1 Find the dual of the following linear programming problem.

$$\begin{array}{l} \text{Maximize } z = 3x - y \\ \text{subject to} \\ \begin{array}{rcrcr} 4x & - & y & \leq & 14 \\ -x & + & 4y & \geq & 10 \\ 2x & + & 3y & = & 32 \\ x, & & y & \geq & 0 \end{array} \end{array}$$

Answer: First I would convert the problem to standard form:

$$\begin{array}{l} \text{Maximize } z = 3x - y \\ \text{subject to} \\ \begin{array}{rcrcr} 4x & - & y & \leq & 14 \\ x & - & 4y & \leq & -10 \\ 2x & + & 3y & \leq & 32 \\ -2x & - & 3y & \leq & -32 \\ x, & & y & \geq & 0 \end{array} \end{array}$$

Then the dual problem is

$$\begin{array}{l} \text{Minimize } z' = 14w_1 - 10w_2 + 32w_3 - 32w_4 \\ \text{subject to} \\ \begin{array}{rcrcrcrcr} 3w_1 & + & w_2 & + & 2w_3 & - & 2w_4 & \geq & 3 \\ -w_1 & - & 4w_2 & + & 3w_3 & - & 3w_4 & \geq & -1 \\ w_1, & & w_2, & & w_3, & & w_4 & \geq & 0 \end{array} \end{array}$$

2 Use the Dual Simplex Method to restore feasibility to the following tableau:

	x	y	u_1	u_2	u_3	
y	0	1	$\frac{3}{14}$	$-\frac{1}{14}$	0	$\frac{3}{7}$
x	1	0	$-\frac{4}{7}$	$\frac{1}{7}$	0	2
u_3	0	0	$\frac{3}{7}$	$-\frac{1}{7}$	1	$-\frac{8}{7}$
	0	0	$\frac{2}{7}$	$\frac{3}{7}$	0	$\frac{52}{7}$

Answer: The departing variable is u_3, the entering variable is u_2.

3 Consider the following linear programming problem.

$$\begin{array}{l} \text{Maximize } z = 4x_1 + 6x_2 + 2x_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & \leq & 10 \\ x_1 & + & 4x_2 & & & \leq & 15 \\ x_1 & & & + & x_3 & \leq & 6 \\ x_1, & & x_2, & & x_3 & \geq & 0 \end{array} \end{array}$$

Applying the Simplex Method to this problem yields the following final tableau

		4	6	2	0	0	0	
$\underline{c}_B$		x_1	x_2	x_3	u_1	u_2	u_3	
0	u_1	0	0	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{7}{4}$
6	x_2	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{9}{4}$
4	x_1	1	0	1	0	0	1	6
		0	0	$\frac{1}{2}$	0	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{75}{2}$

Part a. Answer There is no change to the optimal solution. You can follow the method below to see this.

Part b. Suppose now that the problem is changed to "Maximize $z = 3x_1 + 6x_2 + 2x_3$." Find an optimal solution to this new problem. (No points will be given for starting from scratch!)

Answer: First we need to update the tableau as usual to get

	x_1	x_2	x_3	u_1	u_2	u_3	
u_1	0	0	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{7}{4}$
x_2	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{9}{4}$
x_1	1	0	1	0	0	1	6
	0	0	$-\frac{1}{2}$	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{63}{2}$

After one iteration of the Simplex Method (with entering variable x_3 and departing variable x_1), we get the final tableau shown below.

	x_1	x_2	x_3	u_1	u_2	u_3	
u_1	$-\frac{1}{4}$	0	0	1	$-\frac{1}{4}$	-1	$\frac{1}{4}$
x_2	$\frac{1}{4}$	1	0	0	$\frac{1}{4}$	0	$\frac{15}{4}$
x_3	1	0	1	0	0	1	6
	$\frac{1}{2}$	0	0	0	$\frac{3}{2}$	2	$\frac{69}{2}$

4 Consider the following integer programming problem

$$\begin{array}{l} \text{Maximize } z = x + y \\ \text{subject to} \\ \begin{array}{rcrcl} 13x & + & 5y & \leq & 78 \\ -x & + & y & \leq & 0 \\ x, & & y, & \geq & 0, \text{ integral} \end{array} \end{array}$$

Solve this problem using the Cutting Plane Method. (No points will be given for any other methods!)

