Math 244: Differential Equations for Engineers

Summer 2003, Section C1

Assignment 4: A Tale of Three Tanks (13 points)

Due Monday, June 23, 2003

In this assignment, you will be asked to formulate a physical modeling problem as a systems of differential equations. Please write out your solutions to the problems below on one or more separate sheets of paper. Write neatly and in a well-organized fashion. Write in clear, complete sentences, using diagrams and equations where appropriate. Show all your work, including the methods you use for solving the differential equations you are asked to solve; you will not receive full credit for simply writing down answers.

A production plant for water-based food coloring uses a system of three tanks to produce varied concentrations of dyes. Tank A initially contains 10 gallons of water, while Tank B contains 20 gallons and Tank C contains 30 gallons.

Basic dyes are input from dye sources into each of these three tanks as follows: water containing 4 oz/gal of dye flows into Tank A at a rate of 1 gal/min; water containing 3 oz/gal of dye flows into Tank B at a rate of 2 gal/min; and water containing 5 oz/gal of dye flows into Tank C at a rate of 2 gal/min.

The dye-containing water then circulates among the tanks as follows: water flows from Tank A to Tank B at a rate of 1 gal/min; from Tank B to Tank C at a rate of 1 gal/min; and from Tank C to Tank A at a rate of 1 gal/min.

Finally, water is output from the tanks through stopcocks in the bottom. The output rates are as follows: 1 gal/min flows out of Tank A and 2 gal/min flows out from each of Tanks B and C. Thus in each tank the total input rate equals the total output rate, so the total amount of water in each tank remains constant over time. A flow diagram illustrating the situation is on the back.

But the amount of dye in that water changes, and we want to figure out how it changes over time. Let $Q_1(t)$ denote the amount of dye in Tank A at time t, $Q_2(t)$ the amount in Tank B, and $Q_3(t)$ the amount in Tank C.

a. (7 points) Assume the water in each tank is always well-stirred, i.e. the concentration of dye in each tank is always uniform. Formulate a system of differential equations that model the flow process described above. Is the system of equations homogeneous or nonhomogeneous?

b. (6 points) The initial conditions corresponding to this system are the quantities $Q_1(0), Q_2(0), Q_3(0)$ which give the initial amounts of dye in the tanks. Find values of $Q_1(0), Q_2(0), Q_3(0)$ for which the system is in equilibrium, i.e. the amounts of dye in the tanks do not change over time. (Hint: set up a system of equations involving these three quantities, and solve it using Gaussian elimination).