Stephanie Yates
Daniel Freidenreich

The Origins of Numbers and Their Modern Era Interpretations

Numbers are present in every aspect of our daily lives. Whether we are making a phone call on our cell phones, calculating the waiter's tip at dinner or counting pairs of socks to make sure none of them disappeared in the laundry the use of numbers is evident. It is easy to take them for granted considering how prevalent they are in the world today. But this wasn't always the case. In fact numbers originated out of necessity. As human's mental capabilities increased, so did our comprehension of numbers. Numbers first came about in pre-civilization, the time before history was recorded. This era of numbers dates back to around 600,000 B.C. Later, the Greeks, Babylonians and Egyptians evolved the use of numbers from mere counting to incorporate it into the daily lives of their society members. In addition they also began to utilize numbers to create formulas and to use numbers more abstractly, beyond their everyday usage. This segways into the modern world where axioms and theories were invented to further abstract the idea of numbers. Such advancements lead to mathematics. It was crudely used by the Egyptians, Greeks and Babylonians, but today the scope of mathematics is unparalleled and has helped mankind build the civilization we live in today.

Pre-civilization saw the birth of numbers. It can be divided into three eras, the Paleolithic, Mesolithic and Neolithic eras. Each one saw different advancements in mathematics as human intelligence evolved. The Paleolithic era began around 600,000 B.C. and was the start of counting. The very first use of numbers, was thus to count. But numbers didn't exist yet. Instead humans in this era used tallies. These were in the form of notches carved into stone, wood or ivory (Calinger). Even knots in string, sinews, scratches on cave walls or pebbles for counting were used (Calinger 6). Although the Paleolithic era began in 600,000 B.C. humans didn't make use of tallies until the last 100,000 years of this era when mankind evolved from Neanderthals to Homo Sapien Sapiens (Calinger 7). At this time the human brain quadrupled in size giving humans the ability to comprehend the concept of counting. This humble beginning of numbers seems to reflect the Greek's belief about mankind's evolution of knowledge. According to the Greeks "All [human] work was without thought, until [Prometheus] taught them to see what had been hard to see; where and when the stars rise and set. What's more, [he] gave them numbering, chief of all stratagems. And the painstaking putting together of letters: to be their memory of everything, to be muse's mother, their handmaiden!" (Cuomo 17). But whether you believe in evolution or Greek mythology, this humble beginning led to much further advancements. In the Mesolithic Age around the mid-8th millennium B.C. patterned numbers with cycles were created which formed the basis of the ordinal number system used today where numbers dictate how many and order (Calinger 7). This early number system was called the two-counting system since they had definite symbols for 1, 2 and then many (Calinger 7). By the time the Mesolithic Age ended the number system had been extended to five, ten and even groups of 20 (Calinger 8). In addition descriptive numerical words were created such as one-stick, two-rocks, etc. (Calinger 8). It may have taken hundreds of thousands of years, but the beginnings of counting and numbers were created. The Neolithic Age saw the first uses of mathematics, an enormous step from the creation of numbers and the concept of counting. Math in the Neolithic Age included arithmetic and crude geometry which were to be used towards such applications as architecture, horizon astronomy and rituals (Calinger 13). An example of the early human usage of mathematics is Stonehenge I. In 2800 B.C. the Beakers, an ancient people, built Stonehenge I to mark the locations of stars (Calinger 13). In addition they used rope to measure land and make temples, a rope and stick to draw a circle and two sticks and a rope to make an ellipse (Calinger 13/14). With the advent of agriculture surplus was used for trade which must have led to a further use of numbers and to the creation of social classes. All this came about in 3000 B.C. in Sumerian cities, early ancestors of the Babylonians (Calinger 14). This was the end of pre-civilization and the beginning of the recording of history. In its infancy early humans formed the fundamental grounds for current day number systems and mathematical concepts. In early civilizations this base was further expanded as man's thirst for knowledge expanded.

