Principia Mathematica Historallis Integratus

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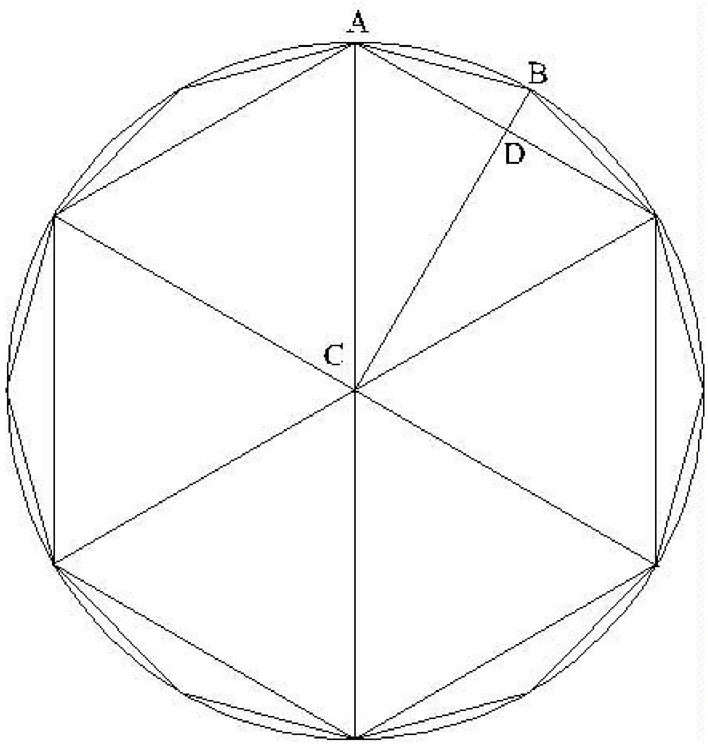
The history of the integral calculus has an interesting development. It begins with ancient Greece and winds up in nineteenth century Europe. But what is the integral calculus? The simplest topic within it is to find the area under a given curve. More advanced applications involve finding surface areas, volumes, centers of mass, etc. More rigorously, the integral involves taking the sum of a function over some infinitesimally small region. By following the evolution of these ideas from geometric to abstract, a picture how mathematical thought has progressed throughout the ages of civilization is painted.

The first civilization to systematize the study of areas was undoubtedly the Greeks. In fact, they were the first civilization to study mathematics as its own subject. Philosophers such as Plato viewed math as being higher than the real world as it was something that could only be understood through the mind. The Greek mathematicians that had the most to do with finding areas are Antiphon, Eudoxus, Euclid, and Archimedes.[3] Each of these men built on the existing knowledge and contributed something new to the field. In fact, Archimedes came tantalizingly close to developing modern calculus. And so our journey to understand the history of integration begins over 2500 years ago.

The first Greek to consider the problem of areas was Antiphon (Ἀντιφῶν) who was born around 430 BCE. His contribution was to be the first to introduce the "method of exhaustion." In this method, simple polygons are used to approximate the area of more complicated curves.[7] Unfortunately, like most mathematicians who invent a new technique, Antiphon did not formulate the method rigorously. That honor belongs to Eudoxus.

Eudoxus (Εὔδοξος) was born on the island of Cnidus in approximately the year 408 BCE and lived until roughly 355 BCE, but these dates are not known exactly. He learned much astronomy by traveling to Egypt and was the first to study spherics, the Greek version of astronomy/astrology.[6] In terms of what is now considered mathematics, Eudoxus was the first to write about proportions. By considering numbers not to be entities of their own right, but simply as ratios of lengths, he paved the way for the rational numbers, the entirety of numbers that all future Greek geometers would accept. While this is quite important, Eudoxus' most lasting act was the logical development of the aforementioned method of exhaustion.[3]

The diagram below shows a simple example of the method of exhaustion.[8] Consider a circle. Now inscribe a triangle in that circle. Now add more triangles to each side to form a hexagon. This hexagon will remain inside the circle, but the area of the hexagon will more closely match the true area of the circle. This process of adding triangles can be continued as many times as one likes to form polygons of increasing numbers of sides. With each cycle, the area of the polygon will become closer to the area of the circle. The process is continued until the difference in area between the circle (which is not known) and the polygon (which can be found through basic geometry) is finite, but small enough to be ignored.[7] Note that Eudoxus does not actually use the concept of a limit. This flaw will be discussed later.



