Math 244: Differential Equations for Engineers

Summer 2003, Section C1

Assignment 2: Modeling with First-Order Equations (13 points)

Due Monday, June 9, 2003

In this assignment, you will be asked to use first-order differential equations to work through a relatively complicated, multi-step modeling process. Please write out your solutions to the problems below on one or more separate sheets of paper. Write neatly and in a well-organized fashion. Write in clear, complete sentences, using diagrams and equations where appropriate. Show all your work, including the methods you use for solving the differential equations you are asked to solve; you will not receive full credit for simply writing down answers.

For parts (d) and (e) of problem 1, you will need to solve equations numerically that are difficult to solve by hand. For parts (d) and (e) of problem 2, you will need to sketch graphs of fairly complicated functions. A graphing calculator with equation-solving capabilities (e.g. TI-92) or a computer algebra system (e.g. Maple) is helpful in completing these portions of the assignment; if you need help finding or using such a system, please ask me about it. When you provide numerical approximations to answers, give the approximations to three decimal places.

1. A basketball player is making a desperate last attempt at a basket in the final seconds of an NBA championship game. He is standing 85 feet away from the basket (almost all the way across the court!). The basket is 10 feet high, and he is throwing the ball from a height of 6 feet, as shown in Figure 1.

The velocity of the ball has two components, a horizontal and a vertical component. Let $v(t)$ and $w(t)$ denote the horizontal and vertical velocity respectively t seconds after the ball leaves the player's hands. If the player throws the ball with initial speed S at an angle of A radians above the horizontal, then the initial values of v and w are $v(0) = S\cos A$ and $w(0) = S\sin A$, as shown in Figure 2.

Assume for now that there is no air resistance, so as the ball flies through the air the only force acting on it is gravity, which affects only the vertical component of the velocity. Let g be the acceleration due to gravity.

a. (1 point) Write down differential equations that the functions $v(t)$ and $w(t)$ satisfy. Then solve these equations, using the initial conditions given, to obtain expressions for $v(t)$ and $w(t)$.

SOLUTION: The only force acting on the ball as it flies through the air is gravity, and gravity affects only the vertical component $w(t)$. The acceleration due to gravity is constant, so we have

$$\frac{dv}{dt} = 0$$

$$\frac{dw}{dt} = -g$$

and integrating both sides of these equations gives $v = c$, $w = -gt + c$.

To determine the values of the constants, we use the fact that $v(0) = S\cos A$ and $w(0) = S\sin A$. These give $v = S\cos A$, $w = -gt + S\sin A$.

b. (1 point) Using the expressions obtained in part (a), find the functions $x(t)$ and $y(t)$ that give the horizontal and vertical position of the ball at time t.

SOLUTION: Since velocity is the derivative of position, we need only integrate both sides of the expressions for v and w to obtain $x = S\cos At + c$, $y = -\frac{1}{2}gt^2 + S\sin At + c$. To solve for the constants, we observe that $x(0) = 0$ and $y(0) = 6$ so $x = S\cos At$ and $y = -\frac{1}{2}gt^2 + S\sin At + 6$.

c. (2 points) Suppose that the acceleration due to gravity is 32 ft/sec^2. Using the equations from part (b), write down a function $f(S, A)$ such that the player makes the basket if and only if $f(S, A) = 0$. Explain why your equation must hold. (Hint: when does the ball cover the horizontal distance to the basket?)

SOLUTION: If the ball goes through the basket, there must be some time t such that $x(t) = 85$ and $y(t) = 10$. That is, such a time t must satisfy the two equations

$$S\cos At = 85$$

$$-16t^2 + S\sin At + 6 = 10$$

Solving the first equation for t yields $t = \frac{85}{S\cos A}$. Substituting this value for t into the second equation yields

$$-16(\frac{85}{S\cos A})^2 + S\sin A(\frac{85}{S\cos A}) + 6 = 10$$

which simplifies to

$$\frac{-115600}{S^2\cos^2 A} + 85\tan A - 4 = 0$$

which is the expression in S and A we want.

d. (1 point) Suppose the player throws the ball at an initial speed of $S = 70$ ft/sec. Use your equation from part (c) to determine the value(s), if any, of A which will allow the player to make the basket.

