1. Find the global extreme values of \( f(x,y) = 2x^3 + x^2y + y^2 \) on the domain given by \( x, y \geq 0 \) and \( 2x + y \leq 1 \).

First we find the critical points of the function by taking derivatives with respect to \( x \) and \( y \), and setting them equal to zero:

\[
\begin{align*}
    f_x &= 6x^2 + 2xy = 0 \\
    f_y &= x^2 + 2y = 0.
\end{align*}
\]

After substituting, we see these two equations have 2 solutions, namely \( x = 0, y = 0 \) and \( x = -6, y = -18 \). The first solution is in our domain, while the second solution is not. Therefore, we consider only the first solution. So we note that \( f(0,0) = 0 \).

Now we test the points on the boundary (See figure 1 in the back). First we test along the line where \( y = 0 \), and \( x \) is between 0 and \( \frac{1}{2} \). \( f(x,0) = 2x^3 \), so \( f'(x,0) = 6x^2 \), which equals zero when \( x = 0 \). We already know that \((0,0)\) is a critical point, so we only need to test the endpoints of \( x \). So all that remains is to note that for the boundary critical point \((\frac{1}{2},0)\), \( f(\frac{1}{2},0) = \frac{1}{4} \).

Now we test along the line where \( x = 0 \), and \( y \) is between 0 and \( 1 \). \( f(0,y) = y^2 \), so \( f'(0,y) = 2y \), which equals zero when \( y = 0 \). Again, we already know that \((0,0)\) is a critical point, so we only need to test the endpoints of \( y \). So all that remains is to note that for \((0,1)\), \( f(0,1) = 1 \).

Finally, we need to test along the line where \( y = 1 - 2x \), where \( x \) is between 0 and \( \frac{1}{2} \). \( f(x,1-2x) = 2x^3 + x^2(1-2x) + (1-2x)^2 = x^3 + (1-2x)^2 \). So \( f'(x,1-2x) = 2x - 4(1-2x) = 10x - 4 \), which equals 0 when \( x = \frac{2}{5} \). So we note that \( f(\frac{2}{5},1-2(\frac{2}{5})) = f(\frac{2}{5}, \frac{1}{5}) = \frac{1}{5} \). We also need to test the end points, but we already found \( f(0,1) \) and \( f(\frac{1}{2},0) \). So we are done finding extreme values.

So the minimum is \( f(0,0) = 0 \) and the maximum is \( f(0,1) = 1 \).
2. Apply the method of Lagrange multipliers to find the maximum and minimum values of \( f(x, y) = x^2 + y^2 \), subject to the constraint \( x^4 + y^4 = 1 \).

First, we set \( g(x, y) = x^4 + y^4 - 1 \), so that \( g(x, y) = 0 \) is our constraint. To use Lagrange multipliers, we set \( \nabla f = \lambda \nabla g \), which gives \( (2x, 2y) = \lambda (4x^3, 4y^3) \). So we have the two equations \( 2x = \lambda 4x^3 \) and \( 2y = \lambda 4y^3 \). This gives rise to 3 cases: Either \( x = 0 \), \( y = 0 \), or neither. (Note that both cannot be 0, because that would violate the constraint. Also note that \( \lambda \) can not equal 0, because if \( \lambda \) were to equal 0, then that would force both \( x \) and \( y \) to equal 0, which we just noted cannot happen.)

If \( x = 0 \), then the constraint tells us that \( y = \pm 1 \). This makes \( f(0, \pm 1) = 1 \). Similarly, if \( y = 0 \), then the constraint tells us that \( x = \pm 1 \). This makes \( f(\pm 1, 0) = 1 \). So the last case to check is when neither are 0.

We divide the first equation by \( 4x \lambda \) and the second equation by \( 4y \lambda \). So we have \( x^2 = \frac{1}{2x} \) and \( y^2 = \frac{1}{2y} \). We plug this back into the constraint equation: \( \left( \frac{1}{2x} \right)^2 + \left( \frac{1}{2y} \right)^2 = 1 \), so \( \lambda = \frac{\sqrt{2}}{2} \). (Note that \( \lambda \) cannot be \( -\frac{\sqrt{2}}{2} \) because \( x^2 = \frac{1}{2x} \) implies \( x^2 \) cannot be negative.) So after plugging in \( \lambda = \frac{\sqrt{2}}{2} \) for \( x^2 \) and \( y^2 \), we get \( f(x, y) = \sqrt{2} \). So the minimum value of \( f \) is 1 and the maximum value is \( \sqrt{2} \).
3. (a) Let \( f(x, y) = xy^2e^{x^2-xy} \) and \( P = (1, 1) \). Find the rate of change of \( f \) in the direction of a unit vector making an angle \( \frac{\pi}{3} \) with \( \nabla f_P \).

