TRANSLATING SOLUTION IN EUCLIDEAN SPACE

JOEL SPRUCK AND LING XIAO

ABSTRACT. In this paper we prove that if 
\[ \liminf_{|x| \to \infty} |Du(x)| > 0, \]
then the entire solution of equation (1.1) in \( \mathbb{R}^2 \) is convex. We also show that if \( u \) is an entire solution of equation (1.1) and \( \Sigma = \{(x, u(x)) | x \in \mathbb{R}^n, n \geq 3\} \) is asymptotic to the cone \( \mathcal{C} = \{(x, \frac{1}{2(n-1)}|x|^2) | x \in \mathbb{R}^n, n \geq 3\} \) in a \( C^2 \) topology. Then \( u = \frac{1}{2(n-1)}|x|^2 + \varphi(x) \) and \( \varphi(x) = o(|x|) \) as \( |x| \to \infty \).

1. INTRODUCTION

In [2], Huisken and Sinestrari proved the following ground breaking result. For any mean convex mean curvature flow in \( \mathbb{R}^{n+1} \), the limit flow obtained by proper blow-up near type II singularities are convex translating solutions. That is, in an appropriate coordinate system, a mean curvature flow of the form \( \mathcal{M} = \{(x, u(x) + t) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n, t \in \mathbb{R}\} \), where \( u \) satisfies
\[ \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{1}{\sqrt{1 + |Du|^2}}. \]

One famous open problem about equation (1.1) can be stated as the following: is every entire convex solution to (1.1) \( k \)-rotationally symmetric? Xujia Wang gave a complete answer to this question in his beautiful paper [5]. He proved that in \( \mathbb{R}^2 \), any entire convex solution to (1.1) must be rotationally symmetric. When \( n > 2 \), he constructed entire convex solution of (1.1) that is not \( k \)-rotationally symmetric. Based on his result, Xujia Wang asked two interesting questions.

**Question1:** Is an entire solution of (1.1) in \( \mathbb{R}^2 \) convex?

**Question2:** If \( |Du(x)| \to \infty \) as \( |x| \to \infty \), is a convex solution of (1.1) rotationally symmetric?

In this paper, we prove:

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Theorem 1.1. Let $u$ be a entire solution of equation (1.1) in $\mathbb{R}^2$, and $\lim \inf_{|x| \to \infty} |Du(x)| > 0$, then $u$ is convex.

In an attempt to answer question 2, we notice that, the uniqueness of the solution is closely related to the decay at infinity. In [5], a key step of proving the uniqueness in $\mathbb{R}^2$ is to prove the following:

Theorem 1.2. Let $u(x) = \frac{1}{2}|x|^2 + \varphi(x)$ be an entire convex solution of (1.1). If $\inf u = 0$, then $|\varphi(x)| = O(|x|^{2/3})$ as $|x| \to \infty$.

In [4], under the assumption that $u(x) \to \frac{1}{2}|x|^2 - \frac{1}{2} \log |x|^2 + O(\frac{1}{|x|})$, as $|x| \to \infty$, Martin, Savas-Halilaj, and Smoczyk showed that the solution to (1.1) is rotationally symmetric. Here following the idea of [5], we prove an analogous result of Theorem 1.2 in $\mathbb{R}^n, n \geq 3$.

Theorem 1.3. Let $u = \frac{1}{2(n-1)}|x|^2 + \varphi(x)$ be an entire convex solution of equation (1.1) and $\inf u(x) = 0$. Moreover, we assume

$$
|D\varphi(x)| = o(|x|), \quad |D^2\varphi(x)| = o(|1|).
$$

Then we have

$$
|\varphi(x)| = o(|x|) \text{ as } |x| \to \infty.
$$

Remark 1.4. First, we note that by Theorem 1.3 in [5], we know $u(x) = \frac{1}{2(n-1)}|x|^2 + \varphi(x)$, where $\varphi(x) = o(|x|^2)$. We also want to point out that, despite we follow the idea in $\mathbb{R}^2$, we need different methods to obtain necessary estimates in higher dimensions, that’s why we have the additional condition (1.2).

We also prove the convexity result in high dimension.

Proposition 1.5. Let $v$ be a solution of equation

$$
\begin{cases}
\delta_{ij} - \frac{v_i v_j}{1 + |Dv|^2} v_{ij} = 1 \text{ in } U \\
v = 0 \text{ on } \partial U.
\end{cases}
$$

and assume that $\partial U$ is convex, then $v$ is a convex function.

Combining Proposition 1.5 and Theorem 1.3 we have

Corollary 1.6. Let $u$ be an entire solution of equation (1.1). Moreover, assume $\Sigma = \{(x, u(x)) | x \in \mathbb{R}^n\}$ is asymptotic to the cone $C = \{(x, \frac{1}{2(n-1)}|x|^2) | x \in \mathbb{R}^n\}$ in a $C^2$
topology. Then
\[ u = \frac{1}{2(n-1)} |x|^2 + \varphi(x) \]
where \[ |\varphi(x)| = o(|x|) \text{ as } |x| \to \infty. \]

2. PROOF OF THEOREM 1.1

**Theorem 2.1.** Let \( u \) be an entire solution of equation (1.1) in \( \mathbb{R}^2 \), and \( \lim \inf_{|x| \to \infty} |Du(x)| > 0 \), then \( u \) is convex.

