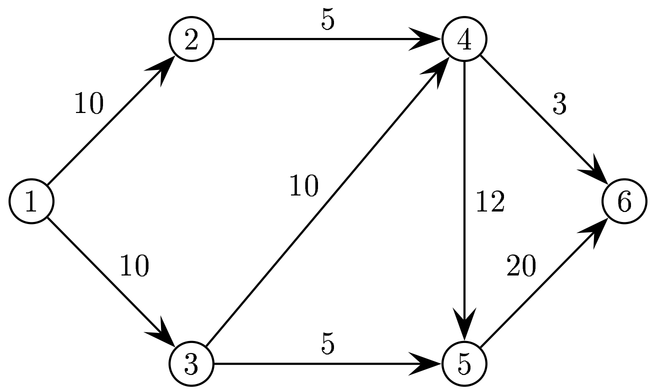
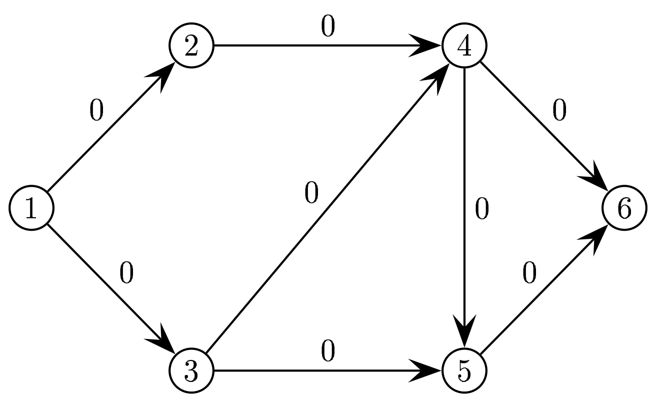
Final Review Solutions

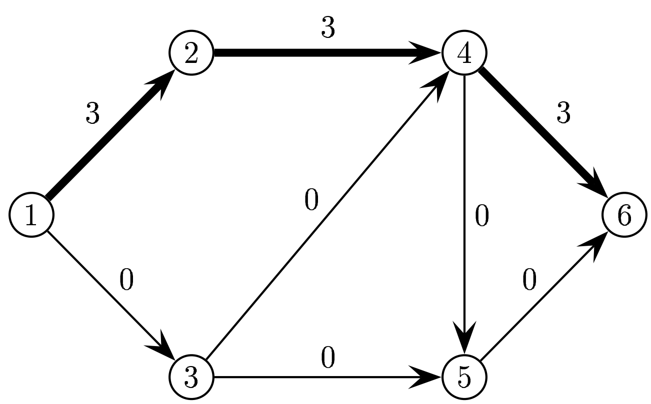
1 Find a maximal flow from the source (vertex 1) to the sink (vertex 6) in the following network using the Ford-Fulkerson Algorithm. Prove that your flow is optimal by exhibiting a cut with the same capacity.



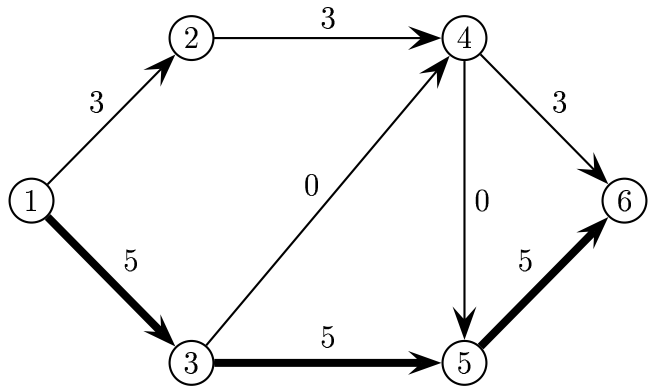
Answer: Let's start with the flow. At the beginning we are piping nothing through:



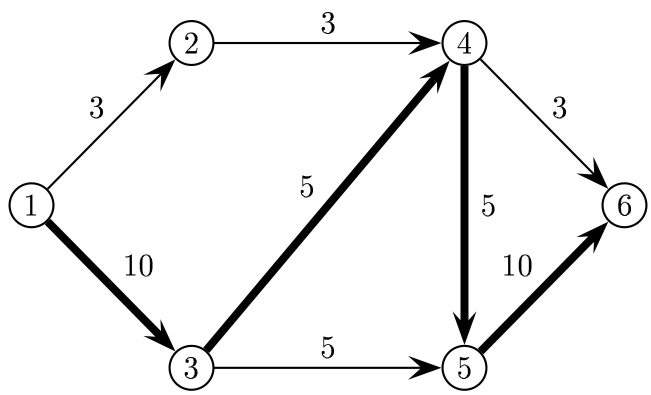
Let's add the augmenting path $1 \to 2 \to 4 \to 6$, which we can pipe 3 down:



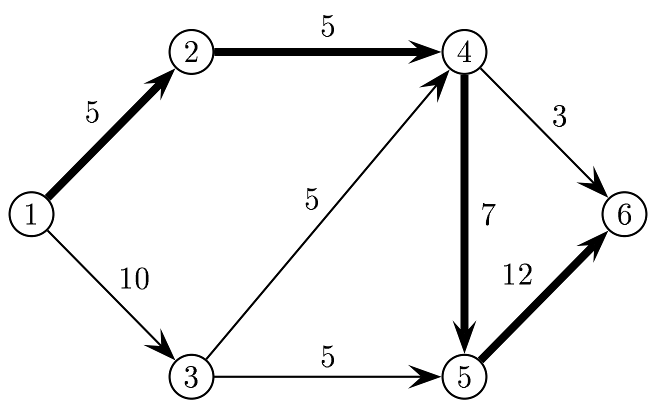
Now we can add the augmenting path $1 \to 3 \to 5 \to 6$, which we can send 5 units down.



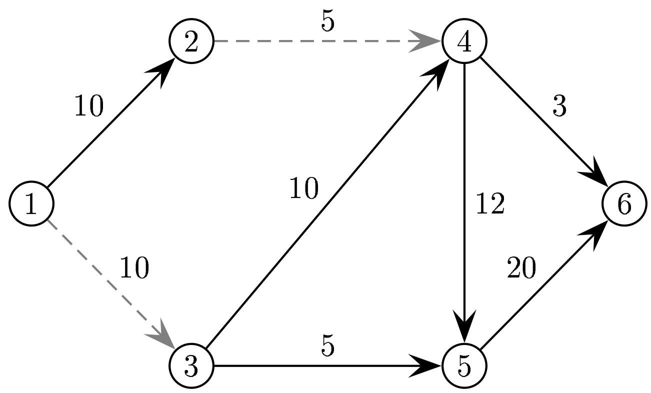
We can also send 5 units down the path $1 \to 3 \to 4 \to 5 \to 6$ to get the flow



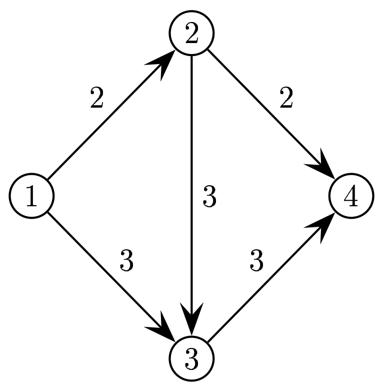
Then we can send 2 units down the path $1 \to 2 \to 4 \to 5 \to 6$ to get



