Math 250 B6 Practice Problems for Exam 2 6/18/03

You must show your work for full credit. Good luck!

1) Use LU-factorization to solve the system $\begin{array}{rrrrl} x_1 & +x_2 & -x_3 & +2x_4 & = 1 \\ -x_1 & +2x_2 & +x_3 & +x_4 & = 2 \\ -2x_1 & +x_2 & +2x_3 & -x_4 & = 1 \end{array}$.

2) Let $A = \begin{bmatrix} -1 & 3 & -2 & 4 & -3 \\ 2 & -6 & 1 & 1 & 6 \\ 1 & -3 & 2 & -4 & 4 \\ 0 & 0 & 1 & -3 & 3 \end{bmatrix}$.

(a) What is $\dim(\text{Col}A)$, $\dim(\text{Row}A)$, $\dim(\text{Null}A)$? (Find the answer without finding the bases.)

(b) Give a basis for Col$A$, Row$A$, and Null$A$.

(c) What is $\dim(\text{Null}A^T)$?

3) Compute the determinant of the following matrices. You should be able to do it by column and row cofactor expansions and by doing elementary row operations first to get an upper triangular matrix.

(a) $\begin{bmatrix} -1 & 2 & 3 & 1 & 0 \\ 3 & -1 & 0 & 0 & -4 \\ 0 & 8 & 0 & 0 & 0 \\ 2 & -5 & 2 & 3 & -2 \\ 0 & 4 & -3 & 2 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 2 & 4 & 2 \\ -3 & 2 & 3 & -1 \\ 6 & 3 & 9 & 2 \\ -9 & 6 & 12 & 3 \end{bmatrix}$

4) Determine if the following subsets of $\mathcal{R}^4$ are subspaces:

(a) $\left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} : u_1^2 = u_3, u_2 = 0, u_4 = 0 \right\}$

(b) $\left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} : 2u_1 = 3u_3, u_2 = 0, u_4 = 0 \right\}$

(c) $\left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} : u_1 + u_2 = 2u_3 + u_4 \right\}$

5) (a) Show that $\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \\ -2 \end{bmatrix} \right\}$ is a basis for the null space of the matris

$\begin{bmatrix} 1 & -2 & 1 & -3 \\ -2 & 3 & -3 & 2 \end{bmatrix}$.

(b) Show that $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \\ -8 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 5 \\ -9 \end{bmatrix} \right\}$ is a basis for the subspace

$$\text{Span}\left\{ \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 3 \\ -7 \end{bmatrix} \right\}.$$

6) Find all (real) eigenvalues and bases for the corresponding eigenspaces for the following matrices:

(a) $\begin{bmatrix} 3 & 2 & 4 \\ -1 & 0 & -1 \\ 3 & 2 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

7) Find all (complex) eigenvalues and bases for the corresponding eigenspaces for the matrix $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$.

8) For the following matrices, find a (real) invertible matrix P and a (real) diagonal matrix D such that $A = PDP^{-1}$, if they exist. If they do not exist, state why.

(a) $\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 2 & -1 & -2 \end{bmatrix}$

9) True or False (Justify it if it's true, find a counterexample if it's false):
(a) For any square matrix A and any scalar c, $\det(cA)=c\det A$.
(b) For any square matrices A and B, $\det(A+B)=\det A+\det B$.
(c) For any square matrix A, $\det A^T = -\det A$.
(d) For any square matrix A, if $A^3 = 0$ then $\det A = 0$.
(e) The null space of an $m \times n$ matrix is a subspace of $\mathcal{R}^n$.
(f) The row space of an $m \times n$ matrix is a subspace of $\mathcal{R}^m$.
(g) Every subspace of $\mathcal{R}^n$ has a unique basis.
(h) For any matrix A, $\text{Row}A = \text{Col}A^T$.
(i) If U and V are both subspaces of $\mathcal{R}^n$, then so is $U \cup V$.
(j) There is only one subspace of $\mathcal{R}^n$ with dimension n.
(k) Every diagonalizable $n \times n$ matrix has n distinct eigenvalues.
(l) If λ is an eigenvalue of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ is the nullity of $A - \lambda I_n$.
(m) Every matrix has an LU-decomposition.
(n) Every linearly independent subset of a subspace of V of $\mathcal{R}^n$ is contained in a basis of V.

(o) The multiplicity of an eigenvalue is equal to the dimension of the corresponding eigenspace.
(p) A scalar λ is an eigenvalue of an $n \times n$ matrix A if $\det(A - \lambda I_n) = 0$.
(q) If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

10) Prove that if A is diagonalizable, then so is A^T.

11) Suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for a 3-dimensional subspace V of $\mathcal{R}^n$. Prove that $\{\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{w}, \mathbf{v} + \mathbf{w}\}$ is also a basis for V.

12) Prove that if A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.

13) Prove that if A is an invertible, diagonalizable matrix, then A^{-1} is diagonalizable.

14) Does the zero subspace have a basis? Justify your answer.

15) Let R be the reduced row echelon form of a matrix A. Prove that $\text{Row}A = \text{Row}R$.