**Proof.** Let
\[
f(x_1, x_2) = \frac{x_1 + x_2}{2} + \left[ \left( \frac{x_1 - x_2}{2} \right)^2 \right]^{1/2},
\]
\[
\phi(r) = \begin{cases} r^4 e^{-1/r^2} & \text{if } r < 0 \\ 0 & \text{if } r \geq 0 \end{cases},
\]
and
\[
g(z) = f(z) \sum_{i=1}^{2} \phi \left( \frac{z_i}{f(z)} \right).
\]
It’s easy to see that \( g \) is smooth when \( z_1 \neq z_2 \). Now denote \( G(S) = g(\kappa(S)) \) and \( F(S) = f(\kappa(S)) \), where \( S \) is a \( 2 \times 2 \) symmetric matrix and \( \kappa(S) \) is the eigenvalues of \( S \). Now let \( S \) be the second fundamental form of \( \Sigma = \{(x, u(x)) \mid x \in \mathbb{R}^2\} \), then it’s easy to see that \( G/F \leq 1 \). Therefore, \( G/F \) achieves its maximum somewhere. By a straight forward calculation we have
\[
\frac{\nabla (G/F)}{F^2} + 2 \left( \nabla F, \nabla \left( \frac{G}{F} \right) \right) + \left( \nabla \left( \frac{G}{F} \right), e_{n+1} \right) = \frac{G^{ij} h_{ij} |A|^2}{F} + \frac{G F^{ij} h_{ij} |A|^2}{F^2} + \left( \frac{G^{pq,rs} h_{pqk} h_{rsk}}{F} - \frac{G F^{pq,rs}}{F^2} \right) h_{pqk} h_{rsk}.
\]
Let’s compute the last term.
\[
G^{pq,rs} h_{pqk} h_{rsk} = g^{pq} h_{pqk} h_{qqk} + 2 \frac{g^2 - g^1}{\kappa_2 - \kappa_1} \frac{h_{12k}^2}{h_{12k}}.
\]
where
\[
g^{pq} = f^{pq} \sum_{i=1}^{2} \left[ \phi \left( \frac{z_i^q}{f} \right) - \frac{z_i^q}{f} \phi \left( \frac{z_i^q}{f} \right) \right]
\]
\[
+ \frac{1}{f} \sum_{i=1}^{2} \phi \left( \frac{z_i^q}{f} \right) \left( \delta_i^p - \frac{z_i^q}{f} f^p \right) \left( \delta_i^q - \frac{z_i^q}{f} f^q \right).
\]

We also have
\[
FG^{pq,rs} - GF^{pq,rs} = \left( fg^{pq} - gf^{pq} \right) + 2 \left\{ f \left( g^2 - g^1 \right) - g \left( f^2 - f^1 \right) \right\}
\]
\[
= I + II.
\]

Next, we calculate term I and term II and get
\[
I = f^{pq} \left[ g - \sum_{i=1}^{2} z_i \phi \left( \frac{z_i^q}{f} \right) \right] + \sum_{i=1}^{2} \phi \left( \frac{z_i^q}{f} \right) \left( \delta_i^p - \frac{z_i^q}{f} f^p \right) \left( \delta_i^q - \frac{z_i^q}{f} f^q \right) - gf^{pq}
\]
\[
= - f^{pq} \sum_{i=1}^{2} z_i \phi \left( \frac{z_i^q}{f} \right) + \sum_{i=1}^{2} \phi \left( \frac{z_i^q}{f} \right) \left( \delta_i^p - \frac{z_i^q}{f} f^p \right) \left( \delta_i^q - \frac{z_i^q}{f} f^q \right),
\]

\[
g^2 - g^1 = \phi \left( \frac{z_2^q}{f} \right) + f^2 \sum_{i=1}^{2} \left[ \phi \left( \frac{z_i^q}{f} \right) - \frac{z_i^q}{f} \phi \left( \frac{z_i^q}{f} \right) \right]
\]
\[
- \phi \left( \frac{z_1^q}{f} \right) - f^1 \sum_{i=1}^{2} \left[ \phi \left( \frac{z_i^q}{f} \right) - \frac{z_i^q}{f} \phi \left( \frac{z_i^q}{f} \right) \right]
\]

and
\[
f(g^2 - g^1) - g(f^2 - f^1)
\]
\[
= f \left[ \phi \left( \frac{z_2^q}{f} \right) - \phi \left( \frac{z_1^q}{f} \right) \right] + f^2 \left[ g - \sum_{i=1}^{2} z_i \phi \left( \frac{z_i^q}{f} \right) \right]
\]
\[
- f^1 \left[ g - \sum_{i=1}^{2} z_i \phi \left( \frac{z_i^q}{f} \right) \right] - g \left( f^2 - f^1 \right)
\]
\[
= f \left[ \phi \left( \frac{z_2^q}{f} \right) - \phi \left( \frac{z_1^q}{f} \right) \right] + \sum_{i=1}^{2} z_i \phi \left( \frac{z_i^q}{f} \right) \left( f^1 - f^2 \right).
\]
Assume \( \kappa_1 > 0 > \kappa_2 \), then we obtain
\[
\Delta \left( \frac{G}{F} \right) + 2 \left\langle \nabla \left( \frac{G}{F} \right), \nabla \left( \frac{G}{F} \right), e_{n+1} \right\rangle + \left\langle \nabla \left( \frac{G}{F} \right), e_{n+1} \right\rangle
= \left( \frac{G_{pq,rs}}{F} - \frac{G_{pq,rs} F}{F^2} \right) h_{pk}^r h_{sk}^r
\]
\[
= \frac{1}{F^2} \left( F G_{pq,rs}^r - G_{pq,rs} F^r \right) h_{pk}^r h_{sk}^r
\]
\[
= \frac{1}{\kappa_1^2} \left[ \frac{\kappa_2}{\kappa_1} \left( \frac{\kappa_2}{\kappa_1} - \frac{\delta_2^p}{\kappa_1} f^p \right) \left( \frac{\delta_2^q}{\kappa_1} f^q - \frac{\kappa_2}{\kappa_1} \right) h_{pk}^r h_{qq}^r \right]
\]
\[
+ \frac{2}{\kappa_1^2} \left[ \frac{\kappa_1}{\kappa_2} \left( \frac{\kappa_2}{\kappa_1} \right) + \frac{\kappa_2}{\kappa_1} \left( \frac{\kappa_2}{\kappa_1} \right) \right] h_{12k}^2 \geq 0.
\]
(2.11)

Here we used \( f_{pq} = 0, \dot{\phi} \left( \frac{\kappa_2}{\kappa_1} \right) = 0 \), and \( f^2 = 0 \) when \( \kappa_1 > 0 > \kappa_2 \). We can see that if \( G/F \) achieves its maximum at an interior point, then by the strong maximum principle, we get \( G/F \equiv \text{constant} \). It’s easy to see that in this case \( G/F \equiv 0 \), thus, \( \kappa_2 \geq 0 \).

If \( G/F \) achieves it’s maximum at infinity, we apply Omori-Yau’s maximum principle and get the following.