The rate of change of \( f \) in the direction of a unit vector making an angle \( \frac{\pi}{3} \) with \( \nabla f_P \) is \( ||\nabla f_P||\cos \frac{\pi}{3} \).

\[
\begin{align*}
  f_x &= xy^2(2x - y)e^{x^2-xy} + y^2e^{x^2-xy} \\
  f_y &= 2xye^{x^2-xy} - x^2y^2e^{x^2-xy},
\end{align*}
\]

so \( \nabla f_P = (2, 1) \). So
\[
||\nabla f_P||\cos \frac{\pi}{3} = \sqrt{5}\frac{\sqrt{3}}{2}.
\]

(b) Find the two points on the ellipsoid \( x^2 + \frac{y^2}{8} + 2z^2 = 10 \), where the tangent plane is normal to \( v = \langle -1, 2, 4 \rangle \).

\( \nabla f \) is normal to the surface \( f \) at each point of \( f \). So we want \( \nabla f \) to be in the same direction as \( v \). So we set \( \nabla f = \lambda v \), where \( \lambda \) is some constant. So \( \langle 2x, \frac{y}{2}, 4z \rangle = \lambda(-1, 2, 4) \). So \( x = \frac{-\lambda}{2} \), \( y = 8\lambda \), and \( z = \lambda \). So we plug these values into the original equation \( f \), and we get that \( \lambda = \pm \sqrt{\frac{20}{41}} \). So we plug these points back into the values of \( x, y, \) and \( z \) to get the two desired points: \((-\sqrt{\frac{10}{41}}, 8\sqrt{\frac{10}{41}}, \sqrt{\frac{10}{41}})\) and \((\sqrt{\frac{10}{41}}, -8\sqrt{\frac{10}{41}}, -\sqrt{\frac{10}{41}})\).
4. Let $D$ be the region bounded by the curves $xy = 3$, $xy = \frac{1}{3}$, $y = 4x$ and $y = \frac{4}{3}$. Find the map that transforms the region $D$ into a rectangular box in the $uv$-plane. After that, compute

$$\int \int_D \frac{ye^y}{x} \, dxdy,$$

using the change of variables formula.

First, we want to take our four equations and put all of the variables on one side. The first two equations are fine. For the last two equations, we divide both sides by $x$. Now substitute $u = xy$, $v = \frac{x}{y}$. This tells us that $\frac{1}{3} \leq u \leq 3$ and $\frac{1}{4} \leq v \leq 4$. We want to find the Jacobian of the transformation $G(u,v)$, so we calculate the Jacobian of $G^{-1}(x,y)$

$$= \left| \begin{array}{cc}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{array} \right| = \left| \begin{array}{cc}
y & x \\
x & -y
\end{array} \right| = 2\frac{y}{x}. \text{ Therefore, the Jacobian of } G = 2\frac{y}{x} = \frac{1}{2v}. \text{ Finally, we note that } \frac{we^w}{x} = ve^v, \text{ so now we can set up the integral:}

$$\int_{u=\frac{1}{3}}^{u=\frac{1}{3}} \int_{v=\frac{4}{3}}^{v=\frac{4}{3}} ve^v \left( \frac{1}{2x} \right) \, dvdu = \frac{1}{2} \int_{u=\frac{1}{3}}^{u=\frac{1}{3}} \int_{v=\frac{4}{3}}^{v=\frac{4}{3}} e^v \, dvdu = \frac{1}{2} \int_{u=\frac{1}{3}}^{u=\frac{1}{3}} \left( e^v \right) du = \frac{1}{2} \left( e^{\frac{4}{3}} - e^{\frac{1}{3}} \right).$$
5. Integrate \( \int \int_D (x^2 + y^2)^{-2} \, dx \, dy \), where \( D \) is the region in the plane given by \( x^2 + y^2 \leq 4, y \geq \sqrt{2} \).

(I am only writing up a solution where we first integrate with respect to \( r \), and then with respect to \( \theta \). If you would like to see how to do it with the variables reversed, feel free to ask.)