The method begun by Eudoxus was used extensively by Euclid (Εὑκλείδης) in his textbook *The Elements.* For example, Proposition 2 of Book 12 of *The Elements* is that the area of a circle is proportional to the square of the diameter. While this is obvious from our modern equation: A=πr^2, the Greeks did not use such algebraic equations. Euclid proved this Proposition by using the method of exhaustion.[3] His reasoning is similar to that of Eudoxus, but he is able to take it a step further. Euclid also applies the Method to various other geometrical shapes, such as cones, pyramids, cylinders, and spheres.[6]

Even if the method of exhaustion was only applied to finding areas and volumes of particular shapes, it would still be interesting. But due to Archimedes, this method was made even more general. Archimedes (Ἀρχιμήδης) lived from 287 BCE to 217 BCE in Syracuse. He was the greatest mathematician of the ancient age, and quite possibly the greatest physicist as well.[6] His use of the method of exhaustion is illustrated in one of his works entitled *The Method*. This work was presumed lost until it was rediscovered by Heiberg in 1906. The scroll in question was a transcription of some of Archimedes' works that had been partially erased by medieval scribes.[1] Heiberg was able to reconstruct much of the work, of which this scroll was the only known copy. It is a very important discovery because it shows Archimedes' method in addition to just listing theorems and proof sketches.[2]

Archimedes applied the Method to several curves, including solid figures such as spheres and cones, as well as plane curves such as parabolas. Here is a summary of Archimedes' method for finding the area bounded by a parabola. Archimedes begins with a parabola and constructs several triangles from it. By using various properties of triangles and segments, Archimedes is able to find that the area under a parabolic arc is 4/3 of the area of a triangle inscribed into that arc. Here is where the method of exhaustion comes into play. Archimedes considered more and more triangles to fill the area under the parabola. What he finds is equivalent to a modern day geometric series. What is quite interesting is that Archimedes uses Proof by Contradiction to show that this series cannot differ from the true value by any finite amount. This would imply that the series converges to that value. However, Archimedes did not use concepts such as limits and convergence. As is usual of the method of exhaustion, Archimedes stops after some finite number of terms and is satisfied that he can make the disagreement between the approximation and the correct value as small as possible (while it of course remains finite).[1]

For all this work, how much did the Greeks actually accomplish? True, the method of exhaustion was a work of creative genius, but it did have two major flaws. First, it was not general. For each different problem, a different ingenious way of drawing triangles or some other polygon needed to be devised. The analytic approach of the modern era is completely general to the point that advanced math courses do not actually use numbers. The second, and larger, flaw was that the method of exhaustion was not at all rigorous by modern standards. Quite simply, there was no inclusion of a limit concept. The geometers all used the same argument. Take some geometrical figure and fill it with some large, but finite number of polygons. The sum of the areas of these polygons would be the area of the figure. But they did not consider this as a series. The math boils down to a convergent series, usually an easy to work with geometric series. Without a concept of infinity, which the Greeks lacked, it would have been impossible for them to rigorize the method of exhaustion.[7]

One of the Greek philosophers who consider the problem of infinity was Zeno (Ζήνων) who was born in 496 BCE. Note that this was before any of the Greek geometers mentioned.[3] Zeno devised four paradoxes which could not be explained in terms of the Greek view of the world. Zeno's first paradox will be sufficient to illustrate the point. Consider a man who wants to walk from where he standing to a wall 2 feet away. But before he can walk to the wall, he must reach the halfway point, a distance of 1 ft. But once there he must walk halfway to the wall, ½ ft. And he must walk half of the remaining distance, ¼ ft. And this must go on, so the man clearly can never reach the wall. Hence motion is impossible. But that is clearly wrong, as people are of course capable of moving around. The flaw in Zeno's argument is not considering a convergent series. If the distances the man walks are written down, the sum is $1 + \frac{1}{2} + \frac{1}{4} + \ldots + 2^{-n}$. This is a geometric series with ratio ½. Since the (absolute value of the) ratio is less than 1, the series converges. Moreover, the sum of a convergent geometric series is $a_1 / (1-r)$ where a_1 is the first term of the series and r is the common ratio. For the series in question, the sum is 1/(1-1/2) = 2, precisely the distance the man must travel. But for any finite number of terms, the sum will be different from 2. It is only in the limiting case of infinity that the sum of the series will be identical to the expected sum.[6] Admittedly, Zeno also used the paradox to show that time could not be continuous, but for the purposes here it suffices to illustrate the Greek refusal to accept infinity.