When \( |\left( \nabla \kappa \right)_{F} | \) is bounded, we know that there exists a sequence \( \{ P_n \} \) such that at \( P_n \), \( \kappa_2/\kappa_1(P_n) = r_n \to r_0 \), and \(-1 \leq r_0 < 0 \). Moreover we have
\[
\frac{1}{n} \geq \left( \sum_{\kappa} \kappa_2 \left( \frac{\kappa_2}{\kappa_1} \right) \left( \frac{\kappa_2}{\kappa_1} - \frac{\delta_2^p}{\kappa_1} f^p \right) \left( \frac{\delta_2^q}{\kappa_1} f^q - \frac{\kappa_2}{\kappa_1} \right) h_{pk}^r h_{qq}^r \right)
\]
\[
+ \frac{2}{\kappa_1^2} \left[ \frac{\kappa_1}{\kappa_2} \left( \frac{\kappa_2}{\kappa_1} \right) + \frac{\kappa_2}{\kappa_1} \left( \frac{\kappa_2}{\kappa_1} \right) \right] h_{12k}^2 \geq 0.
\]
(2.12)

Since
\[
\frac{1}{n} \geq \left| \nabla \kappa_2 \left( \frac{\kappa_2}{\kappa_1} \right) (P_n) \right| = \left| \nabla \left( \frac{H}{\kappa_1} (P_n) \right) \right|,
\]
we get
\[
\frac{\kappa_2}{\kappa_1} \left( \tau_k, e_{n+1} \right) \leq \frac{1}{n} + \left| \left( 1 + r_n \right) h_{11k}^2 \right|,
\]
(2.13)
and by (2.12), the right hand side of (2.14) goes to 0. Therefore we have \( \langle \tau_k, e_{n+1} \rangle = 0 \), which implies \( H = 1 \) at infinity, a contradiction.

Now we assume that \( |\left( \nabla \kappa \right)_{F} (P_n) | \) is unbounded. In this case, since
\[
\left| \left( \nabla \kappa \frac{H}{F} \right) \right| = \left| \frac{\kappa_2 \left( \tau_k, e_{n+1} \right) \left( \frac{h_{11k} + h_{22k}}{\kappa_1} \right)}{\kappa_1} \right| = \left| \frac{h_{11k} + h_{22k}}{\kappa_1} \right| \leq 1
\]
we have \( \langle h_{11k}, h_{22k} \rangle < 0 \).

\[
\frac{\kappa_n^2}{n} \geq \phi \left[ r_n h_{11k} - h_{22k} \right]^2 + 2 \phi \frac{1 + r_n h_{12k}^2}{1 - r_n} - 2 \kappa_1 \left\langle \nabla F, \nabla \left( \frac{G}{F} \right) \right\rangle
\]

(2.15)

\[
= \phi \left[ r_n h_{11k} - h_{22k} \right]^2 + 2 \phi \frac{1 + r_n h_{12k}^2}{1 - r_n} - 2 \kappa_1 \phi \left\langle h_{11k}, \left( \frac{\kappa_n^2}{\kappa_1} \right) k \right\rangle
\]

Thus we get

\[
\frac{\kappa_n^2}{n} \geq \phi \left[ r_n h_{11k} - h_{22k} \right]^2 + 2 \phi \frac{1 + r_n h_{12k}^2}{1 - r_n}.
\]

Since

(2.16) \[
|\kappa_k \langle \tau_k, e_{n+1} \rangle| = | - r_n h_{11k} + h_{22k} | + |(1 + r_n)h_{11k}^2| \leq \frac{\kappa_1}{\sqrt{n}},
\]

same as before, we conclude that \( H = 1 \) at infinity, and this leads to a contradiction. \( \Box \)

Combine Theorem 1.1 with Theorem 1.1 in [5] we have

**Corollary 2.2.** If \( n = 2 \) and \( \lim \inf_{|x| \to \infty} |Du(x)| > 0 \), then any entire solution to (1.1) must be rotationally symmetric in an appropriate coordinate system.

### 3. Proof of Theorem 1.2

In this section, for any constant \( \tau > 0 \) we denote

\[
\Gamma_{\tau, u} = \{ x \in \mathbb{R}^n : u(x) = \tau \},
\]

\[
\Omega_{\tau, u} = \{ x \in \mathbb{R}^n : u(x) < \tau \}.
\]

We will also denote

\[
\mathcal{L}_\sigma[u] =: \sum_{i,j=1}^n \left( \delta_{ij} - \frac{u_i u_j}{\sigma + |Du|^2} \right) u_{ij} = 1.
\]

We are going to study the convex solution to the following equation:

(3.1) \[
\sum_{i,j=1}^n \left( \delta_{ij} - \frac{u_i u_j}{1 + |Du|^2} \right) u_{ij} = 1.
\]
Lemma 3.1. Let \( \{M_t\} \) be a convex solution to the mean curvature flow. Suppose \( M_0 \) is in the \( \delta_0 \) neighborhood of a unit sphere \( S^{n-1} \) and \( \{M_t\} \) shrinks to a point (the origin) at \( t = 1/2 \). Let \( \hat{M}_t = \frac{1}{\sqrt{1-2t}} M_t \) be the normalization of \( M_t \). Then \( \hat{M}_t \) is in the \( \delta_t \)–neighborhood of the sphere centered at the origin, which means,

\[
\hat{M}_t \subset N_{\delta_t}(S^{n-1}), \text{ with } \delta_t \leq C\delta_0 (1/2 - t)^\alpha
\]

for some constant \( \alpha \in (0, 1) \).

Remark 3.2. This can be proved by a Schauder estimate and a modification argument of [1], where we only need to change the scaling factor.

The next lemma plays an important role in our later proof.

Lemma 3.3. Let \( u \) be a convex solution of \( \mathcal{L}_0[u] = 0 \) which attains its minimum 0 at \( y_1 \). Suppose the level set \( \Gamma_{1/2(n-1), u} \subset N_{\delta_0}(S_1(0)) \) for some small \( \delta_0 > 0 \), then \( |y_1| < C\delta_0 \) for some \( C > 0 \) independent of \( \delta_0 \).

