Math 151: Calculus
Workshop #3

1. Let
$$f(x) = \frac{60}{x+3}.$$
Then
$$\lim_{x\to 2} f(x) = 12.$$
In this problem you will attempt to prove this limit statement using the ϵ-δ definition of a limit. Remember that the inequality $|f(x) - 12| < \epsilon$ is equivalent to the inequality $12 - \epsilon < f(x) < 12 + \epsilon$.

   (a) Which limit laws that guarantee the above limit statement?

   (b) Use your calculator, sketch the graph of $f(x)$ for $1 \le x \le 3$. Keeping this picture in mind, find numbers $a < 2$ and $b > 2$ such that $|f(x) - 12| < 0.1$ for all x in the interval $a < x < b$.

   (c) Using your answer to (a), find a number $\delta_1 > 0$ such that $|f(x) - 12| < 0.1$ whenever $|x - 2| < \delta_1$.

   (d) In class: Find a number $\delta_2 > 0$ such that $|f(x) - 12| < 0.01$ whenever $|x - 2| < \delta_2$. (At home: Repeat for two more (smaller and smaller) positive numbers ϵ of your own choosing, obtaining the corresponding δ_3 and δ_4.)

   (e) To conform to the definition of limit, do the following: Let any positive number ϵ be given. In terms of ϵ, find a number $\delta > 0$ such that $|f(x) - 12| < \epsilon$ whenever $0 < |x - 2| < \delta$.

2. For $x \neq 0$, let
$$f(x) = \frac{4x^2 + 2}{x},$$
and note that
$$\lim_{x\to 2} f(x) = 9.$$
The purpose of this problem is to prove the above equation using the ε-δ language.

   (a) Which limit laws that allow us to write the above limit statement?

   (b) Use a graphing calculator to find about the longest interval (a, b) containing the point $x = 2$ so that $f(x)$ is between 8 and 10 for all x in (a, b). Draw a graph showing this.

(c) If the interval (a, b) has to be centered at $x = 2$, what is about the longest that it could be? Give some calculator evidence.

(d) In the formal ϵ, δ definition of $\lim_{x\to 2} f(x) = 9$ that is given in the textbook, if we let $\epsilon = 1$, explain how your answer to part (b) helps you decide which values of δ would satisfy the requirement that $|f(x) - 9| < 1$ for every x which satisfies $0 < |x - 2| < \delta$.

(e) Verify the formal ϵ, δ definition of $\lim_{x\to 2} f(x) = 9$, i.e., given a (general) $\epsilon > 0$, find a value of $\delta > 0$ depending on ϵ so that $|f(x) - 9| < \epsilon$ for every x with $0 < |x - 2| < \delta$. Explain your reasoning.

3. Let $f(x) = x^3 + 5x - 1$ and consider the equation $f(x) = 10$. The purpose of this problem is to find an approximate solution to this equation.

(a) Prove *without the help of a graph* that the equation $f(x) = 10$ has a solution x in the interval $(0, 2)$. *No one can prevent you from looking at a graph, but your reasoning should use the Intermediate Value Theorem and not draw conclusions from a calculator-generated graph.*

(b) Use the Intermediate Value Theorem again to decide whether the equation $f(x) = 10$ has a solution in $[0, 1]$. What does the Intermediate Value Theorem say about solution of the aforementioned equation in $[1, 2]$?

(c) Continue "zooming in on" the solution, deciding at each step whether the solution lies in the left or right half of the latest interval. Stop when you are able to to state the solution with an uncertainty of no more than .05. State it!

This is the so-called "bisection method". You could of course find the solution by zooming in on the graph with your calculator, or by making tables with your calculator with smaller and smaller values of Δ*Tbl. The advantage of the method of bisection is that it is very efficient in the sense that it doesn't have to compute too many values of* $f(x)$. *How many values of* $f(x)$ *did you have to calculate, anyway?*

4. (a) Suppose that F and G are two continuous functions on an interval $a \le x \le b$, and that $F(a) \le G(a)$ but $F(b) \ge G(b)$. Show that the equation $F(x) = G(x)$ is satisfied for some x on the interval. (Hint: apply the Intermediate Value Theorem to a suitable combination of F and G.)

(b) By applying (a), show it is possible to cut any circular cake through its exact center so that the two halves have exactly the same amount (area) of icing, no matter how unevenly the cake may have been iced.