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Solutions to review problems for Midterm #1

First: Midterm #1 covers Chapter 1 and 2. In particular, this means that it does not explicitly cover linear algebra. Also, I promise there will not be any proofs.

1 Consider the following scenario. Model it as a linear programming problem. Be sure to state explicitly what each of your decision variables $x_1, x_2, \ldots$ represent. *Do not attempt to solve the LPP.*
A cattle rancher uses three types of cattle feed: type 1, type 2, and type 3. Type 1 costs \$1.50 per pound, type 2 costs \$3.50 per pound, and type 3 costs \$2.00 per pound. The rancher wants to meet the following minimum daily requirements for each animal. Each day, each animal should have at least 120 mg of vitamin A, 180 mg of vitamin B, and 100 mg of vitamin C.

The following chart shows the number of mg per pound of each vitamin in the three types of feed:

Vitamin	Type 1	Type 2	Type 3
A	8	2	20
B	9	11	5
C	1	10	20

Because of protein content, however, an animal cannot eat more than 15 pounds of type 1, 10 pounds of type 2, and 5 pounds of type 3 cattle feed per day. How many pounds of each type of cattle feed should the rancher purchase per day in order to minimize the cost, while still meeting the nutritional minimum daily requirements?

Answer: Letting

$$\begin{aligned} x_1 &= \text{amount of Type 1 feed,} \\ x_2 &= \text{amount of Type 2 feed,} \\ x_3 &= \text{amount of Type 3 feed,} \end{aligned}$$

the problem can be written as

$$\begin{array}{l} \text{Minimize } z = 1.50x_1 + 3.50x_2 + 2.00x_3 \\ \text{subject to} \\ \begin{array}{rrrcl} 8x_1 & +2x_2 & +20x_3 & \geq & 120, \\ 9x_1 & +11x_2 & +5x_3 & \geq & 180, \\ x_1 & +10x_2 & +20x_3 & \geq & 100, \\ x_1 & & & \leq & 15, \\ & x_2 & & \leq & 10, \\ & & x_3 & \leq & 5, \\ x_1, & x_2, & x_3 & \geq & 0. \end{array} \end{array}$$

Note that this is our answer, the problem specifically told us not to try to solve the problem.

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[2] Convert the following LPP into (a) standard form, and (b) canonical form:

$$
\begin{array}{llll}
\multicolumn{4}{l}{\text{minimize } z = x_1 - 4x_2 + 5x_3} \\
\multicolumn{4}{l}{\text{subject to}} \\
x_1 & & +x_3 & \leq 5 \\
 & x_2 & +3x_3 & \geq 7 \\
2x_1 & +9x_2 & & = 11 \\
 & & & \\
x_1, & x_2 & & \geq 0 \\
 & & x_3 & \text{unconstrained}
\end{array}
$$

Answer: To convert a problem to standard form, we generally have to do three things:

1. Make the problem a maximization problem,
2. Make every constraint except for constraints of the form $x_j \geq 0$ involve a $\leq$,
3. Make every variable constrained to be non-negative.

To do that in this example we have to do quite a bit of work. The final answer is:

$$
\begin{array}{llllll}
\multicolumn{6}{l}{\text{maximize } z = -x_1 + 4x_2 - 5x_3} \\
\multicolumn{6}{l}{\text{subject to}} \\
x_1 & & +x_3^+ & -x_3^- & \leq & 5 \\
 & -x_2 & -3x_3 & +3x_3^- & \leq & -7 \\
2x_1 & +9x_2 & & & \leq & 11 \\
-2x_1 & -9x_2 & & & \leq & -11 \\
 & & & & & \\
x_1, & x_2, & x_3^+, & x_3^- & \geq & 0
\end{array}
$$

The answer for canonical form is:

$$
\begin{array}{llllllll}
\multicolumn{8}{l}{\text{maximize } z = -x_1 + 4x_2 - 5x_3} \\
\multicolumn{8}{l}{\text{subject to}} \\
x_1 & & +x_3^+ & -x_3^- & +u & & = & 5 \\
 & -x_2 & -3x_3 & +3x_3^- & & +v & = & -7 \\
2x_1 & +9x_2 & & & & & = & 11 \\
 & & & & & & & \\
x_1, & x_2, & x_3^+, & x_3^-, & u, & v & \geq & 0
\end{array}
$$

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3 Consider the LPP

$$\begin{array}{lrrcl} \text{maximize } z = 2x_1 + 4x_2 & & & & \\ \text{subject to} & & & & \\ 5x_1 & +3x^2 & +5x_3 & \leq & 15 \\ 10x_1 & +8x_2 & +15x_3 & \leq & 40 \\ & & & & \\ x_1, & x_2 & x_3 & \geq & 0 \end{array}$$

First, sketch the set of feasible solutions. Then find the extreme points of this set. Then find the optimal solution(s).

"Answer:" This is a hard problem to sketch, so I won't do that here. There **will** be an Extreme Point Theorem problem on the midterm, but it will only have two variables. I will post another example problem on the website this weekend. This problem is actually not that hard, so give it a try.