Proof. Let \( F(A_{ij}) = \frac{\sigma_{n-1}(\lambda_i)}{\sigma_{n-2}(\lambda_i)} \), where \( \{A_{ij}\} \) is symmetric matrix and \( \lambda_i \) is the eigenvalue of \( \{A_{ij}\} \). Let \( \tilde{u} = \frac{1}{\sqrt{2(n-1)}} |x|^2 \) be the rotationally symmetric solution to \( \mathcal{L}_0[u] = 0 \). Let \( \omega \), \( \tilde{\omega} \) be respectively the support functions of \( \Gamma_{h,u} \) and \( \Gamma_{\tilde{h},\tilde{u}} \), we also denote \( t = -h \) (regard \( t \in (-\frac{1}{2}, 0) \) as the time). Then we have

\[
\omega_t F(\omega_{ij} + \omega \delta_{ij}) = -1
\]

and

\[
\tilde{\omega}_t F(\tilde{\omega}_{ij} + \tilde{\omega} \delta_{ij}) = -1
\]

Since \( \tilde{\omega}_t = -\frac{1}{\sqrt{|t|}} \) and \( \omega_t = -\frac{1+o(1)}{\sqrt{|t|}} \), (the latter inequality is due to the rescaling mean curvature flow approaching sphere exponentially) we get

\[
\omega_t \tilde{\omega}_t = \frac{1 + o(1)}{|t|}.
\]

Now consider \( \phi = \omega - \tilde{\omega} \), it’s easy to see that

\[
[F(\omega_{ij} + \omega \delta_{ij}) + F(\tilde{\omega}_{ij} + \tilde{\omega} \delta_{ij})](\omega_t - \tilde{\omega}_t)
= F(\omega_{ij} + \omega \delta_{ij})\omega_t - F(\omega_{ij} + \omega \delta_{ij})\tilde{\omega}_t
= - (\omega_t + \tilde{\omega}_t)[F(\omega_{ij} + \omega \delta_{ij}) - F(\tilde{\omega}_{ij} + \tilde{\omega} \delta_{ij})]
= - (\omega_t + \tilde{\omega}_t)[F^{ij}(\phi_{ij} + \phi \delta_{ij})]
\]
Therefore

\[ (3.6) \quad \phi_t = (\omega t \hat{\omega}_t) F^{ij} [\phi_{ij} + \phi \delta_{ij}] . \]

Moreover, the curvature of \( \Gamma_{h,u} \) is equal to that of \( \Gamma_{h,\hat{u}} \) up to a lower order perturbation, which implies \( \phi_{ij} + \phi \delta_{ij} \leq C \delta_0 |t|^\alpha, \alpha > 0 \). We obtain,

\[ (3.7) \quad |\phi_t| \leq C \delta_0 |t|^{\alpha-1} \]

and so \( |y_1| \leq \sup |\phi| \leq C \delta_0 \). \( \square \)

Now, let \( u = \frac{1}{2} (n-1) |x|^2 + \varphi(x) \),

then we have,

\[ u^h = \frac{1}{2(n-1)} |x|^2 + \frac{\varphi(h^{1/2}x)}{h} . \]

For convenient, in the next lemma, we denote \( u^h = u \) and \( \sigma = \frac{1}{h} \). This Lemma is parallel to equation (3.18) in [5].

**Lemma 3.4.** Let \( \Omega = \Omega_{2(n-1)/h}^1 u^h \subset \mathbb{R}^n \). Let \( u_0 \) and \( u^h \) be solutions of \( L_0[u] = 1 \) and \( L_\sigma[u] = 1 \) in \( \Omega \) respectively, with boundary value equals 1/2 \((n-1)\) on \( \partial \Omega \). Then, for any constant \( a > a^* > C \max \{\delta_0, 1/h\} > 0 \) there exists a constant \( C > 0 \), depending on \( a \), such that for any \( a + \inf_{\Omega} u^h < \tau < \frac{1}{2(n-1)} \), we have

\[ (3.8) \quad \text{dist} (\Gamma_{\tau,u_0}, \Gamma_{\tau,u^h}) \leq C(a)\sigma . \]

Here and in the following \( \text{dist}(A, B) \) denotes the Hausdorff distance between \( A \) and \( B \).

**Proof.** By convexity we have

\[ \text{L}_0[u^h] \leq 1 = \text{L}_\sigma[u^h], \]

therefore, due to the comparison principle we have \( u^h \geq u_0 \), which implies \( \Omega_{\tau,u^h} \subset \Omega_{\tau,u_0} \).

We can write the equation \( \text{L}_\sigma[u^h] = 1 \) in the form

\[ \tilde{H} u_\gamma = 1 - \frac{\sigma u_{\gamma\gamma}}{\sigma + u_\gamma^2}, \]

where \( \tilde{H} \) is the mean curvature of the level set \( \Gamma_{\tau,u^h} \), and \( \gamma \) is the unit outward normal to \( \Gamma_{\tau,u^h} \). Therefore, the level set \( \Gamma_{\tau,u^h} \) is moving with velocity \( \frac{\tilde{H} u_\gamma}{\tilde{H} u_\gamma + u_\gamma^2} \), while \( \Gamma_{\tau,u_0} \) is moving with its mean curvature.
Now, suppose \( \inf u_0 = u_0(y_0) = -a_0 \), where \( a_0 > 0 \), and \( h \) is very large so that \( \Gamma_{1/2(n-1),u^h} \) is in a \( \delta_0 \) neighborhood of unit sphere centered at the origin, \( S_1(0) \). Then we have
\[
(3.9) \quad u_0 \geq \frac{1}{2(n-1)}(|x|^2 - (1 + \delta_0)^2) + \frac{1}{2(n-1)} , \quad \text{in } \Omega_{1/2(n-1),u^h},
\]
which implies, \( a_0 < \frac{1}{2(n-1)} \delta_0 \). Moreover, by Lemma 3.3 we have \( |y_0| < C \delta_0 \). Therefore, by assumption (1.2) we have, \( \tau > a > a^* > C \max \{ \delta_0, 1/h \} \) we can write \( \Gamma_{\tau,u^h} \) as a radial graph over \( S_1(y_0) \). Moreover, by Lemma 3.1, it’s easy to see that \( \Gamma_{\tau,u^0} \) is a radial graph over \( S_1(y_0) \) as well (this is for any \( \tau \geq 0 \), but it doesn’t matter here).

