Math 151: Calculus
From the second quiz ...

1. Let $g(x) = \sqrt[5]{x-1}$. Write the linear approximate $L(x)$ for $g(x)$ near $x = 33$. Then use the linearization to approximate $\sqrt[5]{33}$.

   Most of you get the linearized operator right. What goes wrong is with the second part, when approximating $\sqrt[5]{33}$; more or less 95% of the class did this part wrong. Recall the formular for the linearization:

   $$L(x) = g(a) + g'(a)(x-a).$$

   In this problem $a = 33$, so

   $$L(x) = g(33) + g'(33)(x-33).$$

   We have

   $$\begin{aligned} g(33) &= \sqrt[5]{33-1} = \sqrt[5]{32} = 2, \\ g'(x) &= \frac{1}{5}(x-1)^{-4/5} = \frac{1}{5\sqrt[5]{x-1}^4}, \\ g'(33) &= \frac{1}{5\sqrt[5]{33-1}^4} = \frac{1}{80}. \end{aligned}$$

   Hence the linearized operator is

   $$L(x) = 2 + \frac{1}{80}(x-33).$$

   To approximate $\sqrt[5]{33}$, we need to recognize what should we put to x so that $g(x)$ $= \sqrt[5]{33}$. Based on the results I saw, here are a few options:

   - $x = 33$ (most of you pick this choice),
   - $x = \sqrt[5]{33}$ (the next popular choice),
   - and $x = 34$ (very few people pick this).

   Let's see which one is right:

   - $g(33) = \sqrt[5]{33-1} = \sqrt[5]{32}$,

- $g(\sqrt[5]{33}) = \sqrt[5]{\sqrt[5]{33} - 1}$,
- $g(34) = \sqrt[5]{34 - 1} = \sqrt[5]{33}$.

So the right thing to pick is $x = 34$! Head up for the minority who did it right. It follows that

$$\sqrt[5]{33} = g(34) \approx L(34) = 2 + \frac{1}{80}(34 - 33) = 2.0125.$$

You can check with your calculator that $\sqrt[5]{33} \approx 2.0123...$, so the above approximation is quite good.

3. Let $P(x) = x^4 - 3x^2 + 6x - 5$. Assume that we are using Newton's method to find root(s) of $P(x)$.

   (a) Give a recursive formula expressing the $(n+1)$-st approximation x_{n+1} in terms of the n-th approximation x_n.

   You all did this part right.

   (b) Start with $x_0 = -2$, find a root of $P(x)$ accurate to three decimal places.

   Aha, another troublesome question. You all did the first approximation right, but that is not where you stop. If I had asked you to find only the first approximation, that would have been perfect. For the question I asked, you need to keep going until the first three digits after the period stop changing. The correct answer should be -2.492.

   Again, please read the problem carefully.

4. Calculate the following limits:

   (a) $\lim_{x\to 0} \frac{\sin^2 x}{x - x\cos x}$, (b) $\lim_{x\to 0} (\csc x - \cot x)(\sec x - \tan x)$,

   (c) $\lim_{x\to\infty} (x^2 - 3x + 1)^{1/x}$, (d) $\lim_{x\to\infty} (xe^{1/x} - x)(e^x + x)^{1/x}$

   You have lots of trouble with this question. It's completely understandable and don't be ashamed of that. Here are a few comments.

   - If the expression is two complicated, try to break it into parts. I would consider breaking the expression in (b) and (d).
   - Before using L'Hospital rule, remember to check that the limit is in one of the indeterminate form, i.e. $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

- L'Hospital rule is ONLY applicable to QUOTIENT. You can't do it with product, sum nor difference. For those, you need to somehow change the form of the expression to a quotient.
- For limit of the form ∞^0, 0^∞, 1^∞, you might one to use the log trick. This would change the expression form to a product, and then you can switch it to a quotient. Here is an example.

  Find
  $$\lim_{x\to 0}(x+x^2)^{1/x}.$$
  This is of the form 0^∞. Set
  $$y = \ln(x+x^2)^{1/x}.$$
  By logarithm rule,
  $$y = \frac{1}{x}\ln(x+x^2) = \frac{\ln(x+x^2)}{x}.$$
  If we know the limit
  $$\lim_{x\to 0} y = \lim_{x\to 0}\frac{\ln(x+x^2)}{x}$$
  is finite, the original limit will be the exponential of this limit. If this limit is $-\infty$, the original limit will be 0. Otherwise, the original limit doesn't exist. The new limit is of the form $\frac{\infty}{\infty}$ so L'Hospital rule applies.
  $$\lim_{x\to 0} y = \lim_{x\to 0}\frac{\ln(x+x^2)}{x} = \lim_{x\to 0}\frac{\frac{1}{x+x^2}(1+2x)}{1} = \lim_{x\to 0}\frac{1+2x}{x+x^2}.$$
  It turns out that this limit doesn't exist. The left limit gives $-\infty$, while the right limit gives $+\infty$. We conclude that the original one doesn't exist.