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4 Find an optimal solution to the following LPP using the simplex method.

$$\begin{array}{l} \text{maximize } z = x_1 + 3x_2 + 5x_3 \\ \text{subject to} \\ \begin{array}{rrrcl} 2x_1 & -5x_2 & +x_3 & \leq & 3 \\ x_1 & +4x_2 & & \leq & 5 \\ \\ x_1, & x_2, & x_3 & \geq & 0 \end{array} \end{array}$$

Answer: First we have to add slack variables to both inequalities. Then our initial tableau will be

	x_1	x_2	x_3	u_1	u_2	
u_1	2	-5	1	1	0	3
u_2	1	4	0	0	1	5
	-1	-3	-5	0	0	0

First iteration: Entering x_3, departing u_1:

	x_1	x_2	x_3	u_1	u_2	
x_3	2	-5	1	1	0	3
u_2	1	4	0	0	1	5
	9	-28	0	5	0	15

Second iteration: Entering x_2 and departing u_2:

	x_1	x_2	x_3	u_1	u_2	
x_3	13/4	0	1	1	5/4	37/4
x_2	1/4	1	0	0	1/4	5/4
	16	0	0	5	7	50

Therefore the optimal solution is $x_1 = 0, x_2 = 5/4, x_3 = 37/4, x_4 = 0$, and $z = 50$ is the best we can do.

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5 Find an optimal solution to the following LPP using the two-phase simplex method.

$$\begin{array}{l} \text{maximize } z = -x_1 - 2x_2 \\ \text{subject to} \\ \begin{array}{rrrrcr} x_1 & +2x_2 & -x_3 & & = & 3 \\ 3x_1 & +4x_2 & & -x_4 & = & 10 \\ \\ x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array} \end{array}$$

Answer: We have to add artificial variables y_1 and y_2 to the two constraints, so our auxiliary problem is:

$$\begin{array}{l} \text{maximize } z' = -y_1 - y_2 \\ \text{subject to} \\ \begin{array}{rrrrrrcr} x_1 & +2x_2 & -x_3 & & +y_1 & & = & 3 \\ 3x_1 & +4x_2 & & -x_4 & & +y_2 & = & 10 \\ \\ x_1, & x_2, & x_3, & x_4, & y_1, & y_2 & \geq & 0 \end{array} \end{array}$$

This corresponds to the tableau

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	1	2	-1	0	1	0	3
y_2	3	4	0	-1	0	1	10
	0	0	0	0	1	1	0

But before we get started, we have to get 0s in the objective row in the y_1 and y_2 columns. (As we discussed in class, our book does this a little differently, but I think this is easier.) So, we subtract the first and second rows from the third row to obtain the following initial tableau:

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	1	2	-1	0	1	0	3
y_2	3	4	0	-1	0	1	10
	-4	-6	1	1	0	0	-13

Our entering variable will be x_2, and the θ-ratios are 3 and 10/3, respectively, so the departing variable is y_1. After pivoting, this gives the following tableau:

	x_1	x_2	x_3	x_4	y_1	y_2	
x_2	$1/2$	1	$-1/2$	0	$1/2$	0	$3/2$
y_2	1	0	-2	-1	-2	1	4
	-1	0	-2	1	3	0	-4

So our next entering variable will be x_3, and since the θ-ratios are 3 and 4, respectively, the departing variable is y_2. After pivoting, we get

	x_1	x_2	x_3	x_4	y_1	y_2	
x_2	$3/4$	1	0	$-1/4$	0	$1/4$	$5/2$
x_3	$1/2$	0	1	$-1/2$	-1	$1/2$	2
	0	0	0	0	1	1	0

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Important: Remember that the purpose of the first phase of the two-phase method is just to get a basic feasible solution so that we can use the simplex method on the original problem. This can be checked!

For example, here we got $x_1 = 0, x_2 = 5/2, x_3 = 2, x_4 = 0$. We can plug these into our constraints to check the answer:

$$\begin{array}{rrrrcr} 0 & +2(5/2) & -2 & +0 & = & 3 \\ 3(0) & +4(5/2) & -0 & & = & 10 \end{array}$$

Now we are ready to begin the last phase. For this we do the following three things to the final tableau from the first phase:

1. Forget about the columns corresponding to the artificial variables y_1 and y_2,
2. Replace the objective row by the objective row coming from our original problem,
3. Clean up the objective row to get 0s in the places they are needed.

Doing the first two of these in our problem gives the following tableau:

	x_1	x_2	x_3	x_4	
x_2	3/4	1	0	−1/4	5/2
x_3	1/2	0	1	−1/2	2
	1	2	0	0	0

So now we have to get 0s in the x_2 and x_3 columns of the new objective row to complete step three. We do this by subtracting 2 times the first row from the objective row to get our initial tableau

	x_1	x_2	x_3	x_4	
x_2	3/4	1	0	−1/4	5/2
x_3	1/2	0	1	−1/2	2
	−1/2	0	0	1/2	−5

The entering variable is x_1, the departing variable is x_2, and after one pivot we already have our final tableau:

	x_1	x_2	x_3	x_4	
x_1	1	4/3	0	−1/3	10/3
x_3	0	−2/3	1	−1/3	1/3
	0	2/3	0	1/3	−10/3

This gives an optimal solution of $x_1 = 10/3, x_2 = 0, x_3 = 1/3$, and $x_4 = 0$, with $z = -10/3$ being the best we could do. I have several people ask if z could be negative. Yes! Look at what we were trying to maximize: $-x_1 - 2x_2$. Since our variables have to be non-negative, this function will always be negative.