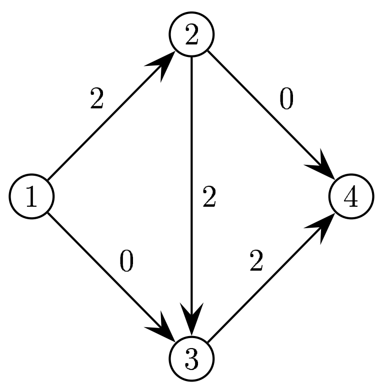
This flow gets 15 units of material from 1 to 6. It is indeed a maximal flow, because there is a cut of capacity 15:



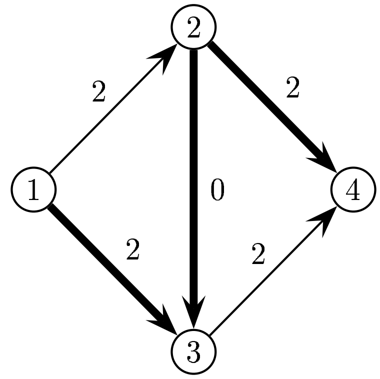
2 Consider the following network with source 1 and sink 4.



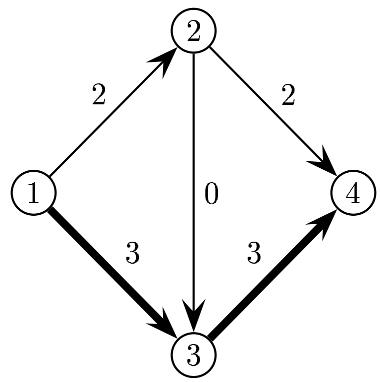
Extend the following flow to an optimal flow using the Ford-Fulkerson Algorithm. Then prove that you have achieved an optimal flow by exhibiting a cut with the same capacity.



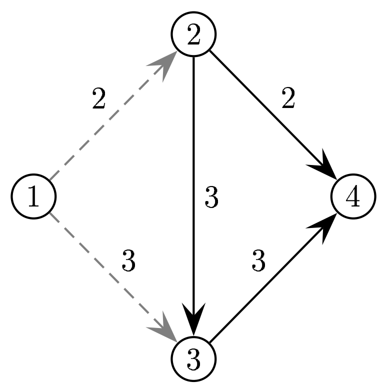
Answer: Let's add the augmenting path $1 \to 3 \to 2 \to 4$. We can send 3 units from 1 to 3, 2 units from 3 to 2 (since we are traveling that arc in the "wrong" direction, we can only send as many units as are flowing in the "right" direction), and 2 units from 2 to 4. So we can send 2 units down this path, which gives us



We can now add the augmenting path $1 \to 3 \to 4$, which we can ship 1 unit down.

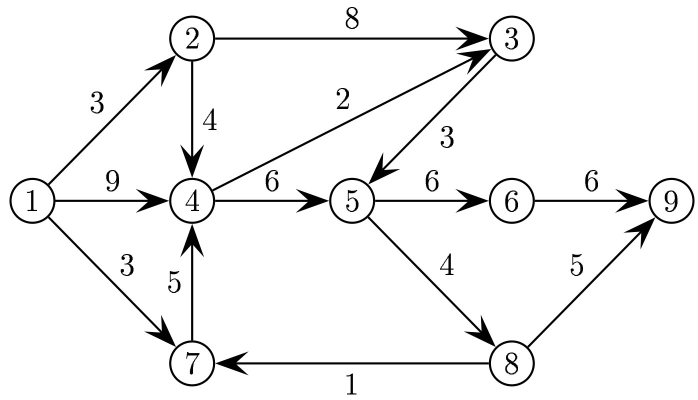


This gives a flow of 5 units, which is maximal because of the cut



3 Use the algorithm from Section 5.5 of your book (which is also the algorithm I presented in

class) to compute the shortest path between vertex 1 and vertex 9 in the following digraph.



Answer: First we set up the initial chart that lists all the arcs we have available, followed by the length of that arc. So for example, the arc from 5 to 8 has length 4 and will be listed under the 5 column as $8 - 4$.