Next, we write \( \partial \Omega \) as a radial graph \( e^{v(z)}z, z \in S_1(y_0) \), then we have
\[
\nu = -z + \nabla v(1 + |\nabla v|^2)^{1/2}.
\]

When \( \partial \Omega \) flows with respect to mean curvature we have
\[
(3.10) \quad b_{ij} = e^v \frac{\sigma_{ij} + \nabla_i v \nabla_j v - \nabla_{ij} v}{(1 + |\nabla v|^2)^{1/2}}.
\]

When \( \partial \Omega \) flows with respect to \( \tilde{H} \) we have
\[
(3.11) \quad g_{ij} = e^{-2v} \left( \sigma_{ij} - \frac{\sigma_{ij} v_i v_j}{1 + |\nabla v|^2} \right) = e^{-2v} a_{ij}.
\]

Now, let \( w = v - \tilde{v} \), since \( \Omega_{\tau,u^h} \subset \Omega_{\tau,u^0} \), we know that \( w \geq 0 \). Moreover, \( w \) satisfies
\[
(3.16) \quad w_t = G_{i,j} w_{ij} + G_i w_i + G_w w + \frac{\sigma_{i,j} \tilde{H}}{\sigma + u_\gamma - \sigma u_{\gamma \gamma}} \times \frac{(1 + |\nabla \tilde{v}|^2)^{1/2}}{e^v}.
\]
It’s easy to see that \( \|v\|_2, \|\tilde{v}\|_2 \leq C(a) \), therefore, there exists \( C_1(a), C_2(a) > 0 \) such that
\[
(3.17) \quad w_i \leq G^{ij} w_{ij} + G^i w_i + C_1(a) w + C_2(a) \sigma.
\]
Consider \( \tilde{w} = e^{-C_1(a) t} w \), then \( \tilde{w} \) satisfies
\[
(3.18) \quad \tilde{w}_t \leq G^{ij} \tilde{w}_{ij} + G^i \tilde{w}_i + e^{-C_1(a) t} C_2(a) \sigma,
\]
We obtain,
\[
(3.19) \quad \frac{d}{dt} (\tilde{w} - C_2(a) \sigma t) - G^{ij} (\tilde{w} - C_2(a) \sigma t)_{ij} - G^i (\tilde{w} - C_2(a) \sigma t)_i \leq 0.
\]
Thus \( \tilde{w} \leq C_2(a) \sigma t \), which implies \( e^\nu - e^{\tilde{v}} \leq C_3(a) \sigma \) when \( \sigma \) is very small. The lemma is proved. \( \square \)

Combining Lemma 3.1 and Lemma 3.4 we get

**Corollary 3.5.**
\[
dist(\Gamma_{\tau,u^h}, S^{\sqrt{2(n-1)(\tau + a_0)}}(y_0)) \leq C(\tau) \sigma + C \delta_0(\tau + a_0)^{1/2+\alpha},
\]
where \( \tau \geq \tau_0 > C \max\{\delta_0, 1/h\} \).

**Remark 3.6.** In [5], it says equation (3.16) + (3.18) gives (3.15). However, the authors don’t see why it’s true. Since
\[
(3.16) \quad \Gamma_{\tau,u^h} \subset \sqrt{2(\tau + a_0)}(N_{\delta_1}(S^1))
\]
with \( \delta_1 \leq C \delta_0(\tau + a_0)^{\alpha} \), which is equivalent to
\[
(3.16') \quad dist(\Gamma_{\tau,u^0}, \sqrt{2(\tau + a_0)} S^1) \leq C \delta_0 \sqrt{2(\tau + a_0)^{\alpha+1/2}};
\]
and
\[
(3.18) \quad dist(\Gamma_{\tau,u^1}, \Gamma_{\tau,u^0}) \leq dist(\Gamma_{\tau,u^1}, \Gamma_{\tau,u^0}) + dist(I, \Gamma_{\tau,u^0})
\]
\[
\leq C \delta_0(\tau + a_0)^{1/2+\alpha} + C \sigma^{2/3}(\tau + a_0)^{-1/6},
\]
where \( a_0 \leq 3 \delta_0 < \tau \). While
\[
(3.15) \quad \Gamma_{\tau,u^h} \subset \sqrt{2\tau}(N_{\delta_\tau}(S^1)),
\]
with \( \delta_\tau \leq C_1(\tau) \sigma^{2/3} + C_2 \delta_0 \tau^{\alpha} \), which is equivalent to
\[
(3.15') \quad dist(\Gamma_{\tau,u^h}, \sqrt{2\tau} S^1) \leq \sqrt{2\tau}(C_1(\tau) \sigma^{2/3} + C_2 \delta_0 \tau^{\alpha}).
\]
However, by (3.16’) and (3.18) we have
\[
\begin{align*}
\text{dist}(\Gamma_\tau,u^h,\sqrt{2}\tau S^1) & \leq \text{dist}(\Gamma_{\tau_0},\sqrt{2(\tau + a_0)}S^1) \\
& + \text{dist}(\Gamma_{\tau_0},\Gamma_{\tau_0}) + \text{dist}(\sqrt{2(\tau + a_0)}S^1,\sqrt{2}\tau S^1) \\
& \leq \sqrt{2\tau}(C_1(\tau)a^{2/3} + C_2\delta_0\tau^\alpha) + \frac{a_0}{\sqrt{2(\tau + a_0)} + \sqrt{2\tau}},
\end{align*}
\]
which is not equivalent to (3.15’).

