x^2 + 2y$

$\boxed{y' = \dfrac{-3x^2+2y}{-2x+4y}}$ or $\boxed{f'(x) = \dfrac{-3x^2+2f(x)}{-2x+4f(x)}}$

(b) $x = -1 \rightarrow$ plug in $x^3 - 2xy + 2y^2 = 6,\ y \geq 1$

$-1 + 2y + 2y^2 = 6,\ y \geq 1$

$\rightarrow$ $\boxed{\text{quadratic formula}}$ $y = 1.43$

$\rightarrow y' = -.018$

$\rightarrow \boxed{y - 1.43 = -.018(x+1)}$

(c) Type 1 $\quad y' = 0 \rightarrow \dfrac{-3x^2+2y}{-2x+4y} = 0 \rightarrow -3x^2 + 2y = 0$

$\rightarrow \boxed{\begin{cases} x^3 - 2x + 2y^2 = 6,\ y \geq 1 \\ -3x + 2y = 0 \end{cases}}$

Type 2 $\quad y' = \infty \rightarrow \cdots \rightarrow -2x + 4y = 0$

$\boxed{\begin{cases} x^3 - 2x + 2y^2 = 6,\ y \geq 1 \\ -2x + 4y = 0 \end{cases}}$

(d) Plug in $\rightarrow y' = \infty \rightarrow$ the tangent line is vertical

$\rightarrow \boxed{x = 2}$

③ $P(x) = x^4 - 3x^2 + 6x - 5$

(a) $$x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}$$

$$P'(x) = 4x^3 - 6x + 6$$

$$\boxed{x_{n+1} = x_n - \frac{x_n^4 - 3x_n^2 + 6x_n - 5}{4x_n^3 - 6x_n + 6}}$$

(b) $x_0 = -2$

$$x_1 = (-2) - \frac{(-2)^4 - 3(-2)^2 + 6(-2) - 5}{4(-2)^3 - 6(-2) + 6}$$

$$= -2.9$$

$$x_2 = \cdots = -2.6$$

(4)

(a) $$\lim_{x \to 0} \frac{\sin^2(x)}{x - x\cos(x)} = \lim_{x \to 0} \frac{\sin^2(x)}{x(1-\cos(x))}$$

$$= \lim_{x \to 0} \underbrace{\frac{\sin^2(x)}{x^2}}_{\downarrow \; 1^2} \cdot \underbrace{\frac{x}{1-\cos(x)}}_{\downarrow \; \frac{1}{0} = \infty} \quad \boxed{= \infty.}$$

(b) $$\lim_{x \to 0} \underbrace{(\csc(x) - \cot(x))}_{\downarrow \; 0} \underbrace{(\sec(x) - \tan(x))}_{\downarrow \text{ plug in } \; 1} \quad \boxed{= 0}$$

→ see the next page

c) $$\lim_{x \to \infty} (x^2 - 3x + 1)^{\frac{1}{x}} \qquad (\infty^0)$$

$$* \lim_{x \to \infty} \ln (x^2 - 3x + 1)^{\frac{1}{x}} = \lim_{x \to \infty} \frac{1}{x} \ln (x^2 - 3x + 1)$$

$$= \lim_{x \to \infty} \frac{\ln(x^2 - 3x + 1)}{x} \qquad \left(\frac{\infty}{\infty}\right)$$

L'Hopital.

$$= \cdots = 0$$

$$\to \lim_{x \to \infty} (x^2 - 3x + 1)^{\frac{1}{x}} = e^0 \boxed{= 1}$$

d) $$\lim_{x \to \infty} \underbrace{(x e^{\frac{1}{x}} - x)}_{\infty - \infty} \underbrace{(e^x + x)^{\frac{1}{x}}}_{\infty^0}$$

$$* \lim_{x \to \infty} (x e^{\frac{1}{x}} - x) = \lim_{y = \frac{1}{x} \to 0} \left(\frac{1}{y} e^y - \frac{1}{y}\right) = \lim_{y \to 0} \frac{e^y - 1}{y} = 1$$

$$* \lim_{x \to \infty} \ln (e^x + x)^{\frac{1}{x}} = \lim_{x \to \infty} \frac{1}{x} \ln(e^x + x) = \lim_{x \to \infty} \frac{\ln(e^x + x)}{x}$$

