Linear approximation and Error Estimation Help

Earlier this semester, we learned that $f'(x) = lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.
When you think about what "limit" means, another way to look at this equation is:
"when Δx is really small, $f'(x) \approx \frac{f(x+\Delta x)-f(x)}{\Delta x}$."

Multiply both sides by Δx, and we see that:
$f(x + \Delta x) - f(x) \approx f'(x) * \Delta x$.

This equation is extremely useful for estimating values of functions, errors, and other such things.

Notice that if you used the other equation for the defintion of derivative:
$f'(x) = lim_{a \to x} \frac{f(a)-f(x)}{a-x}$, following the same steps, you end up with
$f(a) - f(x) \approx f'(x) * (a - x)$, so we could also say
"the linear approximation of f at a is $f(a) \approx f'(x) * (a - x) + f(x)$."

No matter which way you prefer to write it, this equation is the "big idea" for section 3.8. The problems fall into 3 big categories:
(1) estimate the value of a function at a point (e.g. estimate ln(1.1))
(2) compute maximum error and relative error (e.g. when you compute the volume of a sphere after measuring the radius)
(3) compute marginal cost or marginal revenue.

Below is an example of each kind of problem.

example 1: estimate $ln(1.1)$

remember that the "big idea" is $f(x + \Delta x) - f(x) \approx f'(x) * \Delta x$.

first what function do you see? we're asked to estimate ln of something, so your function should be $f(X) = ln(x)$.

next, do you know a value of x, really close to 1.1 where you know what f(x) is? i know that $ln(1) = 0$, so let $x = 1$.

putting together what i have so far, i see that:
$ln(1 + \Delta x) - ln(1) \approx f'(1) * \Delta x$.

first, let's compute $f'(x)$. Since $f(x) = ln(x)$, we know that $f'(x) = \frac{1}{x}$, so now my equation becomes:
$ln(1 + \Delta x) - ln(1) \approx \frac{1}{1} * \Delta x$.

Finally, let's pick Δx. We want to estimate $ln(1.1)$, so it would be really nice if $ln(1+\Delta x) = ln(1.1)$. That is, we want $1 + \Delta x = 1.1$, so $\Delta x = .1$.

Thus, we get $ln(1 + .1) - ln(1) \approx 1 * (0.1)$, or equivalently:
$ln(1.1) \approx 0.1 + ln(1) = 0.1 + 0 = 0.1$.

Summary: Using the "big idea" equation, picking f(x), x, and Δx appropriately, we can plug in and estimate
$f(x+\Delta x) \approx f'(x) * \Delta x + f(x)$, and only have to calculate $f'(x)$ and $f(x)$ for a "nice" value of x, instead of the "yucky" one given in the problem.

example 2: I measure that the radius of a sphere is 4 inches. I could have measured wrong by as much at 1 inch. What is the maximum error when I compute the volume of the sphere? What is the relative error?

Again, the "big idea" is $f(x+\Delta x) - f(x) \approx f'(x) * \Delta x$.

In our equation, we need a function first. Since i'm talking about the volume and radius of a sphere, the equation that makes sense is $V = \frac{4}{3}\pi r^3$. Moreover, since I have V(r) instead of f(x), let's write the "big idea" with those letters and get:
$V(r+\Delta r) - V(r) \approx V'(r) * \Delta r$.

Notice, that I know $V(r) = \frac{4}{3}\pi r^3$, so $V'(r) = 4\pi r^2$.

Now, I measured the radius to be 4, so $r = 4$.

I could have messed up by as much as 1 inch, so $\Delta r = 1$.

Now, the "big idea" equation tells me that:
$V(r+\Delta r) - V(r) \approx V'(r) * \Delta r$, that is
$V(r+\Delta r) - V(r) \approx 4\pi r^2 \Delta r = (4\pi 4^2) * (1) = 64\pi$

Now, notice that when a problem asks for "maximum (propogated) error", they're asking for $f(x+\Delta x) - f(x)$ (or in our case, for $V(r+\Delta r) - V(r)$).