Therefore, we need a better estimate on $a_0$, because we want to show that there exists a $\tilde{\tau} \geq \tau_0 > C \max\{a_0, \delta_0, \sigma\}$ very small, such that
\[
\begin{align*}
\Gamma_{\tilde{\tau},u^h} & \subset \sqrt{2(n - 1)\tilde{\tau}}(N_\delta S_1(y_0))
\end{align*}
\]
with $\delta \leq C\sigma + \delta_0/4$. It's easy to see that equation (3.21) is equivalent to
\[
\begin{align*}
\Gamma_{\tilde{\tau}/2(n - 1),u^h} & \subset \sqrt{\tilde{\tau}}(N_\delta S_1(y_0)).
\end{align*}
\]
First, we assume equation (3.22) is true, then follow Xujia Wang’s argument, we will get $\varphi(x) = O(1)$. For reader’s convenience, we include his argument here. We start at level $\tilde{\tau} - k/2(n - 1)$ for some sufficiently large $k$. Denote $\Omega_k = \sqrt{\tilde{\tau}k}\Omega_{\tilde{\tau} - k/2(n - 1),u^*}$, where $u^* = \frac{|x|^2}{2(n - 1)} + \varphi(x)$ and $\Gamma_k = \partial \Omega_k$. By the property of $\varphi(x)$, we know that $\Gamma_k$ converges to the unit circle as $k \to \infty$. Moreover, it’s easy to verify that
\[
u^\tilde{\tau} - k(x) = \frac{1}{2(n - 1)}\text{ on } \Gamma_k.
\]
Suppose $\Gamma_k$ is in the $\delta_k$ neighborhood of $S_1$, where $\delta_k \to 0$ as $k \to \infty$. Let $y_k$ denote the minimum point of the solution of $L_0[u] = 1$ in $\Omega_k$ and $u = 1/2(n - 1)$ on $\Gamma_k$. By (3.21), $\Gamma_{k - 1}$ is in the $\delta_{k - 1}$ neighborhood of $S_1(y_k)$ with
\[
delta_{k - 1} \leq C\tilde{\tau}^k + \delta_k/4.
\]
By induction we obtain
\[
delta_j \leq 2C\tilde{\tau}^{j + 1} + \delta_k \forall j < k.
\]
Letting $k \to \infty$ and since $\tilde{\tau}^{j + 1} < \tilde{\tau}^j$, we get
\[
\begin{align*}
\Gamma_j & \subset N_{\delta_j}(S_1(y_{j + 1}))
\end{align*}
\]
with $\delta_j \leq 2C\tilde{\tau}^j$. It follows that for $h = \tilde{\tau}^{-j}$ sufficiently large,
\[
\Gamma_{h/2(n - 1),u^*} \subset N_\delta(\sqrt{h}S_1(z_{j + 1}))
\]
with $\delta \leq 2Ch^{-1/2}$ and $z_{j + 1} = h^{1/2}y_{j + 1}$.
Next, we estimate $|z_{j+1} - z_j|$. By Lemma 3.3 we know that $|y_j - y_{j+1}| \leq C\delta_j = C\tilde{\tau}^j$, therefore, we get $|z_{j+1} - z_j| \leq C\tilde{\tau}^{j/2}$. From the above estimate we can see that $\{z_j\}$ is convergent, and by the assumption on $u^*$ we have that $\{z_j\}$ converges to 0 and when $j$ is large $|z_j| \leq C\tilde{\tau}^{j/2}$. We obtain

\begin{equation}
\Gamma_{h/2(n-1), u^*} \subset N_\delta(\sqrt{h}S_1(0)) \tag{3.24}
\end{equation}

where $\delta \leq Ch^{-1/2}$, hence $\varphi = O(1)$.

Now, we are going to prove equation (3.22). Just as before we denote $\sigma = 1/h$. Observe that

\begin{equation}
(\delta_{ij} - u^h_i u^h_j) u^h_{ij} = 1, \tag{3.25}
\end{equation}

which implies

\begin{equation}
\left( \delta_{ij} - \frac{u^h_i u^h_j}{|Du^h|^2} + \frac{\sigma u^h_i u^h_j}{|Du^h|^2(\sigma + |Du^h|^2)} \right) u^h_{ij} = 1. \tag{3.26}
\end{equation}

Therefore,

\begin{equation}
\hat{H}|Du^h| = 1 - \frac{\sigma u^h_i u^h_j u^h_{ij}}{|Du^h|^2(\sigma + |Du^h|^2)} \tag{3.27}
\end{equation}

\begin{align*}
&= 1 - \frac{\sigma}{|Du^h|^2} (\Delta u^h - 1) \\
&= 1 + \frac{\sigma}{|Du^h|^2} - \frac{\sigma}{|Du^h|^2} \Delta u^h,
\end{align*}

to derive the second equality used equation (3.25).

Let $u_1 = |x - y_0|^2 / 2(n-1)$, then $\Gamma_{\tau, u_1}$ is a sphere of radius $\sqrt{2(n-1)\tau}$ centered at $y_0$, and $\Gamma_{1/2(n-1), u_1}$ disappears at $t = 1/2(n-1)$. We also let

\[ D(\Gamma_{\tau, u_1}, x) = \text{signed distance} \]

and when $x \in \Omega_{\tau, u_1}$, $D > 0$ otherwise $D \leq 0$. The idea is we will find the $\min_{0 \leq \tau \leq 1/2(n-1)} \max_{x \in \Gamma_{\tau, u_1}} D(\Gamma_{\tau, u_1}, x)$ and study the moving speed around the point at which the minmax value is achieved.

Case 1. When $\max_{x \in \Gamma_{\tau, u_1}} D(\Gamma_{\tau, u_1}, x) \geq 0$, $\forall \tau \geq \tau_0$. Since

\begin{equation}
\text{dist}(\Gamma_{\tau, u^*}, S) \leq C(\tau)\sigma + C\delta_0\tau^{1/2+\alpha} \tag{3.28}
\end{equation}

we have

\begin{equation}
\sqrt{2(n-1)(\tau + a_0)} - \sqrt{2(n-1)\tau} < C(\tau)\sigma + C\delta_0\tau^{1/2+\alpha}. \tag{3.29}
\end{equation}
Hence,
\(\text{dist}(\Gamma_{\tau,u^h}, S_{\sqrt{2(n-1)}}(y_0)) < C(\tau)\sigma + C\delta_0 \tau^{1/2+\alpha},\)
which yields (3.21).

Therefore, we assume \(\min_{0 \leq \tau \leq 1/2(n-1)} \max_{x \in \Gamma_{\tau,u^h}} D(\Gamma_{\tau,u_1}, x) < 0\)

Case 1' When \(\min_{0 \leq \tau \leq 1/2(n-1)} \max_{x \in \Gamma_{\tau,u^h}} D(\Gamma_{\tau,u_1}, x) = -d_0 < 0\) is achieved at \(\tau = 1/2(n-1)\).