L'Hopital

$$= \cdots = 1$$

$$\to \cdots \; 1 \cdot e^1 \boxed{= e}$$

(4) (continued)

4 (continued)

(e) $\lim_{x\to\infty} \underbrace{x^3 e^{-x}}_{\infty\cdot 0} = \lim_{x\to\infty} \frac{x^3}{e^x} \overset{\text{L'Hopital}}{=} \cdots = \boxed{0.}$

(f) $\lim_{x\to\infty} \left(\underbrace{\arctan(x^2)}_{\frac{\pi}{2}} - \underbrace{\arctan(x)}_{\frac{\pi}{2}} \right) = 0.$

(b) justification of the first limit: $\lim_{x\to 0} \csc(x) - \tan(x) = 0$

$$\lim_{x\to 0} (\csc(x) - \tan(x)) = \lim_{x\to 0} \frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)}$$

$$= \lim_{x\to 0} \frac{1-\cos(x)}{\sin(x)}$$

$$\overset{\text{L'Hopital}}{=} \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = 0.$$

5

$f(x) = \sqrt{17-x^3}$

$f(1.97) = ?$

We'll use linearization.

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

"a" should be close to 1.97 & $f(a)$ & $f'(a)$ should be relatively easy to compute.

Pick $\underline{a=2}$

$$f(2) = \sqrt{17-2^3} = 3$$

$$f'(x) = \frac{-3x^2}{2\sqrt{17-x^3}}$$

$$f'(2) = \frac{-3.2^2}{2\sqrt{17-2^3}} = -2$$

$$L(x) = 3 + (-2)(x-2) = 3 - 2(x-2)$$

$$\boxed{f(1.97) \approx 3 - 2(1.97-2) = 3.015}$$

⑥ $2x^2+y^2=1$

$4x+2yy'=0$

$y'=-\frac{2x}{y}.$

At point where slope of the tangent line is 1, $y'=1$

so

$y'=-\frac{2x}{y}=1 \longrightarrow -2x=y.$

Plug this back to the equation of the ellipse,

$2x^2+(-2x)^2=1$

$x^2=\frac{1}{6}$, $x=\pm\sqrt{\frac{1}{6}}$

$y=-2x=\mp 2\sqrt{\frac{1}{6}}.$

Answer $\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$ and $\left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right).$

(7)

(a) $g(x) = \sin(2x^3)$

Known: $\dfrac{|g(a)-g(b)|}{|a-b|} = |g'(c)| = |6c^2\cos(2c^3)|$

for some c in $[0,1]$.

$|c|$ is at most 1, so $6c^2$ is atmost 6

$|\cos(2c^3)|$ is at most 1

So $|6c^2\cos(2c^3)|$ is at most 6.

Hence

$$\frac{|g(a)-g(b)|}{|a-b|} \le 6 \quad \text{for all } a,b \text{ in } [0,1].$$

Answer : $\boxed{M=6}$

(b) Mean value theorem <u>doesn't</u> apply since f is not differentiable in $(0,5)$.

$$f'(x) = \frac{(2x+3)'(x-2) - (2x+3)(x-2)'}{(x-2)^2}$$

$$= \frac{-7}{(x-2)^2}.$$

$$f(5)-f(0) = \frac{13}{3} - \frac{3}{-2} = \frac{35}{6}$$

positive negative

$f(5)-f(0) = 5f'(c)$ if and only if $\frac{35}{6} = 5\cdot\frac{-7}{(c-2)^2}$.

This equation has no solution, so there is <u>no such c</u>!

(8) $h(x) = x^3 - 3x^2 - 6x + 2$

(a) $h'(x) = 3x^2 - 6x - 6 = 3(x^2 - 2x - 2)$

$h'(x) = 0$,

$x^2 - 2x - 2 = 0$

$$x = \frac{2 \pm \sqrt{2^2 + 8}}{2} = \boxed{1 \pm \sqrt{3} \quad : \text{critical point}}$$

(b)

$$\left.\begin{aligned} f(1+\sqrt{3}) &= \\ f(1-\sqrt{3}) &= \\ f(-3) &= \\ f(4) &= \end{aligned}\right\} \text{compare to get max/min}$$

c) $h''(x) = 6x - 6$

$h''(x) = 0$

$6x - 6 = 0$

$\boxed{x = 1 \quad : \text{inflection point}}$

(9)

(a) rel max: $f(0)=3$ @ $x=0$

rel min: $f(-1)=2$ @ $x=-1$

$f(2)=-4$ @ $x=2$

(b) at end points $(\pm\infty)$, $f(x)$ approaches 4. biggest

critical values are: 3, 2, -4 smallest

absolute min: -4 @ $x=2$

absolute max: not attained (need to be at $\pm\infty$).