1	2	3	4	5	6	7	8	9
$2-3$	$3-8$	$5-3$	$3-2$	$6-6$	$9-6$	$4-5$	$7-1$	
$4-9$	$4-4$		$5-6$	$8-4$			$9-5$	
$7-3$								

Now we choose the shortest arc(s) available to us from node 1. Since there is a tie, we will choose to follow both arcs $1 \to 2$ and $1 \to 7$. We circle these arcs and then right down in the 2 and 7 columns that we can reach these two nodes in 3, and we cross out all the other arcs going to 2 or 7 (in this case just the arc from 8 to 7):

1	$2-3$	3	4	5	6	$7-3$	8	9
($2-3$)	$3-8$	$5-3$	$3-2$	$6-6$	$9-6$	$4-5$	$7-1$	
$4-9$	$4-4$		$5-6$	$8-4$			$9-5$	
($7-3$)								

Now we consider our options again. We look at where we can go from every node we have reached, which right now means from 1, 2, or 7. We can go to node 4 in 9 steps, node 3 in 11 steps, node 4 in 7 steps (through node 2), or node 4 in 8 steps (through node 7). We choose the shortest option, which is node 4 in 7 steps through node 2, and update the chart:

1	$2-3$	3	$4-7$	5	6	$7-3$	8	9
($2-3$)	$3-8$	$5-3$	$3-2$	$6-6$	$9-6$	$4-5$	$7-1$	
$4-9$	($4-4$)		$5-6$	$8-4$			$9-5$	
($7-3$)								

We once again consider our options and find that the shortest option is to go to node 3 in 9 steps through node 4, which gives the following chart:

1	2 − 3	3 − 9	4 − 7	5	6	7 − 3	8	9
2 − 3	3 − 8	5 − 3	**3 − 2**	6 − 6	9 − 6	4 − 5	7 − 1	
4 − 9	**4 − 4**		5 − 6	8 − 4			9 − 5	
7 − 3								

Our shortest option now is node 5 in 12 steps:

1	2 − 3	3 − 9	4 − 7	5 − 12	6	7 − 3	8	9
2 − 3	3 − 8	**5 − 3**	**3 − 2**	6 − 6	9 − 6	4 − 5	7 − 1	
4 − 9	**4 − 4**		5 − 6	8 − 4			9 − 5	
7 − 3								

Then node 8 in 16 steps:

1	2 − 3	3 − 9	4 − 7	5 − 12	6	7 − 3	8 − 16	9
2 − 3	3 − 8	**5 − 3**	**3 − 2**	6 − 6	9 − 6	4 − 5	7 − 1	
4 − 9	**4 − 4**		5 − 6	**8 − 4**			9 − 5	
7 − 3								

Then node 6 in 18 steps:

1	2 − 3	3 − 9	4 − 7	5 − 12	6 − 18	7 − 3	8 − 16	9
2 − 3	3 − 8	**5 − 3**	**3 − 2**	**6 − 6**	9 − 6	4 − 5	7 − 1	
4 − 9	**4 − 4**		5 − 6	**8 − 4**			9 − 5	
7 − 3								

And finally, we go to node 9 in 21 steps:

1	2 − 3	3 − 9	4 − 7	5 − 12	6 − 18	7 − 3	8 − 16	9 − 21
2 − 3	3 − 8	**5 − 3**	**3 − 2**	**6 − 6**	9 − 6	4 − 5	7 − 1	
4 − 9	**4 − 4**		5 − 6	**8 − 4**			**9 − 5**	
7 − 3								

We can read this chart backwards to find that our shortest route was

$$1 \to 2 \to 4 \to 3 \to 5 \to 8 \to 9.$$

4 Use the method we learned to solve the assignment problem to solve the following integer programming problem. Be careful to note that this is a maximization problem.