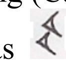
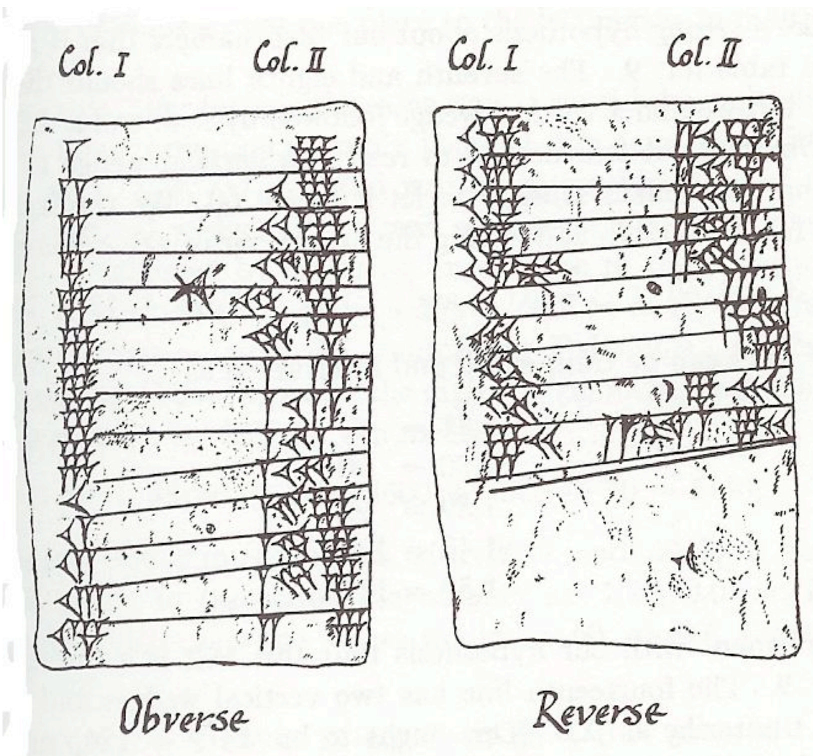
Babylonian mathematics is the accumulation of knowledge of various people's and is one of the early societies to advance mathematics. The first Babylonian people were the aforementioned Sumerians. They advanced the concept of numbers even further than the ancient people who lived in pre-civilization. They used numbers to keep track of populations, food supplies, trade, water for irrigation, inventories of animals, slaves, crops, taxes, and business transactions (Calinger 14). It is plain to see that advancements in mathematics and its usages were tied to the advancements in civilization. As civilization's need of numbers increased so did mathematics. To communicate these new advances in mathematics, the Sumerians invented a written system of counting called cuneiform to replace old pictographs (Calinger 15). This new numeral language was created around 2750 B.C. (Gullberg 35). This language was written with a sharp edged tool that made wedge shaped impressions in clay tablets (Gullberg 35). The Sumerians also had schools where undoubtedly they must have taught mathematics so that it could be used in the various aspects of their lives. A rather impressive creation of the Sumerians was their 354 day calendar based on the movements of the Sun and the moon (Calinger 16). Again, necessity is the mother of invention because the calendar was created to facilitate planting (Calinger 16). So although the calendar wasn't completely accurate it was an inspirational thought. By 2350 B.C. the Sumerians invented a 60 based number system which was developed from two separate 6 and 10 number systems (Calinger 23). Today we know this as the sexagesimal system which is still in use today. We use it in measuring time and angles (Calinger 23). This number system was created to avoid complicated fractions since numbers of base 60 had many divisors (Calinger 24). After the Akkadians invaded the Sumerian lands in 2350 B.C., they adopted their numerical system and adapted it to deal with larger numbers like 100 and 1000 (Gullberg 35). The reason for this extension of the Sumerian numerical system was because they needed the larger numbers for war and to stop cheating in trade (Calinger 25). To aid in calculating such large numbers they invented the abacus and arithmetic by 2100 B.C. (Calinger 25). With the invention of arithmetic the Akkadians could add, subtract, multiply and divide (Calinger 25). Another people that came to the land were called the Amorites and they made their own contributions. For instance they invented a new number system with placeholders and numbers could now be represented by repeating basic units (Calinger 25) Now the Babylonians had the ability to represent all rational numbers (Calinger 27). However, even with the upgrade of the numerical system by the Amorites, their system was still confusing since there were no decimals or commas (Calinger 27) Furthermore the only units of numbers were the wedges for ones and the crescent for tens (Calinger). Therefore all numbers had to be represented by these symbols, and with the lack of commas and decimals, it was hard to distinguish between large numbers and fractions (Calinger 27). Lastly, they had no numerical equivalent of zero in their early number system, another shortcoming (Calinger 29). However, they did create a placeholder form of zero which was written as  (Aaboe 9). Although it had its share of problems the cumulative additions to the mathematical system of each of the Babylonian civilizations created an intuitive mathematical system. The Babylonians wrote out their math on tablets and to perform multiplication they wrote down numbers in columns. Column one would be a number and column two would be the answer after multiplication. Therefore a 9 times table would look as follows (Aaboe 7):

Figure 1. Nine Times Tablet



Written out clearly in our numeric system this table appears as follows.

Figure 2. Nine Times Table 2

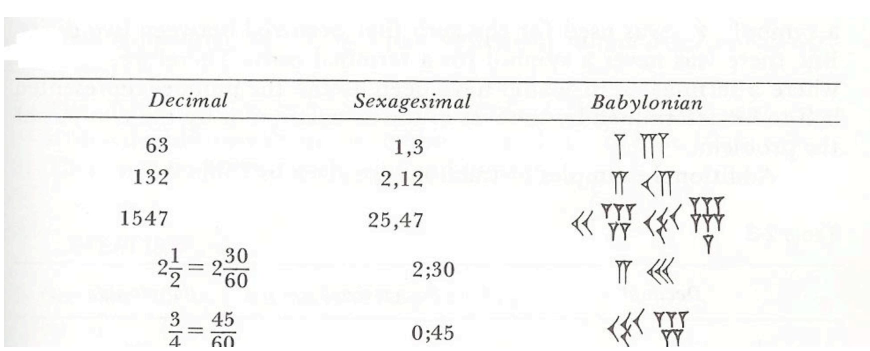
Col I	Col II
1	9
2	18
3	27
4	36
5	45
6	54
7	1,3
8	1,12
9	1,21