To help you out, here is the final tableau for the corresponding non-integer linear programming problem (that is, if we ignore the word "integral"):

	x	y	u_1	u_2	
x	1	0	$\frac{1}{18}$	$-\frac{5}{18}$	$\frac{13}{3}$
y	0	1	$\frac{1}{18}$	$\frac{13}{18}$	$\frac{13}{3}$
	0	0	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{26}{3}$

Answer: We could add cutting planes to deal with either of the two constraints, and at this point we can't tell which one would be better to add, so I will choose the first constraint,

$$x + \frac{1}{18}u_1 - \frac{5}{18}u_2 = \frac{13}{3}.$$

(It turns out that adding a cutting plane to deal with the second constraint is a little less work.) Since x, u_1, and u_2 must all be non-negative, we can round down their coefficients to get a smaller left-hand side:

$$x - u_2 \leq \frac{13}{3}.$$

Now we can round down the right-hand side because $x - u_2$ must be an integer:

$$x - u_2 \leq 4.$$

Finally, we add a new slack variable:

$$x - u_2 + u_3 = 4,$$

and place this constraint into our tableau:

	x	y	u_1	u_2	u_3	
x	1	0	$\frac{1}{18}$	$-\frac{5}{18}$	0	$\frac{13}{3}$
y	0	1	$\frac{1}{18}$	$\frac{13}{18}$	0	$\frac{13}{3}$
u_3	1	0	0	-1	1	4
	0	0	$\frac{1}{9}$	$\frac{4}{9}$	0	$\frac{26}{3}$

But before applying the Dual Simplex Method we have to clean up x's column:

	x	y	u_1	u_2	u_3	
x	1	0	$\frac{1}{18}$	$-\frac{5}{18}$	0	$\frac{13}{3}$
y	0	1	$\frac{1}{18}$	$\frac{13}{18}$	0	$\frac{13}{3}$
u_3	0	0	$-\frac{1}{18}$	$-\frac{13}{18}$	1	$-\frac{1}{3}$
	0	0	$\frac{1}{9}$	$\frac{4}{9}$	0	$\frac{26}{3}$

Now we can apply the Dual Simplex Method. The departing variable will be u_3. The ratio for u_1 is -2 and the ratio for u_2 is $-8/13$, so u_2 will be the entering variable. After pivoting we have the following tableau

	x	y	u_1	u_2	u_3	
x	1	0	$\frac{1}{13}$	0	$-\frac{5}{13}$	$\frac{58}{13}$
y	0	1	0	0	1	4
u_2	0	0	$\frac{1}{13}$	1	$-\frac{18}{13}$	$\frac{6}{13}$
	0	0	$\frac{1}{13}$	0	$\frac{8}{13}$	$\frac{110}{13}$

Unfortunately, x still isn't an integer! So, we have to add another cutting plane for the first constraint,

$$x + \frac{1}{13}u_1 - \frac{5}{13}u_3 = \frac{58}{13},$$

which becomes

$$x - u_3 + u_4 = 4$$

after rounding everything down and adding a new slack variable.
Thus our tableau is now

	x	y	u_1	u_2	u_3	u_4	
x	1	0	$\frac{1}{13}$	0	$-\frac{5}{13}$	0	$\frac{58}{13}$
y	0	1	0	0	1	0	4
u_2	0	0	$\frac{1}{13}$	1	$-\frac{18}{13}$	0	$\frac{6}{13}$
u_3	1	0	0	0	-1	1	4
	0	0	$\frac{1}{13}$	0	$\frac{8}{13}$	0	$\frac{110}{13}$

And after cleaning up the x column we get

	x	y	u_1	u_2	u_3	u_4	
x	1	0	$\frac{1}{13}$	0	$-\frac{5}{13}$	0	$\frac{58}{13}$
y	0	1	0	0	1	0	4
u_2	0	0	$\frac{1}{13}$	1	$-\frac{18}{13}$	0	$\frac{6}{13}$
u_3	0	0	$-\frac{1}{13}$	0	$-\frac{8}{13}$	0	$-\frac{6}{13}$
	0	0	$\frac{1}{13}$	0	$\frac{8}{13}$	0	$\frac{110}{13}$

Now we need to apply the Dual Simplex Method. The departing variable is u_3 and the ratios for u_1 and u_3 are the same, so technically it doesn't matter which one we take as our entering variable. Except that it actually matters quite a bit, because if we take u_1 as our entering variable we get the following tableau:

	x	y	u_1	u_2	u_3	u_4	
x	1	0	0	0	-1	0	4
y	0	1	0	0	1	0	4
u_2	0	0	0	1	-2	0	0
u_1	0	0	1	0	8	0	6
	0	0	0	0	0	0	8

Since this tableau represents the integral solution $x = 4$, $y = 4$, $z = 8$, we are done.
In case we take u_3 as our entering variable, we get the following tableau:

	x	y	u_1	u_2	u_3	u_4	
x	1	0	$\frac{1}{8}$	0	0	0	$\frac{19}{4}$
y	0	1	$-\frac{1}{8}$	0	0	0	$\frac{13}{4}$
u_2	0	0	$\frac{1}{4}$	1	0	0	$\frac{3}{2}$
u_1	0	0	$\frac{1}{8}$	0	1	0	$\frac{3}{4}$
	0	0	0	0	0	0	8

And thus we have to keep adding cutting planes.
(Also, if we had just made the first cutting plane take care of the second constraint, we would have been done almost immediately.)

5 Consider the following scenario. Model it as an integer programming problem. Be sure to state explicitly what each of your decision variables $x_1, x_2, \ldots$ represent. Do **not** attempt to solve the problem.