The Rutgers football program needs to assign positions to four new players. Each can play quarterback (QB), tight-end (TE), wide receiver (WR), and cornerback (CB) moderately well, but the team can have only one new player at each of these positions. The chart below shows the value that each player will add to each position.

	QB	TE	WR	CB
Ben	5	8	4	4
Bob	3	3	4	3
Jim	4	5	3	5
Ross	3	3	5	6

How should the team assign positions to these four players in order to maximize the total value added?

Answer: First we need to convert this problem to a minimization problem. Doing so multiplies our cost matrix by -1 to give

$$\begin{bmatrix} -5 & -8 & -4 & -4 \\ -3 & -3 & -4 & -3 \\ -4 & -5 & -3 & -5 \\ -3 & -3 & -5 & -6 \end{bmatrix}$$

Since this matrix has negative entries, we add the absolute value of the most negative entry to everything in the matrix to get

$$\begin{bmatrix} -5 & -8 & -4 & -4 \\ -3 & -3 & -4 & -3 \\ -4 & -5 & -3 & -5 \\ -3 & -3 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 4 & 4 \\ 5 & 5 & 4 & 5 \\ 4 & 3 & 5 & 3 \\ 5 & 5 & 3 & 2 \end{bmatrix}$$

Now we subtract the smallest entry in each column from everything in that column to get

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 5 & 1 & 3 \\ 1 & 3 & 2 & 1 \\ 2 & 5 & 0 & 0 \end{bmatrix}$$

And we do the same to the rows:

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 4 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix}$$

This gives us the following assignment:

$$\begin{bmatrix} 0 & \textcircled{0} & 1 & 2 \\ 1 & 4 & \textcircled{0} & 2 \\ \textcircled{0} & 2 & 1 & 0 \\ 2 & 5 & 0 & \textcircled{0} \end{bmatrix}$$

5 Consider the following linear programming problem.

$$\begin{array}{l} \text{Maximize } z = 4x_1 + 6x_2 + 2x_3 \\ \text{subject to} \\ \begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & \leq & 10 \\ x_1 & + & 4x_2 & & & \leq & 15 \\ x_1 & & & + & x_3 & \leq & 6 \\ x_1, & & x_2, & & x_3 & \geq & 0 \end{array} \end{array}$$

Applying the Simplex Method to this problem yields the following final tableau

		4	6	2	0	0	0	
$\underset{\sim}{c}_B$		x_1	x_2	x_3	u_1	u_2	u_3	
0	u_1	0	0	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{7}{4}$
6	x_2	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{9}{4}$
4	x_1	1	0	1	0	0	1	6
		0	0	$\frac{1}{2}$	0	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{75}{2}$

This tableau represents the optimal solution

$$\underset{\sim}{x} = \begin{bmatrix} 6 \\ \frac{9}{4} \\ 0 \end{bmatrix}$$

Suppose now that the problem is changed to "Maximize $z = (4+\Delta)x_1 + 6x_2 + 2x_3$" for some real number Δ. For what range of Δ is

$$\underset{\sim}{x} = \begin{bmatrix} 6 \\ \frac{9}{4} \\ 0 \end{bmatrix}$$

still an optimal solution for the new problem? (Hint: you may want to try a specific value or two for Δ first, to get a feel for what I'm asking.)

Answer: We are asked to replace the cost for x_1 by $4+\Delta$. Doing so requires us to recompute the bottom row of the tableau as usual. (We've done several problems like this before, but never with a variable Δ.) When we're done we get the following tableau.

