Math 151: Calculus
Workshop #8

1. An engineer is trying to find the area of a right triangle drawn on the ground. One side of this triangle is known to be exactly 10 feet long. The angle θ adjacent to this side is measured experimentally (with some possible error) and is found to be $45°$. What is the maximum allowable percentage error in measuring the angle if the error in computing the area of the triangle cannot exceed 4%?

2. Consider the equation $x^4 - 18x^2 - 15 = 0$.

   (a) Show that solving this equation by Newton's method leads to the recursion

   $$x_{n+1} = \frac{3[x_n^4 - 6x_n^2 + 5]}{4x_n(x_n^2 - 9)}.$$

   (b) Carry out two recursions starting with $x_0 = 4$. Draw the graph of $y = x^4 - 18x^2 - 15$ and interpret your calculations on the graph, drawing a tangent line at $x = 4$, etc., etc. What do you think happens to x_n as $n \to \infty$?

   (c) At home: Redo (b) for $x_0 = 1/2$ and $x_0 = \sqrt{5}$.

3. (a) Graph the curves $y_1 = \sin x$ and $y_2 = 3\cos x$. Use the cursor and the ZOOM BOX to find the point of intersection of the curves that has the smallest positive x-value. Read off the approximate coordinates (x_0, y_0) of this point.

   (b) Write out the Newton's method iteration to find the solutions to the equation $\sin x - 3\cos x = 0$. Give a formula for x_{n+1} in terms of x_n for any n.

   (c) Starting with the value x_0, calculate x_1 and x_2 using your formula in (b). This is easy if you use the SEQUENCE mode in your calculator (see the manual). Compare your results with the approximate values that you can get for the solution by ZOOMing in on the point of intersection using the calculator.

4. Find the limits for the following, which are in the indeterminate form "$\infty - \infty$".

   (a) $\displaystyle\lim_{x\to 0}\left(\frac{1}{\tan x} - \frac{1}{x}\right)$.

   (b) $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{\sqrt{x}} - \frac{1}{x}\right)$.

   (c) $\displaystyle\lim_{x\to 0}\left(\frac{e^x}{x} - \frac{e^{-x}}{x}\right)$.

5. For functions f and g, we say

- $f(x) >> g(x)$ as $x \to a$ if $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| = +\infty$,
- $f(x) \sim g(x)$ as $x \to a$ if $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right|$ is a positive real number,
- and $f(x) << g(x)$ as $x \to a$ if $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| = 0$.

For each of the following pairs of functions f and g, determine whether $f(x) >> g(x)$, $f(x) \sim g(x)$ or $f(x) << g(x)$ as x approaches the indicated value.

(a) e^x and 3^x as $x \to +\infty$
(b) e^x and x^{30} as $x \to +\infty$
(c) $e^x + e^{-3x}$ and $e^{3x} + e^{-x}$ as $x \to -\infty$
(d) $e^{1/x}$ and $e^{1/(3x)}$ as $x \to +\infty$
(e) $x^{3.0001}$ and $x^3 \ln x$ as $x \to +\infty$
(f) xe^{x^3} and x^3e^x as $x \to +\infty$
(g) $\exp(1/x^2)$ and $\exp(1/x^3)$ as $x \to 0$
(h) x^{-2} and $(x^{-2} + 2x^{-1})$ as $x \to 0^+$

6. Let

$$f(x) = \begin{cases} e^{x^2} + 1 & \text{if } x \leq 1, \\ -x^2 + ex + 2 & \text{if } x > 1. \end{cases}$$

(a) Sketch the graph of f. Is f continuous on $[-2, 2]$? Why? Is f differentiable on $[-2, 2]$? Why?

(b) What are the critical values of f on $[-2, 2]$? Find all relative minima and relative maxima of f on $[-2, 2]$. HINT: Use a graphing calculator to decide if a particular critical point is a relative extremum.

(c) Find the minimum and the maximum values of $f(x)$ on the interval $[-2, 1.3]$.

(d) Does f has a global maximum? A global minimum? Why?

7. If f is a function, then a real number x_0 is called a *fixed point* of f if and only if $f(x_0) = x_0$.

(a) Find all the fixed points of the following functions to two-place accuracy.

$$f(x) = x^2 \qquad g(x) = x^5 \qquad h(x) = \frac{2}{3} \arctan x.$$

Illustrate your answers graphically, using the graphs of $y = x$ and $y = f(x)$.

(b) Suppose that f is a differentiable function and $f'(x) < 1$ for all x. Use the Mean Value Theorem to explain why f can have no more than one fixed point. To which of the functions in (a) does this general statement apply?