If $S = 70$ then A must satisfy the equation

$$-\frac{115600}{4900\cos^2 A} + 85\tan A - 4 = 0$$

Asking Maple to solve this equation for A yields four (approximate) solutions, only two of which fall within the bounds of $0 \leq A \leq \pi/2$– the angles at which the shooter could actually throw the ball forward and upward. These two solutions are (to three decimal places) .346 radians and 1.272 radians.

e. (2 points) Use your equation from part (c) to find the angle A which minimizes the speed S at which the player must throw in order to make the basket. For that optimal angle A, find the required speed S. (Hint: try to collect the terms involving A into a single coefficient, and figure out what should happen to that coefficient in order to minimize the required value of S).

SOLUTION: Solving the equation given in part (c) for S^2 yields

$$S^2 = \frac{115600 \sec^2 A}{85 \tan A - 4}$$

To minimize the required value of S, then, one should minimize the right-hand side as a function of A. To find a minimum of this function, we differentiate it and set the derivative equal to 0. Doing this gives $A \approx 0.809$ radians, and plugging this value of A back in to the equation above yields $S \approx 53.394$ ft/sec.

2. Now assume that as the ball flies through the air it is affected by air resistance. Suppose that the vertical and horizontal components of the acceleration due to air resistance are given by $-\mu v$ and $-\mu w$ respectively, where μ is the coefficient of drag.

a. (1 point) Under this new assumption, write down differential equations that v and w must satisfy, and solve these equations using the same initial conditions as in problem 1 to obtain expressions for $v(t)$ and $w(t)$.

SOLUTION: Now drag is acting on the horizontal velocity, and gravity as well as drag on the vertical velocity, so we have

$$\frac{dv}{dt} = -\mu v$$

$$\frac{dw}{dt} = -g - \mu w$$

These are both first-order linear equations; solving them yields $v(t) = ce^{-\mu t}$ and $w(t) = -\frac{g}{\mu} + ce^{-\mu t}$. Again we use the initial conditions $v(0) = S \cos A$, $w(0) = S \sin A$ to fill in the constants; this gives $v(t) = (S \cos A)e^{-\mu t}$ and $w(t) = -\frac{g}{\mu} + (\frac{g}{\mu} + S \sin A)e^{-\mu t}$.

b. (1 point) Using these expressions, find $x(t)$ and $y(t)$ just as you did in problem 1.

SOLUTION: Again, we need only integrate both sides of the equations for v and w to get equations for x and y, then use the initial conditions $x(0) = 0$, $y(0) = 6$ to fill in the constants. Doing so yields

$$x(t) = -\frac{(S \cos A)e^{-\mu t}}{\mu} + (\frac{S \cos A}{\mu})$$

$$y(t) = -\frac{gt}{\mu} - (\frac{g}{\mu^2} + \frac{S \sin A}{\mu})e^{-\mu t} + (\frac{g}{\mu^2} + \frac{S \sin A}{\mu} + 6)$$

c. (1 point) Suppose that $\mu = 1/5$. As in problem 1, write down a function $f(S, A)$ such that the player makes the basket if $f(S, A) = 0$.

SOLUTION: As before, there must exist some time t such that

$$85 = -5(S \cos A)e^{-t/5} + 5S \cos A$$

$$10 = -160t - (800 + 5S\sin A)e^{-t/5} + (800 + 5S\sin A + 6)$$

(this comes from substituting $1/5$ for μ and 32 for g above). Solving for t in the first equation yields

$$e^{-t/5} = -\frac{85 - 5S\cos A}{5S\cos A} = -\frac{85}{5S\cos A} + 1$$

$$t = -5\ln(-\frac{17}{S\cos A} + 1)$$

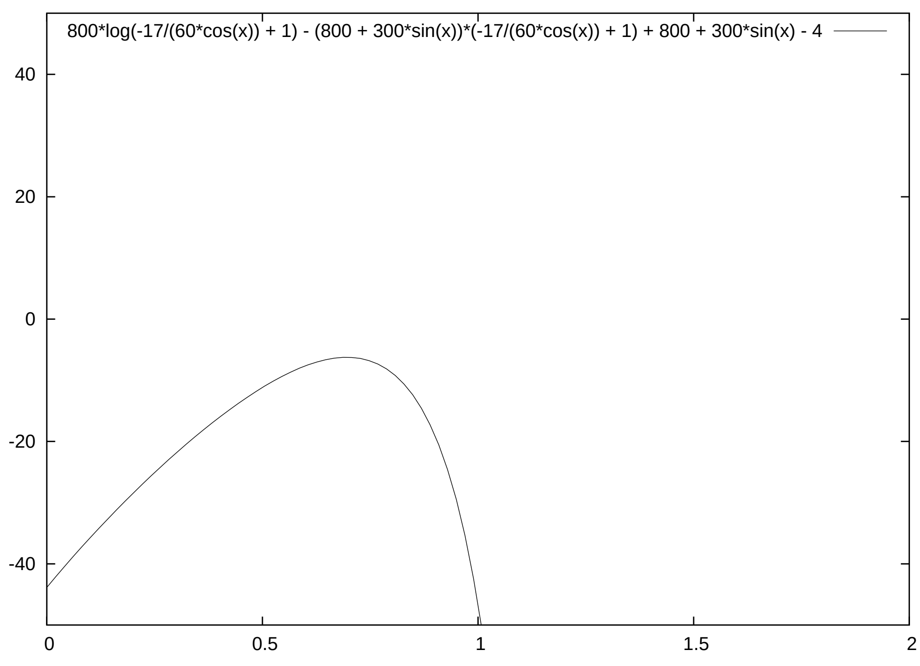
Substituting this into the second equation gives