		$4+\Delta$	6	2	0	0	0	
$\underline{c}_B$		x_1	x_2	x_3	u_1	u_2	u_3	
0	u_1	0	0	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{7}{4}$
6	x_2	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{9}{4}$
$4+\Delta$	x_1	1	0	1	0	0	1	6
		0	0	$\frac{1}{2}+\Delta$	0	$\frac{3}{2}$	$\frac{5}{2}+\Delta$	$\frac{75}{2}+6\Delta$

Our old solution, $\begin{bmatrix} 6 & \frac{9}{4} & 0 \end{bmatrix}^T$, will remain optimal for this new problem if we don't have to apply the Simplex Method to the tableau above. So we need the objective row of this tableau to remain nonnegative. That means that we need

$$\frac{1}{2}+\Delta \geq 0$$

and

$$\frac{5}{2}+\Delta \geq 0.$$

Therefore the range of Δ for which our old solution remains optimal is $\Delta \geq -1/2$.

6 Given that

$$\underline{x} = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$$

is a solution to the primal problem

$$\begin{array}{l} \text{Maximize } z = x_1 + 4x_2 + 5x_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} -x_1 & + & x_2 & + & x_3 & \leq & 4 \\ 3x_1 & + & x_2 & + & x_3 & \leq & 16 \\ & & x_2 & & & \geq & 1 \\ x_1, & & x_2, & & x_3 & \geq & 0 \end{array} \end{array}$$

use the Principle of Complementary Slackness to find a solution to the dual problem

$$\begin{array}{l} \text{Minimize } z = 4w_1 + 16w_2 - w_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} -w_1 & + & 3w_2 & & & \geq & 1 \\ w_1 & + & w_2 & - & w_3 & \geq & 4 \\ w_1 & + & w_2 & & & \geq & 5 \\ w_1, & & w_2, & & w_3 & \geq & 0 \end{array} \end{array}$$

Answer: First we plug our solution to the primal problem into those constraints to check for slack:

$$\begin{array}{rcccccc} -3 & + & 1 & + & 6 & = & 4 \\ 3(3) & + & 1 & + & 6 & = & 16 \\ & & 1 & & & = & 1 \end{array}$$

We didn't find any slack, so the Principle of Complementary Slackness doesn't tell us anything yet. But notice that every entry in our primal problem solution is non-zero, so Complementary Slackness does tell us that there can't be any slack in any of the constraints in the dual problem. So if w_1, w_2, and w_3 represent an optimal solution to the dual problem, we must have

$$\begin{array}{rcccccc} -w_1 & + & 3w_2 & & & = & 1 \\ w_1 & + & w_2 & - & w_3 & = & 4 \\ w_1 & + & w_2 & & & = & 5 \end{array}$$

One could solve this using Linear Algebra, but I'll do it the ad hoc way. From the second and third constraints, $w_3 = 1$. Add the first and third constraints together, we see that $4w_2 = 6$, so $w_2 = 3/2$. Then w_1 must be $7/2$. Our solution is therefore

$$\begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}.$$

7 Consider the following linear programming problem.

$$\begin{array}{l} \text{Maximize } z = 3x_1 - x_2 + 2x_3 + 4x_4 \\ \text{subject to} \\ \begin{array}{rcccccccc} & & x_2 & + & 7x_3 & + & 2x_4 & \geq & 3, \\ x_1 & + & 2x_2 & + & x_3 & & & = & 9, \\ 2x_1 & + & 3x_2 & + & x_3 & - & 4x_4 & \leq & 7, \end{array} \\ x_j \geq 0,\ j = 1, 2, 3, 4. \end{array}$$

When solving it, the following tableau came up

	x_1	x_2	x_3	x_4	u_1	u_2	
x_4	0	$\frac{1}{4}$	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	$\frac{11}{4}$
u_1	0	$-\frac{1}{2}$	$-\frac{13}{2}$	0	1	$-\frac{1}{2}$	$\frac{5}{2}$
x_1	1	2	1	0	0	0	9
	0	8	2	0	0	16	38

Could it be correct? Explain.

Answer: No. The trick is to recompute the objective row using the rest of the tableau. The mistake is in the u_2 column. In the tableau, this entry is 16, while it should be

$$\begin{bmatrix} 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{2} \\ 0 \end{bmatrix} - 0 = -1.$$