(c) y-int: 3

x-int: one in $(0,2)$ and another one in $(2,\infty)$.

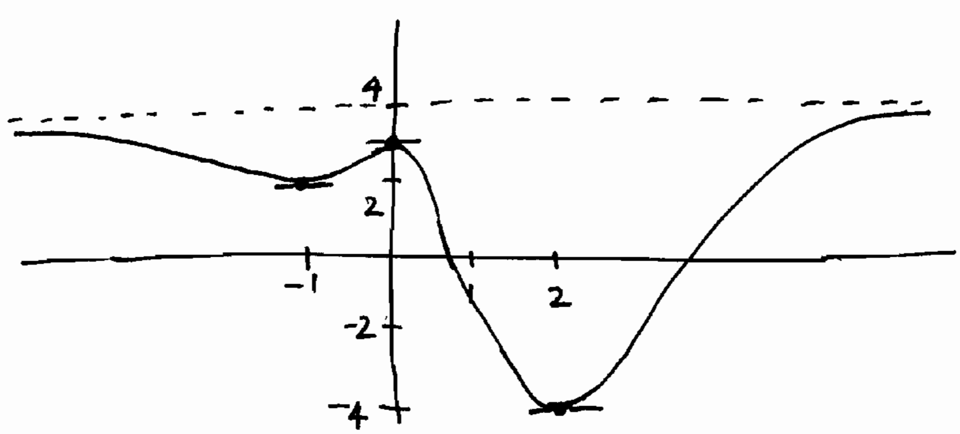
(d) No VA

One HA: $y=4$.

(e) $f'(-1)=f'(0)=0$. By MVT, there is one zero of f'' in $(-1,0)$. Similarly, there is one in each of $(-\infty,-1)$, $(0,2)$ & $(2,\infty)$.

(f) Check IVT

(g)



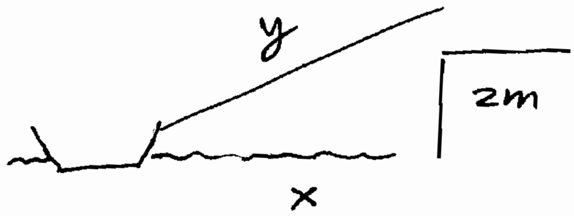
(10) $r'(t) = \sin t + t^3 - 2t + 1$

$r(t) = -\cos t + \frac{t^4}{4} - t^2 + t + C$

$r(0) = 3 \rightarrow C = 3$

$$\boxed{r(t) = -\cos t + \frac{t^4}{4} - t^2 + t + 3}$$

(11)



$$\frac{dy}{dt} = .5\ m/s$$

$$x = 5\ m$$

$$\frac{dx}{dt} = ?$$

$$x^2 + 4 = y^2$$

$$2xx' = 2yy'$$

$$x' = \frac{yy'}{x}$$

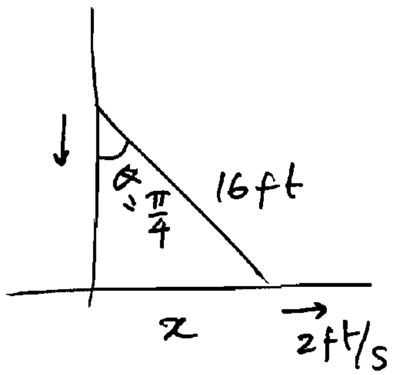
$x = 5 \longrightarrow 5^2 + 4^2 = y^2$

$$y^2 = 25 + 4 = 29$$

$$y = \sqrt{29}\ m$$

$$x' = \boxed{\frac{\sqrt{29} * (.5)}{5}\ m/s.}$$

12



$\frac{dx}{dt} = 2$ ft/s

$\theta = \frac{\pi}{4}$

$\frac{d\theta}{dt} = ?$

$x = 16\sin(\theta)$

$x' = 16\cos(\theta) \cdot \theta'$

$\theta' = \frac{x'}{16\cos(\theta)} = \frac{2}{16\cos(\frac{\pi}{4})} = \frac{2}{16 \cdot \frac{\sqrt{2}}{2}} = \boxed{\frac{1}{4\sqrt{2}} \text{ rad/s}}$

(13)

Volume $= 10 \text{ cm}^3$

dimensions? which minimize the surface area.

