1. Let $D$ be the region between the curves $y = 5 - x^2$ and $y = x^2 - 3$. Sketch the region and compute the integral of $f(x, y) = x^2$ over this region.

Intersections:

$5 - x^2 = x^2 - 3$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

(a) $\int_{-2}^{2} \int_{x^2-3}^{5-x^2} x^2 \, dy \, dx$

\[
= \int_{-2}^{2} \left[ x^2 y \right]_{x^2-3}^{5-x^2} \, dx
\]

\[
= \int_{-2}^{2} x^2 (5 - x^2 - (x^2 - 3)) \, dx
\]

\[
= \int_{-2}^{2} (5x^2 - x^4 - x^4 + 3x^2) \, dx
\]

\[
= \int_{-2}^{2} 8x^2 - 2x^4 \, dx
\]

\[
= \left[ \frac{8}{3} x^3 - \frac{2}{5} x^5 \right]_{-2}^{2}
\]

\[
= \frac{8}{3} (2)^3 - \frac{2}{5} (2)^5 - \frac{8}{3} (-2)^3 + \frac{2}{5} (-2)^5
\]

\[
= 16 \cdot \frac{8}{3} - \frac{4 \cdot 32}{5} = 128 \left( \frac{1}{3} - \frac{1}{3} \right) = \frac{256}{15}
\]
2. Compute the volume of the region $\mathcal{R}$ sitting above the triangle bounded by $x = 0$, $y = 0$ and $y = 1 - x$ in the $xy$-plane, and between the planes $x + y + z = 5$ and $2x + y + 3z = 6$.

\[
2 = 5 - x - y \quad \quad z = \frac{1}{3}(6 - 2x - y)
\]

\[
\int_0^1 \int_0^{1-x} \int_0^{5-x-y} \frac{1}{\frac{1}{3}(6-2x-y)} \, dz \, dy \, dx \quad (\text{x})
\]

\[
= \int_0^1 \int_0^{1-x} 5 - x - y - \frac{1}{3} (6 - 2x - y) \, dy \, dx \quad (\text{v})
\]

\[
= \int_0^1 \int_0^{1-x} 3 - \frac{1}{3} x - \frac{2}{3} y \, dy \, dx \quad (\text{v})
\]

\[
= \int_0^1 \frac{1}{3} (4x^2) - \frac{1}{6} xy - \frac{1}{3} y^2 \bigg|_0^{1-x} \, dx
\]

\[
= \int_0^1 \frac{1}{3} (1-x) - \frac{1}{3} x + \frac{1}{3} x^2 - \frac{1}{3} (1-x)^2 \, dx \quad (\text{vi})
\]

\[
= \left[ -\frac{3}{2} (1-x)^2 - \frac{1}{6} x^2 + \frac{1}{9} x^3 + \frac{1}{9} (1-x)^3 \right]_0^1
\]

\[
= \frac{1}{9} - \frac{1}{6} + \frac{3}{2} - \frac{1}{9} = \frac{8}{6} = \frac{4}{3} \quad (\text{vii})
\]
3. Find the integral of \( f(x, y, z) = x + z \) over the region inside the hemisphere of radius 4 where \( y \geq 0 \), and above the plane \( z = 2 \).

\[
\begin{align*}
\mathcal{V} = 2 & \rightarrow \rho \cos \varphi = 2 \\
4 = \rho & = \frac{2}{\cos \varphi} \\
(\cos \varphi) & = \frac{1}{2} \quad (2) \quad \varphi = \frac{\pi}{3}, \\
\rho & = \sqrt{3}.
\end{align*}
\]

\[
\begin{align*}
& \rightarrow \quad \int_0^{\pi/3} \int_0^{\pi} \int_0^4 \left( \rho \cos \varphi \sin \psi + \rho \sin \varphi \right) \rho^2 \sin \varphi \ d\rho \ d\varphi \ d\psi, \\
& = \int_0^{\pi/3} \int_0^{\pi} \int_0^4 \rho^3 \cos \varphi \sin^2 \psi + \rho^2 \cos \varphi \sin \psi \ d\rho \ d\varphi \ d\psi, \\
& = \int_0^{\pi/3} \int_0^{\pi} \left( 6 - \frac{4}{\cos \varphi} \right) \left( \cos \theta \sin^2 \psi + \cos \varphi \sin \psi \right) \ d\theta \ d\varphi \\
& \quad \left. \left[ 6 - \frac{4}{\cos \varphi} \right] \left( \sin \theta \sin^2 \psi + \cos \varphi \sin \psi \right) \right|_0^\pi \\
& = \pi \int_0^{\pi/3} 6 \cos \varphi \sin \psi \ d\psi - 4 \frac{\sin \varphi}{\cos^2 \varphi} \ d\varphi, \\
& = \pi \int_0^{\pi/3} \left[ -3\cos^2 \varphi \sin \psi \right]_0^\pi + \left[ -3 \frac{\sin \varphi}{\cos^2 \varphi} \right]_0^\pi \\
& = 18 \pi.
\end{align*}
\]