Math 244: Differential Equations for Engineers

Summer 2003, Section C1

Assignment 5: A Tale of Three Tanks, Continued (13 points)

Due Monday, June 30, 2003

In this assignment, you will be asked to formulate a physical modeling problem as a system of differential equations. Please write out your solutions to the problems below on one or more separate sheets of paper. Write neatly and in a well-organized fashion. Write in clear, complete sentences, using diagrams and equations where appropriate. Show all your work, including the methods you use for solving the differential equations you are asked to solve; you will not receive full credit for simply writing down answers.

The setup for this problem is the same as for the last homework assignment, with one slight change. Here it is in full:

A production plant for water-based food coloring uses a system of three tanks to produce varied concentrations of dyes. Tank A initially contains **100** (not 10) gallons of water, while Tank B contains 20 gallons and Tank C contains 30 gallons.

Basic dyes are input from dye sources into each of these three tanks as follows: water containing 4 oz/gal of dye flows into Tank A at a rate of 1 gal/min; water containing 3 oz/gal of dye flows into Tank B at a rate of 2 gal/min; and water containing 5 oz/gal of dye flows into Tank C at a rate of 2 gal/min.

The dye-containing water then circulates among the tanks as follows: water flows from Tank A to Tank B at a rate of 1 gal/min; from Tank B to Tank C at a rate of 1 gal/min; and from Tank C to Tank A at a rate of 1 gal/min.

Finally, water is output from the tanks through stopcocks in the bottom. The output rates are as follows: 1 gal/min flows out of Tank A and 2 gal/min flows out from each of Tanks B and C. Thus in each tank the total input rate equals the total output rate. But the amount of dye in that water changes, and we want to figure out how it changes over time. Let $Q_1(t)$ denote the amount of dye in Tank A at time t, $Q_2(t)$ the amount in Tank B, and $Q_3(t)$ the amount in Tank C.

(Flip over to the other side for the problems)

a. (7 points) In the last homework you formulated a system of differential equations describing the amounts of dye in the tanks. With the new capacity for Tank A, that system should be:

$$\frac{dQ_1}{dt} = 4 + \frac{Q_3}{30} - \frac{2Q_1}{100}$$
$$\frac{dQ_2}{dt} = 6 + \frac{Q_1}{100} - \frac{3Q_2}{20}$$
$$\frac{dQ_3}{dt} = 10 + \frac{Q_2}{20} - \frac{3Q_3}{30}$$

This system is not homogeneous. Formulate the homogeneous version. For that homogeneous system, calculate the characteristic polynomial whose roots are the eigenvalues of the system. Sketch a graph of that polynomial and use a calculator or computer algebra system to approximate its roots. How many real roots does it have, and are they repeated? Based on that information, what can you say about the general solution to the homogeneous system?

b. (3 points) Now find a particular solution to the full (nonhomogeneous) system of equations using the method of undetermined coefficients.

c. (3 points) Suppose all three tanks start with pure (undyed) fresh water, so $Q_1(0) = Q_2(0) = Q_3(0) = 0$. Using the results of parts (a) and (b), what can you say about the form of the solution to this system of equations satisfying these initial conditions? How will that solution behave as $t \to \infty$?