This flaw is of course present in the method of exhaustion. So while Archimedes may have gotten extremely close to discovering the integral, he was also extremely far away from it. But does that make the Greek work irrelevant? Of course not. The Greek method of exhaustion is similar enough to the modern method of approximating areas of curves with simple shapes (rectangles, trapezoids, parabolas, etc.) that it should be taught to students, at least in a simple form.[3] It would be a great disservice to ignore Archimedes and Eudoxus when doling out credit for devising integral calculus. In fact, after the Greeks, it would be almost 2000 years before the concept of integrals was taken up in any sort of satisfactory manner, and another 200 years after that before it was rigorized.[4]

In the 17th and 18th century, integration along with the rest of calculus was still in its infancy. Some of the earliest work on finding areas was done by Johannes Kepler (1571-1630), who thought of areas of plane figures and volumes of solids as an infinite number of infinitesimal elements. This idea was most likely influenced by the ancient Greek's method of exhaustion. For example Kepler thought of a circle as the union of an infinite number of triangles, each with a vertex at the center of the circle and a base on the circumference. Similarly he thought of the sphere as infinitely many small cones. Other work on integration was carried out by Galileo Galilei (1564-1642) who showed that under uniform acceleration the area under the time-velocity curve is distance, a result that foreshadowed the Fundamental Theorem of Calculus.[3]

Bonaventura Cavalieri(1598-1647), incorporated the ideas of Kepler and Galileo into his work, and continued the study of indivisibles by introducing his namesake principle, which states that if two figures have the same height, then their volumes will be proportional to the areas of their bases. He used this principle to find the areas of quadrilaterals, circles and other planar figures as well as the volumes of several solids, including the right circular cone. His method was subsequently adapted by various mathematicians including Gilles Roberval (1602-1675) who applied it to rectify (find the arc length of) the cycloid. Concurrently, other mathematicians made strides in the method of indivisibles by approximating curves with rectangles, a novel approach that contrasted the Greek's method of exhaustion that did not use a standard partitioning figure when finding the area under a curve.[3]

All of the aforementioned mathematicians did not really capture the scope and generality of the methods they were employing on specific problems. It was the work of two men, Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716) that made calculus a field in its own right. Like other mathematicians before them, Newton and Leibniz recognized that, in some sense that they could not formalize, integration could be thought of as the reverse of differentiation. Newton used this idea extensively in his work, and it helped him circumvent the use of indivisibles. However, Newton discovered that he could not, for obvious reasons, formalize all of calculus without a workable limit concept, which he did not have. In its place he offered fluxions, to which he assigned the vague definition, "the prime ratio of nascent increments."(Kline 1, 363) Newton's sometimes circular, always vague definitions of fluxions led to subsequent criticism from some of his contemporaries.[3]

Leibniz's work on calculus was equally informal as Newton's. Leibniz discovered calculus while studying sequences of functions. His notation was in a sense clearer than Newton's, though he was by no means close to providing the rigorous underpinnings for calculus. He is responsible for introducing the Integral sign as well as the differential (dx). Truly, Leibniz thought of integration as an exact analogue of its discrete counterpart, the sum, while he thought of differentiation as a type of subtraction, hence he was led to posit correctly that integration and differentiation are inverse processes. Throughout the 17th and 18th centuries, mathematicians crept closer to the idea of Riemannian Integration that we are familiar with today. It wasn't until the 19th century, though, that the limit concept was introduced, and integration was subsequently rigorized.[3]

In 1823, Augustin Louis Cauchy (1789-1857) provided the first rigorous treatment of integration. Cauchy insisted that it was not enough to simply state that integration is antidifferentiation, but that an integral must be defined and then proven to exist. He began by defining the integral of a continuous function f(x) over the interval $[x_0,X]$ as $\sum_1^n f(\xi_i)(x_{i+1} - x_i)$] where $x_1,...,x_{n-1}= X$ is a partition of the interval $[x_0,X]$. His definition implicitly assumes that the largest interval over any partition goes to zero with n. Equipped with the definition for the integral of any continuous f(x), he goes on to give the first proof of the Fundamental Theorem of Calculus. Cauchy also gave a formal treatment for improper integrals such as the integral of 1/x over any interval containing zero.[4]

The next big step in Riemannian Integration came from its namesake in an 1854 paper on trigonometric series. He generalized Cauchy's work to include the integrals of bounded functions. He defines the oscillation of f(x) in an interval as the difference between the greatest and least value of the function on that interval. His work then shows that a function is integrable iff the sum of all intervals where the oscillation of f(x) is bigger than some fixed number λ , approaches zero with the size of these intervals. The importance of this definition is that it allowed Riemann to integrate functions with an infinite number of discontinuities. The modern notion of Riemann sums was not actually completed by Riemann, but instead was finished by Gaston Darboux (1842-1917), who defined the Upper Riemann and Lower Riemann sums, and showed that a function is integrable only when it's Upper and Lower Riemann sums approach the same value as the maximum subinterval approaches zero in any partition over the interval of choice.[4]

