Problem 3

(a) What are the domain and range of

$$f(x) = \left(\arctan(\ln(\sqrt{x}-1))\right)^3$$

(i) Domain

Observe that $f(x)$ is well defined ~~iff~~ if and only if $\sqrt{x}$ and $\ln(\sqrt{x}-1)$ are well defined. Therefore the domain of f is given by the inequalities:

$$\begin{cases} x \geq 0 \\ \sqrt{x}-1 > 0 \end{cases}$$

Solving this system yields $x > 1$

Answer: $\boxed{D = \{x > 1\} = (1, \infty)}$

(ii) Range

Given $x \in (1,\infty)$, $\sqrt{x}-1$ is an arbitrary number in $(0, \infty)$, which is the domain of ln. Thus, $\ln(\sqrt{x}-1)$ is an arbitrary number in $\mathbb{R}$.

It follows that $\arctan(\ln(\sqrt{x}-1))$ is an arbitrary number in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the range of arctan.

Finally, the range of $f(x)$ is thus

$$\boxed{R = \left(-\frac{\pi^3}{8}, \frac{\pi^3}{8}\right)}$$

(B) If $y=f(x)$, write x interms of y.

We have

$$y = \left(\arctan\left(\ln\left(\sqrt{x}-1\right)\right)\right)^3$$

$$y^{\frac{1}{3}} = \arctan\left(\ln\left(\sqrt{x}-1\right)\right)$$

$$\tan\left(y^{\frac{1}{3}}\right) = \ln\left(\sqrt{x}-1\right)$$

$$\exp\left(\tan\left(y^{\frac{1}{3}}\right)\right) = \sqrt{x}-1$$

$$\exp\left(\tan\left(y^{\frac{1}{3}}\right)\right)+1 = \sqrt{x}$$

$$\left(\exp\left(\tan\left(y^{\frac{1}{3}}\right)\right)+1\right)^2 = x$$

Answer:

$$\boxed{x = \left(\exp\left(\tan\left(y^{\frac{1}{3}}\right)\right)+1\right)^2}$$