If you think about it, "error" is a good name for this. V(r) tells me the volume of a sphere with the radius i measured. $V(r+\Delta r)$ tells me the volume of a sphere with as different of a radius of possible from what i measured. So, $V(r+\Delta r) - V(r)$ tells me "what's the biggest possible difference between the real volume and the volume i computed?", or "how big could i have messed up?", i.e. $V(r+\Delta r) - V(r)$ is the maximum error.

So, the maximum error is about 64π.

Now, we need to find the relative error. The book defines relative error to be (maximum propogated error)/(f(x)), or in our case, (maximum propogated error)/V(r).

We already found that the maximum error is 64π, so let's find V(r). $V(r) = \frac{4}{3}\pi r^3$, and $r = 4$, so $V(4) = \frac{4}{3}\pi 4^3 = \frac{256}{3}\pi$.

Now, put this together and we get that the relative error $= \frac{64\pi}{\frac{256}{3}\pi} = \frac{3}{4}$.

Maximum error tells me "how much is my calculation off by?". Relative error tells me "how big a deal is my bad measurement?". In my example the relative error was $\frac{3}{4}$. That means my volume calculation could be off by 0.75, or 75 percent! that's pretty huge! usually relative error will be a smaller decimal (like 0.05 or something like that).

Summary: Again, the "big idea" equation is the answer to our question. In example one, we just needed to estimate the value of a function at a point, so we wanted to pick a function and estimate $f(x + \Delta x)$. When we find error, the equation, x and Δx are usually given, and we want to compute $f(x + \Delta x) - f(x)$ (which we estimate by computing $f'(x) * \Delta x$). This is the maximum error. Otherwise, we also might want to compute relative error, which is max error divided by f(x).

example 3: It costs a company $f(x) = 6x^2 - 7x + 5$ dollars to make x pencils. Find an equation to estimate the marginal cost. Estimate the marginal cost of producing 101 pencils.

Again, the "big idea" is $f(x + \Delta x) - f(x) \approx f'(x) * \Delta x$.

Here's the other BIG idea for marginal cost problems: **Marginal means** $\Delta x = 1$.

So, when we talk about "marginal cost", my "big idea" equation turns into $f(x+1)-f(x) \approx f'(x)*1$, or even simpler, $f(x + 1) - f(x) \approx f'(x)$.

When someone asks for the marginal cost, they want to know "how much does it cost to make just one more". If it costs me f(x) to make x pencils, and f(x+1) to make x+1 pencils, then f(x+1)-f(x) is how much MORE it costs to make x+1 pencils than it costs to make x pencils.

we can estimate the marginal cost with our equation. Since $f(x + 1) - f(x) \approx f'(x)$, when you are told to "estimate marginal cost", then they're really just saying "compute f'(x)".

Now, we also need to estimate the marginal cost of making 101 pencils. That is, we want to estimate $f(101) - f(100)$ (i.e. how much more does it cost to make that 101st pencil instead of just making 100 of them?)

we know $f(x + 1) - f(x) \approx f'(x)$, so we know $f(101) - f(100) \approx f'(100)$. Thus, to estimate the marginal cost of making 101 pencils, we actually compute $f'(100)$. At first it might look a little strange that we plug in one less than the number we care about, but again, if you go back to the "big idea" equation, it should make a little more sense.

Summary: Marginal just means $\Delta x = 1$ in the "big idea" equation. So estimating marginal cost is the same as computing the derivative. Estimating marginal cost at a particular value for x is the same as computing $f'(x - 1)$.

grand summary of section 3.8: $f(x+\Delta x) - f(x) \approx f'(x) * \Delta x$. (or equivalently $f(a) - f(x) \approx f'(x) * (a - x)$), really is the key to all the problems you're asked to do to "estimate", "compute error" or "compute marginal cost (or revenue)". Practice makes perfect though. If you haven't yet, find at least half a dozen problems of each of these kinds and try to work them out for yourself until you start to really see the patterns for yourself! Good luck!