6. Let $h(x) = x^3 - 3x^2 - 6x + 2$. Find the exact values of the following quantities:
   (a) critical points of h,
   (b) maximum and/or minimum value(s) of h in the interval $[-3, 4]$,
   (c) and points of inflection of the graph of h.

7. Assume that f is a differentiable function on the whole real line. Suppose for some reason you know that
   - $f(-1) = 2$, $f(0) = 3$, $f(2) = -4$,
   - $f'(-1) = f'(0) = f'(2) = 0$,

- $f' < 0$ in $(-\infty, -1) \cup (0, 2)$ and $f' > 0$ in $(-1, 0) \cup (2, \infty)$.
- $\lim_{x \to \pm\infty} f(x) = 4$.

(a) Find all relative maxima and minima of f.

(b) Decide whether f has an absolute maximum and/or an absolute minimum. If so, what are they?

(c) What are the intercepts of f?

(d) Does the graph of f have any asymptotes? If yes, what are their equations?

(e) Assume that f is twice differentiable, i.e. f'' exists everywhere. Why does the graph of f have at least four inflection points?

(f) Show that there exists some $c \in [-1, 0]$ such that $f'(c) = 1$. Can c be 0 or 1?

(g) Sketch the graph of f. Make sure to show important points and lines.

My appology, I made some typos in these two problems. "Inflection point" should be used instead of "reflection point". However, for cubic polynomial, reflection point is the same as inflection point. The graph of a cubic polynomial is always symmetric about the point of inflection. It always exists for cubic polynomials though.

Here is some observations:

- Some of you didn't distinguished between local extrema, global extrema, and absolute extrema.
  - Critical points only gives you local extrema. If you check the second derivative, it would tell you whether they are (local) maxima or minima.
  - Absolute extrema are the max/min of the function in some interval. Remember the closed interval test that we did in class. That is what you need to locate absolute extrema. (You need to compare the values of the function at the critical numbers and at the endpoints of the interval. The smallest value is the absolute minimum, the largest is the absolute maximum.) If you are asked to find max/min in an interval, you need to find the absolute max/min.
  - Global extrema are extrema in the whole real line. This is somewhat similar to absolute extrema, except that the endpoints are now $\pm\infty$. This time, you cannot calculate the values of the function at those 'points'. You need to take limit instead.
- Some of you don't recognize the difference between a vertical and a horizontal asysmtotes. Please review.

8. Find $r(t)$ given that $r(0) = 3$ and $r'(t) = \sin t + t^3 - 2t + 1$.

   Most of you start correctly, but somehow end up with a wrong answer. What you need to remember for this type of problem is that antiderivatives are determined up to a constant.

   For example, in this problem, I first look for one particular antiderivative. I break the expression into different parts. An antiderivative of $\sin t$ is $-\cos t$. For the rest, I can take $\frac{1}{4}t^4 - t^2 + t$. So an antiderivative of $\sin t + t^3 - 2t + 1$ would be $-\cos t + \frac{1}{4}t^4 - t^2 + t$. Then all antiderivative would be $-\cos t + \frac{1}{4}t^4 - t^2 + t + C$, where C is some constant.

   In particular, since $r(t)$ is an antiderivative of $r'(t)$, $r(t)$ must be of the above form, i.e.

   $$r(t) = -\cos t + \frac{1}{4}t^4 - t^2 + t + C$$

   To find C, we use the condition $r(0) = 3$. This gives

   $$-\cos 0 + 0 - 0 + 0 + C = 3,$$

   so $C = 4$. The final answer is

   $$r(t) = -\cos t + \frac{1}{4}t^4 - t^2 + t + 4.$$

9. A person pulls a boat into a dock by a rope attached to the bow of the boat and passing through a pulley of the dock that is $2m$ higher than the bow of the boat. How fast is the boat moving towards the dock when it is 10 m away from the dock? It's known that the person pulls the rope at a constant rate of 0.5 m/s.

10. A plank 16 ft long rests against a vertical wall. If the bottom of the plank is sliding away from the wall at a speed of 1 ft/s, how fast is the angle between the top of the plank and the wall changing when the angle is $\pi/3$?

    The last two problems are really easy, but amazingly many of you didn't get it completed. Please review related rate.