This table looks like a modern one for a nine times table until line seven is reached and column II reads 1,3. We know that 7 * 9 = 63, so why is 1,3 written? The reason is their 60 based system. The comma divides the ones place from the first power of sixty. Therefore 1,3 actually means 1*60 + 3 = 63 (Aaboe 7). Line 8 reads 1, 12 which means 1*60 + 12 = 72 (Aaboe 8). Similarly if you wanted to compute 15*21 you would write the answer as 5,15 which equals 5*60 + 15 = 315. Each comma means a higher power of 60. So 5,4,12 means $5*60^2 + 4*60 + 12 = 18{,}252$.

Although it looks very different this system is actually very similar to the one we use today. Our system is base ten, therefore the number 315 written out is really 3*100 + $1*10 + 5 = 3*10^2 + 1*10 + 5$. To reveal the similarities even further if we look at a previous calculation we would see: $5*60^2 + 4*60 + 12 = 18{,}252$. Written in base ten this can be written as $5*10^2*62 + 4*10*6 + 12$. By factoring out six from the base sixty, their numbers appear to look like our system, just multiplied by powers of six. Thus their system is very close to ours just with a base of 60 and not 10. This also shows that their mathematical thought process is very similar to our own which shows that our mathematical thought has been relatively similar for over 4000 years!

Similary their division was also written in powers of sixty. For instance 7, 30 used to represent 7*60 + 30. But when the tablet is division and not multiplication, this number actually means $7{,}30 / 60^2 = 1/8$ (Aaboe 15). Written out as before in multiplication this becomes $(7*60 + 30) / 60^2 = 7/60 + 30/60^2 = 1/8$ (Aaboe 15). The table below summarizes the multiplication and division of the Babylonians (Bunt, Jones & Bedient 45).

Figure 3. Decimal, Sexagesimal and Babylonian



Decimal	*Sexagesimal*	*Babylonian*
63	1,3	
132	2,12	
1547	25,47	
$2\frac{1}{2} = 2\frac{30}{60}$	2;30	
$\frac{3}{4} = \frac{45}{60}$	0;45	

The Babylonians weren't the only ones developing mathematics, there is also evidence of the Egyptians making great strides in math as well. Ancient papyri scrolls from 1650 B.C. and 1850 B.C. reveal problems and tables mathematical solutions for employers of labor, granary overseers and brewers for making beer and bread 9Calinger 43). These papyri reveal that the Egyptians used math for very practical purposes; in fact their math was for practical usage. Unlike the Greeks who will be mentioned shortly, Greeks did not prove their math, they only devised and utilized it. Unlike the Babylonian number system which was based on 60, the Egyptians based theirs on 10 (Calinger 44). After creating the basis for their number system, they proceeded to expand their number system to include multiples of their base numbers such as 20, 30, 400, 500 (Calinger 44). This Egyptian number system of base ten and its multiples was called the hieratic system (Calinger 44). Like the Akkadians, they had arithmetic and could use addition, subtraction, multiplication and division (Calinger 46) But they represented multiplication with addition and subtraction which one would think would make their mathematics a bit more difficult to record and use. For example, suppose the Egyptians wanted to multiply 47 * 33, the process would look something like this (Robins & Chute 16).

Figure 4. Egyptian Multiplication

/1	47	or	/1	47
2	94		/10	470
4	188		/20	940
8	376		/2	94
16	752			
/32	1504			
Total 33	1551		Total 33	1551

The process of Egyptian multiplication is as follows. There are two numbers being multiplied for the calculation of 47*33, 47 is the first number and 33 is the second. Two columns are made. The first column is a series of numbers such as 1, 2, 4, 8, 16, 32 (Robins & Chute 16). The second column is a list of multiples of the first number times the numbers in the series. So row one is 1 and 47 since 1*47 = 47. Row two contains 2 and 94 since 2*47 = 94. Once these columns are setup in this fashion, addition comes into play. The end goal is to collect numbers in column one that will add up to the second number, in this case 33. The dashes next to the numbers 1 and 32 represent that these numbers have been chosen to add up to 33 (Robins & Chute 16). The next step is to add up in column two the corresponding multiples of 47. The result of the addition of the numbers in column two is the answer to the problem 47*33. In this way multiplication was carried out utilizing addition.

In a similar fashion to their multiplication, their method of division also used addition. Suppose the Egyptians wanted to calculate 47/33. They would write it as a multiplication of a whole number and a fraction 47*/33 (Robins & Chute 17). This means 47* 1/33. The process of division would appear as follows (Robins & Chute 17).

