Math 354 Spring 2005

Review problems for Midterm #2

What to study: Finding duals (3.1), the Duality Theorem and other theorems from 3.2, how to convert the solution of the primal problem to the solution of the dual problem (3.3), the Dual Simplex Method (3.4), Sensitivity Analysis (3.6), writing down integer programming problems (4.1), cutting plane methods (4.2).

PLEASE NOTE: The collection of problems below is a rough guide only. It is meant to assist by giving example problems from the sections above. You are expected to understand the concepts as well as the exercises. The list of sections above is complete.

1 Find the dual of the following linear programming problem.

$$\begin{array}{l} \text{Maximize } z = 3x_1 + x_2 \\ \text{subject to} \\ \begin{array}{rcrcl} x_1 & + & 3x_2 & \geq & 10 \\ 3x_1 & + & 2x_2 & = & 20 \\ x_1 & & & \leq & 5 \\ x_1, & & x_2 & \geq & 0 \end{array} \end{array}$$

2 Consider the following linear programming problem

$$\begin{array}{l} \text{Maximize } z = 15x_1 + 4x_2 \\ \text{subject to} \\ \begin{array}{rcrcl} 5x_1 & + & 2x_2 & \leq & 10 \\ x_1 & & & \leq & \frac{3}{2} \\ 4x_1 & + & 4x_2 & \leq & 14 \\ x_1, & & x_2 & \geq & 0 \end{array} \end{array}$$

(a) Find the dual of this problem.

(b) Use the fact that $\left[\frac{3}{2}\ \frac{5}{4}\right]^T$ is an optimal solution to the primal problem and the Principle of Complementary Slackness to find an optimal solution to the dual problem. (No points will be given for solving the dual problem by any other method.)

(c) Without referring to (b), but using the fact that $\left[\frac{3}{2}\ \frac{5}{4}\right]^T$ is an optimal solution to the primal problem, find an optimal solution to the dual problem. To do this construct the tableau that represents this optimal solution. Use this tableau to give the optimal solution to the dual problem.

3 Consider the following primal problem

$$\begin{array}{l} \text{Maximize } z = x_1 + 3x_2 + 5x_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} 2x_1 & - & 5x_2 & + & x_3 & \leq & 3 \\ x_1 & + & 4x_2 & & & \leq & 5 \\ x_1, & & x_2, & & x_3 & \geq & 0 \end{array} \end{array}$$

and its dual

$$\begin{array}{rcrcl} \multicolumn{5}{l}{\text{Minimize } z' = 3w_1 + 5w_2} \\ \multicolumn{5}{l}{\text{subject to}} \\ 2w_1 & + & w_2 & \geq & 1 \\ -5w_1 & + & 4w_2 & \geq & 3 \\ w_1 & & & \geq & 5 \\ w_1, & & w_2 & \geq & 0 \end{array}$$

Show that $\underline{x} = \begin{bmatrix} 0 \\ \frac{5}{4} \\ \frac{37}{4} \end{bmatrix}$ is an optimal solution to the primal problem and that $\underline{w} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ is an optimal solution to the dual problem. (Hint: do **not** attempt to solve either problem. Instead use a theorem from section 3.2.)

4 Use the dual simplex method to restore feasibility in the following tableau.

	x_1	x_2	u_1	u_2	u_3	u_4	
x_2	0	1	-2	0	3	0	2
x_1	1	0	1	0	2	0	1
u_2	0	0	-4	1	5	0	0
u_4	0	0	1	0	-5	1	-1
	0	0	2	0	4	0	5

5 Consider again our saw mill problem, which corresponds to the linear programming problem

$$\begin{array}{rcrcl} \multicolumn{5}{l}{\text{Maximize } z = 120x_1 + 100x_2} \\ \multicolumn{5}{l}{\text{subject to}} \\ 2x_1 & + & 2x_2 & \leq & 8 \\ 5x_2 & + & 3x_2 & \leq & 15 \\ x_1, & & x_2, & \geq & 0 \end{array}$$

To solve it we introduce slack variables u_1 and u_2 to get the following initial tableau

		120	100	0	0	
$\underline{c}_B$		x_1	x_2	u_1	u_2	
0	u_1	2	2	1	0	8
0	u_2	5	3	0	1	15
		-120	-100	0	0	0

Then we do two iterations of the Simplex Method and get the following final tableau

		120	100	0	0	
$\underline{c}_B$		x_1	x_2	u_1	u_2	
100	x_2	0	1	$\frac{5}{4}$	$-\frac{1}{2}$	$\frac{5}{2}$
120	x_1	1	0	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{2}$
		0	0	35	10	430

(a) Suppose that we now decrease b_2 from 15 to 10. Find an optimal solution for this new problem. (No points will be given for starting again from scratch! The point of this problem is for you to use the techniques from section 3.6.)

(b) Now consider changes to the cost vector **c**. Use sensitivity analysis to give bounds on how much one can change c_1, c_2, c_3 and c_4 (individually) so that the optimal solution does NOT change.

(c) Now suppose that we increase c_1 from 120 to 200. Find an optimal solution for this problem. (Again, no points will be given for starting over.)

6 Suppose that there are three jobs to be assigned and we have three workers available, and so we want to match each of these three workers to exactly one of the three jobs. A point scale has been set up rating the value of assigning a particular worker to a particular job, which is presented in the following chart:

Worker	Job 1	Job 2	Job 3
1	6	9	5
2	5	5	5
3	7	3	7

Formulate, but **do not solve**, an integer linear programming problem whose solution would reveal how to assign each worker to a job in such a way to maximize the total number of value points.

7 Solve the following integer programming problem using the Cutting Plane Method.

$$\begin{array}{l} \text{Maximize } z = x + 4y \\ \text{subject to} \\ \begin{array}{rcrcl} x & + & 6y & \leq & 36 \\ 3x & + & 8y & \leq & 60 \\ x, & & y, & \geq & 0, \text{ integral.} \end{array} \end{array}$$