Even after Riemann generalized Cauchy's concept of the integral, mathematicians still discovered functions that could not be integrated. One such example was provided by Lejeune Dirichlet (1805-1859). He considered a function that takes the value one on the Rationals and zero on the Irrationals, called the characteristic function of the Rationals, since it picks out exactly those points in the real numbers that are rational. In some vague sense mathematicians realized that Riemann Integration was incapable of handling Dirichlet's function basically because of the large number of discontinuities in any interval one might choose to integrate over. The man who resolved this limitation of Riemannian Integration is Henri Lebesgue (1875-1941), but it is necessary to introduce another concept, that of measure before we explain his contribution.[4]

It was mentioned before that Dirichlet's function is not Riemann integrable because there are just "too many" discontinuities, Dirichlet and other mathematicians had an intuitive feeling about this, but it was measure that allowed mathematicians to eventually quantify what exactly was meant by a function having a "too many" discontinuities. The idea stemmed from a very natural extension of the notion of the length of interval. In its infancy, measure (initially called content) was defined by covering a set E, a subset of the interval [a,b], with subintervals that contained points from E, and then taking the greatest lower bound of the sum of the lengths of these subintervals. Although this definition of content was incomplete it did allow mathematicians to give some restrictions to the kind of functions that were Riemann Integrable, which after all was the point of introducing content in the first place. The work on content was expounded upon by Giuseppe Peano (1858-1932) and Camille Jordan (1838-1922) who both introduced the concept of inner and outer content of a set. Peano extended content to include regions in R^2 and also gave a necessary and sufficient condition for the integrability of a function f. His result stated that, f is Riemann integrable iff the region bounded above by the graph of f has equal inner and outer content. Borel also contributed to the theory of content, first by giving it the name measure by which it is known today, and then by applying the work of Georg Cantor (1845-1918) to the definition of the measure of a set, and defined the measure of the union of sets as the sums of the individual measures. He also showed that sets of non-zero measure are non-denumerable, that is they can't be put into a one-to-one correspondence with the natural numbers.[4]

The work on measure set the stage for Henri Lebesgue a pupil of Emile Borel (1871-1956) and a professor at the College de France, whose thesis presented ideas that went beyond existing concepts. In his 1902 thesis, Lebesgue introduced an alternative form of integration. Lebesgue's insight is best summed up in his own words, "One could say that according to Riemann's procedure one tried to add the indivisibles by taking them in the order in which they were furnished by the variation in x, like an unsystematic merchant who counts coins and bills at random in the order in which they come to hand, while we operate like a methodical merchant who says: I have $m(E_1)$ pennies which are worth $1*m(E_1)$, I have $m(E_2)$ nickels worth $5*m(E_2)$, I have $m(E_3)$ dimes worth $10*m(E_3)$, etc… The two procedures [the other is counting the coins individually, akin to a Riemann sum] will certainly lead the merchant to the same result because no matter how much money he has there is only a finite number of coins or bills to count. But for us who must add an infinite number of indivisibles the difference between the two methods is of capital importance." (Lebesgue 181-182)[5]. Formally Lebesgue's definition is to consider a bounded and measurable function defined on set E. Now partition the interval [f(a),f(b)] into [f(a),y_1], ...,[y_{n-1},f(b)] with f(a)=y_0 and f(b)=y_n. Then, look at the set $e(i) = \{x \mid y_{i-1} \le f(x) \le y_i\}$. Now define in a fashion completely analogous with Riemann integration, S = $\sum_1^n y_i m(e_i)$ and s = $\sum_1^n y_{i-1} m(e_i)$. Lebesgue shows that S is lower bounded by some number J and s is upper bounded by some number I and I=J. This number is called the Lebesgue integral of f(x) on E. Unlike Riemannian integration, Lebesgue integration can be extended to functions that are unbounded, and can for example easily be used integrate Dirichlet's function, which has a Lebesgue integral of 0 on any interval.[4]

What is fascinating is how abstract Lebesgue's notion of integration is. The Greeks were tied to the physical world, and the method of exhaustion must work because all Greek mathematics was finite. By the 19th century, mathematics had moved far beyond simply a tool for science. It was possible to define something as abstract as Lebesgue measure and integration. This is but one of many examples of how human thought has evolved throughout civilized history.

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