Figure 5. Egyptian Division

/1	33
/$\overline{3}$	11
/$\overline{11}$	3
Total $1 + \overline{3} + \overline{11}$	47

Once again two columns are used. Column I represents a divisor a reciprocal of a divisor of 33. Column II consists of the result of the multiplicand of the number in column I and 33. For example: row one contains 33 and 1. This means 33/1. So the divisor, 1 and the result, 33. Row two contains /3 and 11. This is the multiplicand, written 1/3 and 11 since 33/3 or 33 * (1/3) = 11. The rows are all written as divisors and multiplicands until in column two the numbers add up to the first number in the calculation which in this case is 47. In this example all the numbers have dashes next to them in column one since all the numbers corresponding to them add up to 47 in column two. Once the numbers in Column I are added, the answer to the problem, 47* 1/33 is obtained. The method of Egyptian division is similar to how we count change at a cash register. If you owe $5.50 and you give a $10 bill, then the sales clerk will add 50 cents to make $6 and then add another $4 to make your change for ten giving you $4.50 in change. This is how the Egyptians performed their mathematics. They would try to find factors that when added would get closer and closer to the first number in this equation. In the above example, factors were found to get closer to 47. So first 33 was found, but 14 units were still needed. So the factor eleven was used, but 3 more units were needed, so three was used to reach 47. The corresponding divisors were then added together to get the answer. This system requires more intuition and less procedure than does division today.

The Egyptians alongside of arithmetic did develop some geometry as well. It came about in measuring land after the flooding of the Nile to calculate taxes (Calinger 49). In addition geometry was used in their artwork such as pottery, jewelry etc (Calinger 50) Patterns of squares, triangles, triangles and squares inscribed in circles were also used (Calinger 50). However, as mentioned before, unlike the Greeks, the Egyptians didn't use axioms or proofs to prove that their geometrical figures were correct (Calinger 50). Finally, like the Babylonians, the Egyptians created a calendar in 2776 B.C. . However unlike the Babylonian calendar, it was much more accurate by incorporating 365 days in a year (Calinger 54). It is quite impressive that they were able to compute such an accurate calendar considering that they lived in the same time period as the Babylonians whose calendar was less accurate. Egyptian mathematics was created out of necessity, but the Greeks created math for more than just to meet the growing needs of their civilization, but also to better understand the universe and to satisfy their curious nature.

The Greeks were a people smitten with curiosity, seeking to understand the universe, and to do this they turned to mathematics. They sought after it not only to aid themselves in their daily lives, but to solve the mysteries of the universe by using math logic. These feelings of the Greeks can be summarized by Philolaus when he said, 'all things which can be known have number; for it is not possible that without number anything can either be conceived or known' (Heath 67). In other words, to them everything was reducible to numbers. An example of this was Aristotles's belief that monetary exchange was a form of mathematics and that trading could thus be broken down into arithmetic equations (Cuomo). He believed this to be true because in order for things to be traded they must be believed by the traders to be equal in value (Cuomo). Thus mathematical equations should be able to be derived in order to determine this ratio of values of goods. The Greek's study of math was very intense. They divided math into three subjects in order to more thoroughly evaluate each one. Therefore math was divided into arithmetic (theory of numbers and calculations), geometry (measurement and volume), and physical subjects like astronomy (applied mathematics) (Heath 13, 16, 17). These in-depth studies were backed by thoroughly worked out proofs in order to prove the validity of their findings (Cuomo 5).

Although the Greeks did delve into math with great zeal, the average citizen was unaware of the depth of research the mathematicians of Greece performed. The majority of Greek citizens were copious users of mathematics, just not of the more advanced topics the mathematicians were involved in. In school Greek students learned how to count by counting the letters in words (Heath 19). Children even used math in games such as cubic dice and knucklebones, a game which modern day jacks was derived from (Heath 19 & Hoodmuseum). In addition the adult community also was exposed to math on a daily basis. Although original Greek documents about mathematics are gone, indirect sources of information, like their daily usages of it are available (Cuomo 5) For instance the Greeks used math to compile inventories, record tribute, calculate the cost expense of building a temple and even architectural designs of towns using geometry (Cuomo 5). Cites were built built more efficiently using geometry to make them more organized and evenly spaced (Cuomo 8). Buildings were also constructed using math. In 540 B.C. building like the Parthenon in Athens and temples were built using mathematics to make them stand out from the other city structures (Cuomo 9). A good example of this is that the width of a temple was the mean of the average of its height and length (Cuomo 9). An even more impressive architectural achievement using mathematics was the creation of the Eupalinus tunnel. It was started at two different ends of a mountain and built under it with the two sides meeting in the middle (Cuomo 9/10). There is evidence of the use of math because there were distance numbers recorded at regular intervals in the tunnel (Cuomo 11). The planning of meeting in the middle and starting at two different places was quite an impressive mathematical feat. Such amazing architectural feats were possible because of the depth of research the Greek mathematicians performed. Plato best described the process of mathematics as follows: "They used claims that were not proved but only assumed to be valid, and on which other claims were then based. Thus, proving a mathematical statement consisted of working back from the statement to other known statements of which the first was a consequence, until one found an accepted hypothesis which needed no further proof and could then serve as the starting-point (reversing the logical process that led to it)," (Cuomo 26). In short, the Greeks began with an idea, most likely a problem they currently faced and then worked toward a mathematical solution and then checked to see if it worked. These tireless efforts are what made the Greeks such phenomenal mathematicians in their time.