* $V = \pi r^2 \cdot h = 10 \text{ cm}^3$
* $S =$ shear + bottom → minimize

$= 2\pi r h + \pi r^2$

* h in terms of r:

$$h = \frac{V}{\pi r^2} = \frac{10}{\pi r^2}.$$

* S in terms of r

$$S = 2\pi r h + \pi r^2 = 2\pi r \cdot \frac{10}{\pi r^2} + \pi r^2$$

$$S = \frac{20}{r} + \pi r^2$$

* critical points

$$S' = -\frac{20}{r^2} + 2\pi r = 0 \Leftrightarrow \frac{20}{r^2} = 2\pi r$$

$$\Leftrightarrow \frac{10}{\pi} = r^3 \Leftrightarrow r = \sqrt[3]{\frac{10}{\pi}} \text{ cm}$$

* end points

$0 \leq r \leq +\infty$

$$S(r) = \frac{20}{\sqrt[3]{\frac{10}{\pi}}} + \pi \left(\sqrt[3]{\frac{10}{\pi}}\right)^2 \leftarrow \text{mini}$$

"$S(0)$" = "$S(+\infty)$" $= \infty$ (in the sense of limit)

* answer: $\boxed{r = \sqrt[3]{\frac{10}{\pi}} \text{ cm}, \quad h = \frac{10}{\pi r^2} = \frac{10}{\pi\left(\sqrt[3]{\frac{10}{\pi}}\right)^2} \text{ cm.}}$

(14) $f'(x) = x^2(x-3)$

(a) We need to check the sign of $f'(x)$

critical points : $x=0$, $x=3$

x	$-\infty$		0		3		$+\infty$
$f'(x)$		$-$	0	$-$	0	$+$	
$f(x)$		$\searrow$		$\searrow$		$\nearrow$	

$f(x)$ is increasing in $(3,+\infty)$
decreasing in $(-\infty, 3)$

(b) From (a), $f(x)$ has a local minimum @ $x=3$, but no local maximum.

(c) Also at $x=3$, $f(x)$ attains its minimum for all time for $f(x)$ decreases all the way from $-\infty$ to 3 and increases all the way from 3 to $+\infty$.

No absolute max exists

(d) $f''(x) = 2x(x-3) + x^2 = 3x^2 - 6x$

$f''(x) = 0 \Leftrightarrow 3x^2 - 6x = 0 \Leftrightarrow x = 0$ or $x = 2$.

x	$-\infty$		0		2		$+\infty$
$f''(x)$		$+$	0	$-$	0	$+$	
$f(x)$		$\cup$		$\cap$		$\cup$	

$f(x)$ is concave upward in $(-\infty, 0)$ and $(2,\infty)$,
downward in $(0,2)$.

(15)

Let $(a, \frac{1}{a-1})$ be the tangent point.

Then we can determine the slope in two ways:

* slope = derivative @ a:

$$m = -\frac{1}{(a-1)^2}.$$

* ~~slope~~ slope $= \frac{\text{rise}}{\text{run}}$

$$m = \frac{\frac{1}{a-1} - 2}{a-(-5)} = \frac{\frac{1-2(a-1)}{a-1}}{a+5}$$

$$m = \frac{-2a+3}{(a-1)(a+5)}$$

So

$$\frac{-2a+3}{\cancel{(a-1)}(a+5)} = -\frac{1}{(a-1)^{\cancel{2}}}$$

$$\frac{-2a+3}{a+5} = -\frac{1}{a-1}$$

$$(-2a+3)(a-1) = -(a+5)$$

$$-2a^2+3a+2a-3 = -a-5$$

$$2a^2-6a-2 = 0$$

$$a^2-3a-1 = 0 \rightarrow a = \frac{3 \pm \sqrt{3^2+4}}{2}$$

$$a = \frac{3 \pm \sqrt{13}}{2}.$$

Equations of the tangent lines:

$$\boxed{y-2 = -\frac{1}{\left(\frac{3\pm\sqrt{13}}{2}-1\right)^2}(x+5)}$$