$$800\ln(-\frac{17}{S\cos A} + 1) - (800 + 5S\sin A)\cdot(-\frac{17}{S\cos A} + 1) + (800 + 5S\sin A + 6) - 10 = 0$$

which is the equation we want.

d. (1 point) Suppose the player releases the ball at initial speed $S = 60$ ft/sec. Sketch a graph of $g(A) = f(60, A)$ for $0 \leq A \leq \pi/2$. Are there any values of A which allow the player to make the basket?

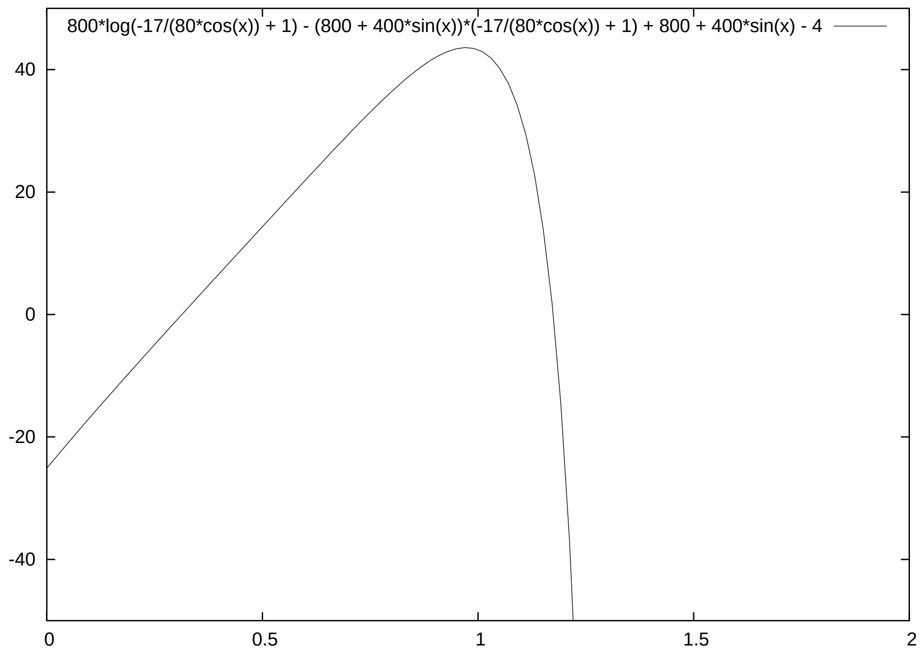
SOLUTION: A plot of $f(60, A)$ for $0 \leq A \leq \pi/2$ (actually $0 \leq A \leq 2$) is shown below.



It is readily apparent from this plot that $f(60, A)$ has no zeroes in the range $0 \leq A \leq \pi/2$. Therefore, at this speed there is no angle which allows the player to make the basket.

e. (1 point) Suppose now that $S = 80$ ft/sec. Sketch a graph of $h(A) = f(80, A)$ for $0 \leq A \leq \pi/2$. Are there any values of A which allow the player to make the basket?

SOLUTION: A plot of $f(80, A)$ is shown below.



In this case, the function $f(80, A)$ clearly does have zeroes, so there are values of A allowing the player to make the basket.

f. (1 point) Given your results from parts (d) and (e), what can you say about the minimum speed at which the player now must throw the ball in order to make the basket?

SOLUTION: The minimum speed required has now increased (which we would expect given the application of drag); it is more than 60 ft/sec but less than 80 ft/sec.