The depth of Greek mathematics was far reaching, but at its heart was arithmetic, the foundation for much of mathematics. To facilitate this foundation the Greeks needed a mathematical language. Like the Egyptians, they had two numerical languages. The first was known as the herodianic system and the second was called the ionic system (Bunt, Jones & Bedient 66). The herodianic system consisted of five symbols and these symbols could be combined to make other numbers. Figure 6 (Bunt, Jones & Bedient 66).

Figure 6. The Herodianic System

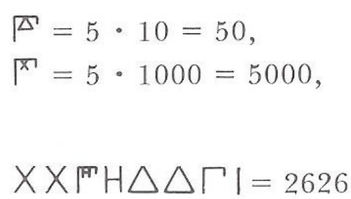
$$\mathrm{I} = 1, \quad \Gamma = 5, \quad \Delta = 10, \quad \mathrm{H} = 100, \quad \mathrm{X} = 1000, \text{ and } \mathrm{M} = 10{,}000.$$

The symbols for the herodianic system could be combined as below in order to create other numbers. In some cases, one symbol could be written multiple times and other combinations required multiplication and addition (Bunt, Jones, Bedient 67).

Figure 7. Combining Symbols Repeatedly

$$\Delta\Delta\Delta\Delta = 40$$

Figure 8. Combining Numbers using Multiplication and Addition



The examples in figure 8 demonstrate how the Greeks performed multiplication and addition. To multiply two symbols, one symbol would be written smaller and put within the other symbol. Addition was assumed when symbols were aligned next to each other without spaces. However, some of this changed with the onset of the Ionic system. There were 27 symbols in this ancient language. It used the 24 letters of the Greek alphabet and then added three more symbols from an older alphabet (Bunt, Jones, Bedient 67). The new Greek alphabet used under the Ionic system appears below (Bunt, Jones, & Bedient 67).

Figure 9. The Greek Alphabet also used Numerically under the Ionic System

1	2	3	4	5	6	7	8	9
α	β	γ	δ	ϵ	ς	ζ	η	θ
10	20	30	40	50	60	70	80	90
ι	κ	λ	μ	ν	ξ	ο	π	ϙ
100	200	300	400	500	600	700	800	900
ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ
1000	2000	3000	4000	5000	6000	7000	8000	9000
,α	,β	,γ	,δ	,ϵ	,ς	,ζ	,η	,θ

In order to distinguish between the alphabet and numerical calculations, the Greeks put dashes above the symbols or apostrophes after the symbol (Bunt, Jones, Bedient 67). Their new alphabet brought a new way to perform calculations and to write numbers. Numbers were still written as if added together by being put side by side. For example in the figure below the number 3738 is written by putting the symbols that mean 3000, 700, 30 and 8 next to each other. That is they write the number 3738 as if it were 3000 + 700 + 30 + 8. Multiplication was carried out by placing one number above the other as follows in figure 10 (Bunt, Jones, Bedient 68).

Figure 10. Numbers Written in the Ionic System

$$\overset{\beta}{\mathrm{M}} = 2 \cdot 10{,}000 = 20{,}000.$$

$$\iota\gamma\ (= \overline{\iota\gamma} = \iota\gamma') = 13, \qquad \varsigma\epsilon = 65, \qquad \sigma\lambda\zeta = 237, \qquad \tau\alpha = 301,$$

$$,\gamma\psi\lambda\eta = 3738, \qquad ,\epsilon\eta = 5008, \qquad \overset{\lambda\eta}{\mathrm{M}},\alpha\varphi o\delta = 381{,}574.$$

The Greeks added numbers together like we do in modern times. They placed numbers above each other over a horizontal bar to add as follows below (Bunt, Jones & Bedient 68).

Figure 11. Greek Addition

$$\begin{array}{r} \kappa\alpha \\ \upsilon\xi\alpha \\ \underline{o\epsilon} \end{array} \qquad\qquad \begin{array}{r} ,\alpha\chi\xi\theta \\ \\ \underline{\overset{\epsilon}{\mathrm{M}},\epsilon\phi\lambda\beta} \end{array}$$

By analyzing the mathematics of these ancient civilizations the evolution of mathematical thought can be explored. The Babylonians used a mathematical system practically identical to that of today in terms of their addition and multiplication except that they used a sexagesimal system. The Egyptians and Greeks developed a decimal system and Greek mathematics was performed very similarly to today except that they wrote out their calculations differently and did not have a distinct mathematical language. The Egyptians possessed a mathematical thought process that was a far cry from that of today and although it was being created around the same times as the Babylonians and the Greeks, it developed very differently. Modern mathematics can be seen as a conglomeration of ideas from these ancient civilizations. Mathematics is carried out similar to that of the Babylonians, written more like the Greeks and utilizes the base ten system utilized by the Egyptians. However, mathematics today is more than just arithmetic. This arithmetic is now based on a system of set principles to make mathematics more formal and justified.