The Rutgers football program is attempting to create their 2006 schedule. They are allowed to play up to four non-conference games, which must be chosen from the list below. Also, they can not play the same opponent twice.

Opponent	Chance of Winning	Revenue (in thousands of $)
Army	60%	$400
Buffalo	65%	$100
Kent State	60%	$100
New Hampshire	70%	$80
Notre Dame	20%	$1,000
Michigan State	25%	$500
Missouri	20%	$450
Ohio State	15%	$850
Navy	50%	$500
Villanova	55%	$100

The Scarlet Knights, despite winning the first intercollegiate football game ever played (against Princeton in 1869 by the score of 6-4, only shortly after Princeton beat Rutgers 40-2 in baseball), have played in only one bowl game, the Garden State Bowl in 1978. To give you an idea of how bad they have been doing, Southern Methodist University has been to four bowl games since 1978.

To give Rutgers a shot to break their bowl-drought, they need a schedule for which they can expect at least two victories against their non-conference opponents. (By a principle called Linearity of Expectation, the number of games they can expect to win is the sum of the probabilities of winning each game individually, so for example, if they scheduled Army, Notre Dame, Missouri, and Ohio State, they could expect to win only $.60 + .20 + .20 + .15 = 1.15$ games.) Beyond this, they would like to maximize revenue.

Answer: First we have to decide on variables. There are 10 possible games, so our variables will be $x_1, x_2, \ldots, x_{10}$, where

$$x_i = \begin{cases} 1 & \text{if Rutgers plays game } i, \\ 0 & \text{otherwise.} \end{cases}$$

(Variables like this are often called "indicator variables.")
Now we must translate the constraints into mathematical terms. We were told that Rutgers can play at most four non-conference games, so this means that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \leq 4.$$

We were also told that Rutgers must be able to expect at least two victories from their non-conference schedule. By Linearity of Expectation, they can expect

$$.60x_1 + .65x_2 + .60x_3 + .70x_4 + .20x_5 + .25x_6 + .20x_7 + .15x_8 + .50x_9 + .55x_{10}$$

victories, so this must be at least 2.

Finally, we were told to maximize revenue. The revenue from their schedule will be

$$400x_1 + 100x_2 + 100x_3 + 80x_4 + 1000x_5 + 500x_6 + 450x_7 + 850x_8 + 500x_9 + 100x_{10}.$$

Below I have this in the usual table layout

Maximize $z = 400x_1 + 100x_2 + 100x_3 + 80x_4 + 1000x_5 + 500x_6 + 450x_7 + 850x_8 + 500x_9 + 100x_{10}$
subject to

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} &\leq 4 \\ .60x_1 + .65x_2 + .60x_3 + .70x_4 + .20x_5 + .25x_6 + .20x_7 + .15x_8 + .50x_9 + .55x_{10} &\geq 2 \\ x_i &\leq 1 \text{ for } i = 1, 2, \ldots, 10 \\ x_i \geq 0, \text{ integer for } i = 1, 2, \ldots, 10 & \end{aligned}$$

6 Consider the following primal problem

$$\begin{array}{l} \text{Maximize } z = x_1 + 4x_2 + 5x_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} -x_1 & + & x_2 & + & x_3 & \leq & 4 \\ 3x_1 & + & x_2 & + & x_3 & \leq & 16 \\ & & x_2 & & & \geq & 1 \\ x_1, & & x_2, & & x_3 & \geq & 0 \end{array} \end{array}$$

and its dual

$$\begin{array}{l} \text{Minimize } z = 4w_1 + 16w_2 - w_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} -w_1 & + & 3w_2 & & & \geq & 1 \\ w_1 & + & w_2 & - & w_3 & \geq & 4 \\ w_1 & + & w_2 & & & \geq & 5 \\ w_1, & & w_2, & & w_3 & \geq & 0 \end{array} \end{array}$$

Without using the Simplex Method on either problem, find optimal solutions to the two problems from the following list. Explain your reasoning.

$$\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} \quad \begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 15 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix} \quad \begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} \quad \begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 12 \\ 7 \end{bmatrix}$$

Answer: The Theorem we need to use says that if $\underline{x}$ is a feasible solution to the primal problem and $\underline{w}$ is a feasible solution to the dual problem and the objective values of both solutions (in their respective problems) are the same, then they are both optimal solutions.

We can start this by ruling out the two solutions with negative entries:

$$\begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 12 \\ 7 \end{bmatrix}$$

For the rest, we can make a chart as follows

possible optimal solution	objective value in primal problem	objective value in dual problem
$\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$	23	infeasible
$\begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$	infeasible	infeasible
$\begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 15 \end{bmatrix}$	infeasible	infeasible
$\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$	37	infeasible
$\begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$	$14\frac{1}{2}$	37

Note that as soon as we find the two 37s we can stop, since the Weak Duality Theorem (or is it a corollary to the Weak Duality Theorem?) now tells us that $\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$ is an optimal solution to the primal problem and $\begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$ is an optimal solution to the dual problem.