Since \(\Gamma_{1/2(n-1),u^h} \subset N_{\delta_0}(S_1(0))\), and by Lemma 3.3 \(|y_0| < C\delta_0\), we have \(d_0 < C\delta_0\).

Therefore
\[
\sqrt{2(n-1)(\tau + a_0)} - \sqrt{2(n-1)\tau} < C(\tau)\sigma + C\delta_0 \tau^{1/2+\alpha} + C\delta_0
\]
for any \(\tau \geq \tau_0\). In particular, we have at \(\tau = \tau_0\), we have
\[
a_0 \leq C(\tau_0)\sigma + C\delta_0 \tau_0^{1+\alpha} + C\delta_0 \tau_0^{1/2}.
\]
We can choose \(\tilde{\tau} > C\tau_0\) small such that
\[
\sqrt{2(n-1)(\tilde{\tau} + a_0)} - \sqrt{2(n-1)\tilde{\tau}} < C(\tilde{\tau})\sigma + \frac{1}{8}\delta_0,
\]
thus (3.21). Note also that the same argument holds whenever \(d_0 < C\delta_0\).

Case 2' When \(\min_{0 \leq \tau \leq 1/2(n-1)} \max_{x \in \Gamma_{\tau,u^h}} D(\Gamma_{\tau,u_1}, x) = -d_0 < 0\) is achieved at \(0 < \tau^* < 1/2(n-1)\). In a small neighborhood of this point, \(\Gamma_{\tau,u^h}\) can be rewritten as a radial graph of \(S_1(y_0)\), denoted by \(\tilde{\rho}\), and we also denote \(\Gamma_{\tau,u_1}\) as a radial graph \(\rho\). Since
\[
\min_{x \in \Gamma_{\tau,u^h}} \max_{x \in \Gamma_{\tau,u_1}} (\rho - \tilde{\rho}) = \rho(z_0, 1/2(n-1) - \tau^*) - \tilde{\rho}(z_0, 1/2(n-1) - \tau^*) = -d_0
\]
we have
\[
\frac{d}{dt}(\rho - \tilde{\rho})|_{(z_0, 1/2(n-1) - \tau^*)} = -H + \frac{\tilde{H}}{1 + \frac{\sigma}{|Du^h|^2} - \frac{\sigma}{|Du^h|^2} \Delta u^h} = 0.
\]
It’s easy to see that the mean curvature of \(\Gamma_{\tau^*,u^h}\) is \(\frac{1}{\sqrt{2(n-1)\tau^*}}\). On the other hand, \(S_{\sqrt{2(n-1)\tau^*+d_0}}(y_0)\)
is an interior ball of \(\Gamma_{\tau^*,u^h}\), therefore, at \((z_0, 1/2(n-1) - \tau^*)\)
\[
\tilde{H} \leq \frac{1}{\sqrt{2(n-1)\tau^* + d_0}}.
\]
By our assumption on \(\varphi\), we can see that
\[
|\Delta u^h - 1| < C_1.
\]
Moreover, \(u^h_{ij}\eta_i\eta_j \geq c_0|\eta|^2\), for some \(c_0 > 0\) and \(|Du^h|(0) = 0\), we get
\[
|Du^h| > C_2|x|.
\]
Thus, by (3.27)

\[(3.39) \quad \tilde{H} u_{\tau} \geq 1 - \frac{C_1 \sigma}{C_2 |x|^2}.\]

Since \( u(x) = \frac{|x|^2}{2(n-1)} + \varphi(x) \) and \(|\varphi(x)| = o(|x|^2)\), for any \( \lambda_0 > 0 \) small, there exists \( R_0 > 0 \) such that \(|\varphi(x)| \leq \lambda_0 |x|^2 \) when \(|x| > R_0\). When \(|x| < R_0\), \(|\varphi(x)| < C\) is bounded.

Now, assume at \( \tau^* \), \(|h^{1/2}x| > R_0\), which implies \(|x| > R_0\sigma^{1/2}\). We have

\[(3.40) \quad \tau^* = \frac{|x|^2}{2(n-1)} + \frac{\varphi(h^{1/2}x)}{h} \leq \frac{|x|^2}{2(n-1)} + \lambda_0 |x|^2,\]

which implies,

\[(3.41) \quad |x|^2 \geq \frac{2(n-1)\tau^*}{1 + 2(n-1)\lambda_0}.\]

Since \( \tilde{H} u_{\tau} \geq 1 - \frac{C_\sigma}{|x|^2} \), where \( C \) is independent of \( h \). WLOG, we assume that \( R_0^2 \sigma > 2C_\sigma \).

Under this assumption, we have

\[(3.42) \quad \frac{1}{1 - \frac{C_\sigma}{|x|^2}} < 1 + \frac{2C_\sigma}{|x|^2}.\]

Combining with equations (3.35) and (3.36) we obtain,

\[(3.43) \quad 1 + \frac{2C_\sigma}{|x|^2} \geq 1 + \frac{d_0}{\sqrt{2(n-1)\tau^*}},\]

thus, by (3.41)

\[(3.44) \quad \frac{2C[1 + 2(n-1)\lambda_0]\sigma}{\sqrt{2(n-1)\tau^*}} \geq d_0.\]

When \( \tau^* \geq d_0 \), we have \( d_0 \leq C\sigma^{2/3} \). When \( \tau^* < d_0 \), since

\[(3.45) \quad \tau^* \geq \frac{|x|^2}{2(n-1)} - \lambda_0 |x|^2,\]

we have

\[(3.46) \quad \tau^* \geq \left( \frac{1}{2(n-1)} - \lambda_0 \right) R_0^2 \sigma.\]

Therefore, \( d_0 \leq C\sigma^{1/2} \).