The birth of modern set theory is attributed to the famous mathematician, Georg Cantor. Prior to Cantor, many mathematicians, including the Greeks, had considered the idea of infinity and the definition of a set was still rather vague. For example, Bernard Bolzano in 1847 considered sets to be defined as, “an embodiment of the idea or concept which we conceive when we regard the arrangement of its parts as a matter of indifference” (Mactutor). Cantor’s greatest discoveries occurred between the years 1874 and 1884 and his first ideas of set theory can actually be found in papers on trigonometric series. In his paper 13 published in 1874 in Crelle’s Journal, he shows that it possible to distinguish different kinds of infinity. This idea was met with a great deal of controversy from other mathematicians, especially Leopold Kronecker who did not even believe in the irrational numbers. According to Cantor, “…there are at least two different orders of infinity: the set of real algebraic numbers can be put in one-to-one correspondence with the set of natural numbers whereas the same is not true of the set of real numbers.” (Johnson 23) Algebraic numbers are the real roots of algebraic equations of the type $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots a_{n-1}x+a_n=0$, where n is a natural number, a_i an integer, i=0,1…n, and $a_0>0$. Hence the infinity ascribed to the natural number is smaller than the infinity ascribed to the real numbers because the set of real algebraic numbers is a subset of the real numbers and the idea of orders of infinity was first introduced. It is believed that originally Cantor thought that the real numbers could be put into a one-to-one correspondence with the natural numbers.

Cantor continued writing essays and papers that were met with great opposition, especially by Leopold Kronecker. He moves on to defining equivalent sets as those that can be put in one-to-one correspondence, proves that the natural numbers have the smallest infinite power, where power refers to the number of elements in the set, defining the term countable set as a set that can be put in a one-to-one correspondence with the natural numbers, and that set of rational numbers also has the smallest infinite power since it has the power of the natural numbers. His fifth essay in this series of essays and papers was published separately and is known as *Grundlagen einer allegemeinen Mannichfaltigkeitslehre*. This was a substantial piece of work in the development of mathematics as he defines the improper and proper infinites. The improper infinite “is used for a magnitude which either increases above all limits or decreases to an arbitrary smallness, but always remains finite; so that it may be called a variable finite." The proper infinite "is typified by the infinite real integers, and to emphasize this, the old symbol "∞," which was and is used also for the improper infinite, was here replaced by "ω." It was also in this work that he discusses well-ordered sets and defined ordinal numbers as the order types of well-ordered sets. Cantor defines a well-ordered set as "any well-defined set whose elements have a given definite succession such that there is a first element, a definite element that follows every other (if it is not the last), and to any finite or infinite set a definite element belongs which is the next following element in the succession to them all" (Johnson 30). It is believed also that one of Cantor's original intentions was to be able to define the real numbers as a well-ordered set, which he discovered to be impossible.

While Cantor did not explicitly define axioms to base his theories on an analysis of his theorems bring forth three axioms that his theorems were derived from: the axiom of extensionality of sets; the axiom of abstraction; the axiom of choice. The axiom of extensionality states that two sets are identical if and only if they have the same members. The axiom of abstraction states that given any property, there exists a set whose members are just those entities having that property. The axiom of choice states that, given a collection of mutually disjoint, nonempty sets, there exists a set that has as its elements exactly one element from each set in the given collection of sets. It was these axioms that brought forth the development of paradoxes that would lead to the introduction of new, more restricted axioms.

Cesare Burali-Forti published the first paradox in 1897. Although Burali-Forti's definition of perfectly ordered set was not the equivalent of Cantor's definition of well-ordered set, the paradox could still be established on the basis of Cantor's definition of well-ordered set. It is perhaps due to this original error that this paradox did not receive the reaction many might have expected, however many more paradoxes followed this one. Cantor himself revealed a paradox that was similar to Burali-Forti's:

> "Consider the cardinal number of the set of all sets. It is clear that this is the greatest possible cardinal number. But by the standard theorem of intuitive set theory, the set of all subsets of a set has a greater cardinal number than the set itself. Therefore, the cardinal number of the set all sets is greater than the greatest possible cardinal number." (Johnson 50)

This is an obvious contradiction. The problems in the logic seemed to from stretching the idea of sets too far. These paradoxes were also developed from the results of the set theory, thus they were not as shattering as the paradox that was discovered by Bertrand Russell in 1902. Russell's paradox was based on the concept of set as it was defined. Consider the set $A=\{X \mid X \text{ is not a member of } X\}$ (Johnson 51). Is A an element of A? When assuming that A is or is not a member of A one is led to a contradiction, thus it is the construction of A that leads to the paradox and it was in effort to not allow A to be a set that the modern set axioms were defined.