The hint suggests that we use polar coordinates: \( x = r \cos \theta, y = r \sin \theta \). First we find bounds for \( \theta \). The first equations tells us that \( r^2 \leq 4 \), so \( r \leq 2 \). From the second equation, we get that \( r \sin \theta \geq \sqrt{2} \). So \( \sin \theta \geq \frac{\sqrt{2}}{2} \), which occurs when \( \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \). We also just got our bounds for \( r \), namely \( \frac{\sqrt{2}}{\sin \theta} \leq r \leq 2 \). (See figure 2 for a picture of the region.) Now we need to turn the function inside the integral into polar coordinates: \( (x^2 + y^2)^{-2} = (r^2)^{-2} = r^{-4} \). Finally, we can set up the integral:

\[
\frac{3\pi}{4} \int_{\theta = \pi/4}^{3\pi/4} \frac{1}{2} \int_{r = \sqrt{2}/\sin \theta}^{2} r^{-4} r \, dr \, d\theta
\]

\[
= \frac{3\pi}{4} \int_{\theta = \pi/4}^{3\pi/4} \frac{1}{2} \left[ -\frac{1}{3} r^{-3} \right]_{r = \sqrt{2}/\sin \theta}^{2} \, d\theta
\]

\[
= -\frac{1}{2} \frac{3\pi}{4} \int_{\theta = \pi/4}^{3\pi/4} \left( \frac{1}{4} - \frac{1}{2} \sin^2 \theta \right) \, d\theta
\]

\[
= -\frac{1}{2} \frac{3\pi}{4} \int_{\theta = \pi/4}^{3\pi/4} \frac{\cos 2\theta}{4} \, d\theta
\]

\[
= -\frac{1}{2} \frac{3\pi}{4} \left( \frac{\sin 2\theta}{8} \right)_{\theta = \pi/4}^{3\pi/4}
\]

\[
= -\frac{1}{2} \frac{3\pi}{4} \left( \frac{1}{8} - \frac{1}{8} \right)
\]

\[
= \frac{1}{8}
\]
6. Find \( \iiint_V z(x^2 + y^2 + z^2)^{-\frac{3}{2}} \, dx \, dy \, dz \) if 
\[ V = \{(x, y, z) | x^2 + y^2 + z^2 \leq 64, z \geq 4 \} \].

Since \( V \) is bound by a sphere and a plane (see figure 3), we will use spherical coordinates for this problem. The first equation that bounds \( V \) is \( x^2 + y^2 + z^2 \leq 64 \), so in spherical coordinates that is \( \rho^2 \leq 64 \), so \( \rho \leq 8 \). The second equation is \( z \geq 4 \), so in spherical coordinates that is \( \rho \cos \phi \geq 4 \). None of these equations affect \( \theta \), so we have that \( 0 \leq \theta \leq 2\pi \).

Now we get bounds for \( \phi \). From the two equations bounding \( V \) written in spherical coordinates, we get that \( \cos \phi \geq \frac{1}{2} \). This occurs when \( 0 \leq \theta \leq \frac{\pi}{3} \).

The two equations also gave us the bounds for \( \rho \), namely \( 4 \sec \phi \leq \rho \leq 8 \).

Now that we have the bounds, we can rewrite the function in terms of spherical coordinates: 
\( z(x^2 + y^2 + z^2)^{-\frac{3}{2}} = \rho^{-2} \cos \phi \). Finally, we can set up the integral:
\[
\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{3}} \int_{\rho=4 \sec \phi}^{8} \rho^{-2} \cos(\rho^2 \sin \phi) \rho \, d\rho \, d\phi \, d\theta
\]
\[
= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{3}} \int_{\rho=4 \sec \phi}^{8} \cos(\rho^2 \sin \phi) \rho \, d\rho \, d\phi \, d\theta
\]
\[
= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{3}} (8 \cos \phi \sin \phi - 4 \sin \phi) \, d\phi \, d\theta
\]
\[
= \int_{\theta=0}^{2\pi} (4 \sin 2 \phi - 4 \sin \phi) \, d\phi \, d\theta
\]
\[
= \int_{\theta=0}^{2\pi} [-2 \cos 2 \phi + 4 \cos \phi]_{\phi=0}^{\frac{\pi}{3}} \, d\theta
\]
\[
= \int_{\theta=0}^{2\pi} (1 + 2 + 2 - 4) \, d\theta
\]
\[
= 2\pi.
\]