Math 151: Calculus
Workshop #3 :: Addendum

1. Let

$$f(x) = \frac{60}{x+3}.$$

Then

$$\lim_{x\to 2} f(x) = 12.$$

In this problem you will attempt to prove this limit statement using the ϵ-δ definition of a limit. Remember that the inequality $|f(x)-12| < \epsilon$ is equivalent to the inequality $12-\epsilon < f(x) < 12+\epsilon$.

(e) To conform to the definition of limit, do the following: Let any positive number ϵ be given. In terms of ϵ, find a number $\delta > 0$ such that $|f(x)-12| < \epsilon$ whenever $0 < |x-2| < \delta$.

Prelure: In an ϵ-δ argument, there are a few things you have to keep in mind. 1. Never assign to ϵ a specific value. 2. You can assume any positive upper bound of ϵ and/or δ.

Solution: Fix some $\epsilon > 0$. We need to find $\delta > 0$ such that

$$|f(x)-12| < \epsilon \text{ whenever } 0 < |x-2| < \delta. \tag{1}$$

To make this easier to process, we first attempt to simplify $|f(x)-12| < \epsilon$ as follows:

$$\begin{aligned} |f(x) - 12| &< \epsilon, \\ 12-\epsilon < f(x) &< 12+\epsilon, \\ 12-\epsilon < \frac{60}{x+3} &< 12+\epsilon. \end{aligned}$$

To solve this inequality, it'll be helpful if we know the sign of each of the three sides. Since we are interested in small ϵ and so small δ, we can assume that $\epsilon < 12$ and $\delta < 5$. Under the first assumption, $12-\epsilon$ and $12+\epsilon$ are both positive. Under the second condition, $x+3 > (2-\delta)+3 = 5-\delta > 0$, so $x+3$ is positive. In other words, all three sides in the above inequality are positive. Consequently, that inequality is equivalent to

$$\frac{1}{12-\epsilon} > \frac{x+3}{60} > \frac{1}{12+\epsilon},$$

$$\frac{60}{12-\epsilon} > x+3 > \frac{60}{12+\epsilon},$$
$$\frac{60}{12-\epsilon} - 3 > x > \frac{60}{12+\epsilon} - 3.$$

We come to the conclusion that $|f(x) - 12|$ is the same as

$$\frac{60}{12+\epsilon} - 3 < x < \frac{60}{12-\epsilon} - 3.$$

We next rewrite (1) as

$$\frac{60}{12+\epsilon} - 3 < x < \frac{60}{12-\epsilon} - 3 \text{ whenever } 2-\delta < x < 2+\delta. \tag{2}$$

Recall that we are looking for δ such that this relation holds. We thus require

$$\frac{60}{12+\epsilon} - 3 < 2-\delta < 2+\delta < \frac{60}{12-\epsilon} - 3. \tag{3}$$

Obviously, if this relation holds, (2) and so (1) hold.

We are now in the position to solve (3) for δ. Observe that (3) consists of two inequalities

$$\frac{60}{12+\epsilon} - 3 < 2-\delta$$

and

$$2+\delta < \frac{60}{12-\epsilon} - 3.$$

The former gives

$$\delta < 5 - \frac{60}{12+\epsilon}$$

while the latter gives

$$\delta < \frac{60}{12-\epsilon} - 5.$$

Taking all requirements we have so far for δ into account, we end up with

$$\delta < \min\left(5, 5 - \frac{60}{12+\epsilon}, \frac{60}{12-\epsilon} - 5\right).$$

Note that as $\epsilon > 0$, the right hand side is always positive, so it is always possible to pick a positive δ satisfying the above inequality.

We have shown that for any given ϵ positive, $\epsilon < 12$, there is a $\delta > 0$ such that whenver $|x-2| < \delta$, $|f(x) - 12| < \epsilon$. By definition of limits, we conclude that

$$\lim_{x \to 2} f(x) = 12.$$