The most generally accepted set of axioms are the Zermelo-Fraenkel axioms, though other approaches have been taken and are sometimes used, such as the Gödel-

Bernays set of axioms (Hayden, Kennison 152). The Zermelo-Fraenkel axiom system, developed by Ernst Zermelo, and improved by Adolf Fraenkel, Thoralf Skolem, and John von Neumann around 1922, includes the following axioms (Hanjal, Hamburger 111-112):

1. Axiom of the Empty Set: $\exists x\forall u(u\notin x)$.
2. Axiom of Extensionality: $\forall x\forall y(\forall u(u\in x\Leftrightarrow u\in y)\Rightarrow x=y)$.
3. Axiom of Pairing: $\forall x\forall y\exists z\forall u(u\in x\Leftrightarrow u=x\vee u=y)$.
4. Axiom of Power Set: $\forall x\exists y\forall u(u\in y\Leftrightarrow\forall v(v\in u\Rightarrow v\in x))$.
5. Axiom of Union: $\forall x\exists y\forall u(u\in y\Leftrightarrow\exists v(u\in v\wedge v\in x))$.
6. Axiom of Infinity:
   $\exists x(\exists u(u\in x\wedge\forall v(v\notin u))\wedge\forall u(u\in x\Rightarrow\exists v(v\in x\wedge\forall w(w\in v\Leftrightarrow w\in u\vee w=u))))$.
7. Axiom of Replacement: $\forall x(\exists u(u\in x)\Rightarrow\exists v(v\in x\wedge\forall w(\neg(w\in x\wedge w\in v))))$.

From these axioms, the different number systems and the arithmetic within them can be defined.

Using these axioms mathematicians began by defining the natural numbers, also known as the set *N*. To do this a constructive approach was used. " In 1908, Zermelo proposed to use Ø, {Ø}, {{Ø}}… as the natural numbers. Later von Neumann proposed an alternative, which has several advantages and has become standard. The guiding principle behind von Neumann's construction is to make each natural number be the set of all smaller natural numbers. Thus we define the first four natural numbers as follows: $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{0,1\} = \{\emptyset,\{\emptyset\}\}$, $3 = \{0,1,2\} = \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$." (Enderton 67) This leads to the definition that for any set a, $a^{+}=a\cup\{a\}$ and a natural number is a set that belongs to every inductive set. The set *N* is the set of all natural numbers.

Addition on *N* is defined through relations where a relation, which is a set of ordered pairs and using recursion, which is defining a value based on previous values. The recursive function A_m:*N*_*N* is defined by $A_m(0) = m$ and $A_m(n^{+}) = A_m(n)^{+}$. So for natural numbers, $+ = \{\langle\langle m,n\rangle,p\rangle | m\in N \;\&\; n\in N \;\; p=A_m(n)\}$. Multiplication on *N* also is defined by using recursion. The recursive function M_m:*N*_*N* is defined by $M_m(0)=0$ and $M_m(n^{+})=M_m(n)+m$. Thus multiplication is defined by $m\bullet n=M_m(n)$.

The next step in the progression towards defining the real numbers is to define the integers, or the set *Z*. To do this mathematicians considered the set *N*x*N* and defined the equivalence relation ~ as $\langle a,b\rangle\sim\langle c,d\rangle$ if and only if a+d=b+c. The set *Z* is equal to the set (*N*x*N*)/~ of all equivalence classes of differences. For example, the integer 4 is the equivalence class $[\langle 4,0\rangle] = \{\langle 4,0\rangle, \langle 5,1\rangle, \langle 6,2\rangle\ldots\}$. By our basic knowledge of subtraction, we see that 4-0=5-1=6-2=4, thus all these ordered pairs are in the equivalence class of the integer 4. Using this definition of the integers, addition is defined as $[\langle m,n\rangle] +_Z [\langle p,q\rangle] = [m+p, n+q\rangle]$, and multiplication is defined by $[\langle m,n\rangle] \bullet_Z [\langle p,q\rangle] = [\langle mp+nq, mq+np\rangle]$. For example, $4_Z + (-2_Z) = [\langle 4,0\rangle] + [\langle 0,2\rangle] = [\langle 4,2\rangle]=2_Z$ and $4_Z \bullet (-2_Z)=[\langle 4,0\rangle]\bullet[\langle 0,2\rangle]=[\langle 0+0, 8+0\rangle]=(-8)_Z$. (Enderton 91-95)