If at \( \tau^* \), \(|x| < R_0\sigma^{1/2}\). Since

\[(3.47) \quad \tau^* = \frac{|x|^2}{2(n-1)} + \frac{\varphi(h^{1/2}x)}{h} \geq \frac{|x|^2}{2(n-1)} - C(\lambda_0)\sigma,\]
we get
\[(3.48) \quad |x|^2 \leq 2(n-1)\tau^* + C(\lambda_0)\sigma.\]

Then, we have
\[(3.49) \quad (\sqrt{2(n-1)\tau^* + d_0})^2 < 2(n-1)\tau^* + C(\lambda_0)\sigma,
\]
(this is because \(S_{\sqrt{2(n-1)\tau^* + d_0}}^\tau \subset \Omega_{\tau^*,a^h}\) hence, \(d_0 < C\sigma^{1/2}\). In all, we always have
\[(3.50) \quad d_0 < C\sigma^{1/2}.
\]

Finally, we prove by contradiction. Suppose, \(u(x) = \frac{|x|^2}{2(n-1)} + \varphi(x)\), and \(\varphi(x) = O(|x|)\) (we include the case when \(\frac{\varphi(x)}{|x|} \to \infty\)). Suppose \(\lim_{|x| \to \infty} \frac{\varphi(x)|x|}{|x|} > c_1 > 0\), we have there exists \(h\) very large such that
\[(3.51) \quad \frac{1}{2(n-1)} = \frac{|x|^2}{2(n-1)} + \frac{\varphi(h^{1/2}x)}{h} \geq \frac{|x|^2}{2(n-1)} + c_2|x|\sigma^{1/2}
\]
or
\[(3.52) \quad \frac{1}{2(n-1)} = \frac{|x|^2}{2(n-1)} + \frac{\varphi(h^{1/2}x)}{h} \leq \frac{|x|^2}{2(n-1)} - c_2|x|\sigma^{1/2}.
\]
Therefore, \(\delta_0 \geq c_2\sigma^{1/2}\), where \(c_2 = c_2(n, c_1)\), and \(d_0 < C\delta_0\), then we go back to case 1', which leads to a contradiction. Therefore, we have \(\varphi(x) = o(|x|)\).

4. Convexity

In this section, we will study the following equation
\[(4.1) \quad \begin{cases}
\left(\delta_{ij} - \frac{v_i v_j}{1 + |Dv|^2}\right) v_{ij} = 1 \text{ in } U \\
v = 0 \text{ on } \partial U.
\end{cases}
\]

We will prove the following result.

**Proposition 4.1.** Let \(v\) be a solution of equation (4.1) and assume that \(\partial U\) is convex, then \(v\) is a convex function.

**Proof.** Step 1. Let’s consider
\[(4.2) \quad \begin{cases}
\left(\delta_{ij} - \frac{v_i^\epsilon v_j^\epsilon}{1 + |Dv|^2}\right) v_{ij}^\epsilon + 1 - \epsilon v^\epsilon = 0 \text{ in } U \\
v^\epsilon = 0 \text{ on } \partial U.
\end{cases}
\]
We will show that the solution to equation (4.2) are concave for any \( \epsilon > 0 \) small instead. Let

\[
C^\lambda(x, y) = v^\epsilon(z) - \lambda v^\epsilon(x) - (1 - \lambda)v^\epsilon(y),
\]

for \( x, y \in \bar{U}, \lambda \in [0, 1] \) and \( z = \lambda x + (1 - \lambda)y \). By Kennington’s concavity maximum principle (Theorem 3.1 [3]) we know that the negative infimum is not attained in \( U \times U \).

Step 2. If \( v^\epsilon \) is not concave, then \( C^\lambda(x, y) \) has a negative minimum on \( \partial(U \times U) \). Assume \( \inf C^\lambda(x, y) = C^\lambda(x_0, y_0) \) and not both \( x_0, y_0 \) are interior points. Without loss of generality, we assume \( x_0 \in \partial U, y_0 \in U \), and \( z_0 = \lambda x_0 + (1 - \lambda)y_0 \). Then we would have

\[
\frac{\partial C^\lambda}{\partial \lambda} = (x_0 - y_0) \cdot Dv^\epsilon(z_0) + v^\epsilon(y_0) = 0,
\]

which implies

\[
(x_0 - y_0) \cdot Dv^\epsilon(z_0) < 0,
\]

leads to a contradiction. Thus, \( \inf_{\partial(U \times U)} C^\lambda(x_0, y_0) \) is achieved when \( x_0 = y_0 = z_0 \in \partial U \). We conclude that

\[
C^\lambda(x_0, y_0) \geq 0 \text{ on } \bar{U} \times \bar{U},
\]

therefore, \( v^\epsilon \) is concave.

Step 3. Next, we want to show that there is a subsequence which we still denote by \( \{v^\epsilon\} \), such that \( v^\epsilon \to -v \) as \( \epsilon \to 0 \) uniformly on \( \bar{U} \), where \( v \) is the solution of equation (4.1). Choose a large ball \( B(p, R) \) containing \( U \) with \( \text{dist}(p, U) \geq R/2 \), and we also assume that \( p \) is the origin. Set \( w = 2R^2 - |x|^2 \), then we have

\[
\left( \delta_{ij} - \frac{w_i w_j}{1 + |Dw|^2} \right) w_{ij} + 1 - \epsilon w
= \left( \delta_{ij} - \frac{4x_i x_j}{1 + 4|x|^2} \right) (-2\delta_{ij}) + 1 - \epsilon(2R^2 - |x|^2)
\]

\[
= -2n + \frac{8|x|^2}{1 + 4|x|^2} + 1 - \epsilon(2R^2 - |x|^2)
< -2n + 3 < 0
\]

when \( \epsilon > 0 \) small. Since \( w > 0 \) on \( \partial U \), by the maximum principle we have \( 2R^2 \geq w > v^\epsilon > 0 \) in \( U \). Since \( U \) is convex, the higher order estimates can be obtained by standard arguments. Therefore, there exists a subsequence \( v^\epsilon \to -v \) as \( \epsilon \to 0 \) uniformly in \( \bar{U} \). \( \square \)

Combine Proposition 4.1 with Theorem 1.3 we get:
Corollary 4.2. Let $u$ be an entire solution of equation (1.1). Moreover, assume $\Sigma = \{(x, u(x))| x \in \mathbb{R}^n\}$ is asymptotic to the cone $C = \{(x, \frac{1}{2(n-1)} |x|^2)| x \in \mathbb{R}^n\}$ in a $C^2$ topology. Then

$$u = \frac{1}{2(n-1)} |x|^2 + \varphi(x)$$

where

$$|\varphi(x)| = o(|x|) \text{ as } |x| \to \infty.$$