**Math 244: Differential Equations for Engineers**

**Summer 2003, Section C1**

**Assignment 3: Modeling with Second-Order Equations (13 points)**

**Due Monday, June 16, 2003**

In this assignment, you will be asked to use second-order differential equations to work through a relatively complicated, multi-step modeling process. Please write out your solutions to the problems below on one or more separate sheets of paper. Write neatly and in a well-organized fashion. Write in clear, complete sentences, using diagrams and equations where appropriate. Show all your work, including the methods you use for solving the differential equations you are asked to solve; you will not receive full credit for simply writing down answers.

1. A mass weighing 6 lb stretches a spring 4 in. Suppose the mass is pulled downward an additional 3 in and then released. Assume first that there is no damping, and no external force acts on the mass.

a. (2 points) Determine the position $u$ of the mass at any time $t$.

SOLUTION: The general differential equation describing undamped free motion is $mu'' + ku = 0$. We have $m = 6/32$ lb/ft-sec^2 and $k = 6/(4/12) = 18$, so our initial value problem is $6/32u'' + 18u = 0$, $u(0) = 1/4$, $u'(0) = 0$ (this last because there is no initial push on the mass).

Multiplying the equation through by $32/6$ gives $u'' + 96u = 0$. The characteristic equation is $r^2 + 96 = 0$ which has roots $\pm\sqrt{96}i$. Thus the general solution to the equation is $u = c_1 \cos(4\sqrt{6}t) + c_2 \sin(4\sqrt{6}t)$.

The initial conditions then give $c_1 = \frac{1}{4}$ and $c_2 = 0$, so we end up with

$$u = \frac{1}{4}\cos(4\sqrt{6}t)$$

b. (2 points) Find the frequency, period, amplitude, and phase of the motion.

SOLUTION: From the equation above, we see that the frequency of the motion (oscillations per $2\pi$ time units) is $4\sqrt{6}$, the period is $\frac{2\pi}{4\sqrt{6}}$, the amplitude is $1/4$ feet ($=$ 3 inches) and the phase is 0.

2. Now suppose the mass is damped by attaching it to a viscous damper. Suppose the damper exerts a force of $a$ lb when the velocity of the mass is 5 ft/sec.

a. (2 points) If $a = 1$, find the position of the mass $u$ at any time $t$.

The drag coefficient given by the damper is $1/5$ lb/ft-sec, so our new initial value problem is $\frac{6}{32}u'' + \frac{1}{5}u' + 18u = 0$, $u(0) = 1/4$, $u'(0) = 0$. Multiplying through the equation by $32/6$ gives $u'' + \frac{16}{15}u' + 96u = 0$. This equation has characteristic polynomial $r^2 + \frac{16}{15}r + 96$, whose roots are $-\frac{8}{15} \pm \sqrt{96 - \frac{64}{225}}i$. Thus the general solution is $u = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t$ where $\lambda = -\frac{8}{15}$ and $\mu = \sqrt{96 - \frac{64}{225}}$.

Again the initial conditions give $c_1 = \frac{1}{4}$. Since now $u' = c_1(\lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t) + c_2(\lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t)$, we have $\lambda c_1 + \mu c_2 = 0$ so $c_2 = -\frac{\lambda c_1}{\mu} = \frac{2}{15\sqrt{96-\frac{64}{225}}} = \frac{1}{2\sqrt{1346}}$. Thus we have

$$u = \frac{1}{4}e^{-\frac{8}{15}t}\cos(\sqrt{96 - \frac{64}{225}}t) + \frac{1}{2\sqrt{1346}}\sin(\sqrt{96 - \frac{64}{225}}t)$$

b. (3 points) Find the value of $a$ such that the system is critically damped.

SOLUTION: In general we have $\gamma = a/5$. Critical damping occurs when $\gamma^2 - 4km = 0$. This yields the equation

$$\frac{a^2}{25} - 4 \cdot 18 \cdot \frac{6}{32} = 0$$

Solving this yields $a = \pm\sqrt{337.5}$ and as the drag coefficient must be positive, we get $a = \sqrt{337.5}$ as the right value for critical damping.

3. Finally, suppose the mass is undamped, but is acted on by an external force of $3\cos\omega t$ lb.

a. (2 points) If $\omega = 2$, find the position of the mass $u$ at any time $t$.

Now our initial value problem is $\frac{6}{32}u'' + 18u = 3\cos 2t$. We know that the general solutions to the homogeneous equation are $u = c_1\cos(4\sqrt{6}t) + c_2\sin(4\sqrt{6}t)$. So it remains only to find a particular solution of the nonhomogeneous equation. To do this, we use the method of undetermined coefficients (you could also use variation of parameters).

Since $\cos 2t, \sin 2t$ are not solutions of the homogeneous equation we may guess $Y = A\cos 2t + B\sin 2t$. Then $Y' = -2A\sin 2t + 2B\cos 2t$ and $Y'' = -4A\cos 2t - 4B\sin 2t$, so we have

$$\frac{6}{32}(-4A\cos 2t - 4B\sin 2t) + 18(A\cos 2t + B\sin 2t) = 3\cos 2t$$

which yields the two equations $\frac{69}{4}A = 3$ and $\frac{69}{4}B = 0$. We conclude that $Y = \frac{4}{23}\cos 2t$ is a particular solution, so our general solution is

$$u = c_1\cos(4\sqrt{6}t) + c_2\sin(4\sqrt{6}t) + \frac{4}{23}\cos 2t$$

Now our initial condition $u(0) = \frac{1}{4}$ yields $c_1 + \frac{4}{23} = \frac{1}{4}$ so $c_1 = \frac{1}{4} - \frac{4}{23} = \frac{7}{92}$, and $u'(0) = 0$ again yields $c_2 = 0$. Thus the solution to our initial value problem is

$$u = \frac{7}{92}\cos(4\sqrt{6}t) + \frac{4}{23}\cos 2t$$

b. (2 points) Find the value of $\omega$ such that resonance occurs.

SOLUTION: Resonance occurs when the frequency of the periodic forcing is the same as the natural (unforced) frequency of the spring's oscillation; thus $\omega = 4\sqrt{6}$ is the value such that resonance occurs.