From the integers mathematicians then proceeded to define the rationals, set *Q*, as the set (*Z*×*Z*)/◊ of all equivalence classes of fractions where here ~ is the relation defined as $\langle a,b\rangle$◊$\langle c,d\rangle$ if and only if ad=cb. Thus $\langle 1,2\rangle$◊$\langle 6,12\rangle$ since 1(12)=6(2)=_. Thus addition for the rationals is defined by the equation $[\langle a,b\rangle] +_{Q} [\langle c,d\rangle] = [\langle ad+cb,bd\rangle]$ and multiplication by $[\langle a,b\rangle]\bullet_{Q}[\langle c,d\rangle] = [\langle ac,bd\rangle]$. For instance, _ $+_{Q}$ _= $[\langle 1,2\rangle]+_{Q}[\langle 1,3\rangle]=[\langle 3+2,6\rangle]=[\langle 5,6\rangle]=5/6$ and _ $\bullet_{Q}$ _=$[\langle 1,2\rangle]\bullet_{Q}[\langle 1,3\rangle]=[\langle 1,6\rangle]=1/6$. (Enderton 102-103) It was easily proven that these definitions were sufficient to provide algorithms for the basic arithmetic of these number systems. Also, these definitions were suitable for the development and proofs of arithmetic properties such as commutativity and associativity. This was extremely important given the significance that rigor had at this point in mathematics.

The next system to be considered was that of the real numbers, which had been mathematicians' goal from the beginning of the axiom systems. However, this proved to still be a sizeable problem since the real numbers were much more complex than the rationals or integers. It was at this stage that Richard Dedekind's concept called a Dedekind cut became of immense value in the study of set theory. A Dedekind cut is defined as a subset x of *Q* such that $\varnothing \neq x \neq Q$, x is "closed downward," i.e. $q\in x \,\&\, r<q \Rightarrow r\in x$, and x has no largest member. Dedekind first had the idea of this "cut" while he was thinking how to teach differential and integral calculus in November of 1858 and thought to use this concept to represent the real numbers. "His idea was that every real number r divides the rational numbers into two subsets, namely those greater than r and those less that r." (Mac Tutor, Dedekind) Today the set *R* of real numbers is equal to the set of all Dedekind cuts and a real number is defined to be a Dedekind cut. (Enderton 113) Using these definitions, mathematicians have been able to again define formulas for addition of multiplication and they are as follows (Enderton 118):

$x+_{R}y=\{q+r \mid q\in x \,\&\, r\in y\}$ and

If x and y are nonnegative real numbers, then $x\bullet_{R}y=0_{R}\cup\{rs \mid 0\leq r\in x \,\&\, 0\leq s\in y\}$.

If x and y are both negative real numbers, then $x\bullet_{R}y=|x|\bullet_{R}|y|$.

If one of the real numbers x and y is negative and one is nonnegative, then

$x\bullet_{R}y=-(|x|\bullet_{R}|y|)$.

These formulas are noticeably more complicated signifying a great difference between the real number system and that of the naturals, integers, and rationals.

In an effort to clearly define the structure of these number systems and the arithmetic within them mathematicians created a formal, yet highly complex new construction. There now exists a very detailed formulation for each type of number, however in many ways it is just as foreign as the arithmetic during ancient periods. Though intuitive feelings of basic arithmetic agree with the results given by these current constructions, it is obvious that most people would be baffled by the complexities that underwrite the mathematics students learn throughout their school lives. Though attempts were made to introduce a set theoretical background into elementary schools when teaching addition and other forms of arithmetic, these attempts in general failed. Much like arithmetic with ancient Egyptians and the Babylonians, these formal concepts of arithmetic are not known or considered by mainstream society and are left to only a select group.

References

Aaboe, Asger. Episodes from The Early History of Mathematics. Random House Inc. New York. 1964.

Bunt, L., Jones, P., Bedient, J. The Historical Roots of Elementary Mathematics. Prentice-Hall Inc. Englewood Cliffs, New Jersey.1976.

Calinger, Ronald. A Contextual History of Mathematics. Prentice-Hall Inc. Upper Saddle River. 1999.

Coming of Age in Ancient Greece.
http://hoodmuseum.dartmouth.edu/exhibitions/coa/re_high_games.html

Cuomo, S. Ancient Mathematics. Routledge. New York. 2001.

Enderton, Herbert B. Elements of Set Theory. Academic Press. San Diego. 1977

Gullberg, Jan. Mathematics: From the Birth of Numbers. W.W. Norton and Company. 1997.

Hanjnal, András and Hamburger, Peter. Set Theory. Cambridge University Press. Cambridge. 1999.

Hayden, Seymour and Kennison, John F. Zermelo-Fraenkel Set Theory. Charles E. Merrill Publishing Company. Columbus. 1968.

Heath, Thomas. A History of Greek Mathematics: Volume I From Thales to Euclid. Dover Publications, Inc. New York. 1981

Johnson, Phillip E. A History of Set Theory. Prindle, Weber & Schmidt, Inc. Boston. 1972.

Kline, Morris. Mathematical Thought From Ancient to Modern Times Volume 3. Oxford University Press. New York. 1972

MacTutor History of Mathematics Archive. http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Dedekind.html

Robins, G., Shute, C., The Rhind Mathematical Papyrus an Ancient Egyptian Text. Dover Publications Inc. New York. 1987.