Math 244: Differential Equations for Engineers

Summer 2003, Section C1

Assignment 6: Masses, Springs, and the Laplace Transform

Due Thursday, July 10, 2003

Please write out your solutions to the problem below on one or more separate sheets of paper. Write neatly and in a well-organized fashion. Write in clear, complete sentences, using diagrams and equations where appropriate. Show all your work, including the methods you use for solving the differential equations you are asked to solve; you will not receive full credit for simply writing down answers.

A mass weighing 8 lb stretches a spring 4 in. Assume that there is no damping and that the mass is in equilibrium at time $t = 0$.

The mass is acted on by an external force given by a function $f(t)$ defined piecewise as follows:

$$f(t) = \begin{cases} 10 & 0 \le t < 20 \\ 50 - 2t & 20 \le t < 25 \\ 0 & t \ge 25 \end{cases}$$

1. (3 points) Write $f(t)$ as a single function using the step functions $u_c(t)$ described in Section 6.3 of the textbook.

SOLUTION: Using the procedure described in class, we write

$$\begin{aligned} f(t) &= 10 + u_{20}(t)(50 - 2t - 10) - u_{25}(t)(50 - 2t) \\ &= 10 + u_{20}(t)(40 - 2t) - u_{25}(t)(50 - 2t) \end{aligned}$$

2. (3 points) Using this expression for $f(t)$, write down an initial value problem that the displacement of the mass, $u(t)$, must satisfy.

SOLUTION: The standard model for undamped spring-mass problems yields an equation of the form $mu'' + ku = f(t)$. Here we have $m = 8/32 = 1/4$ lb/ft-sec^2, and $k = 8/(1/3) = 24$, so our equation is

$$\frac{1}{4}u'' + 24u = 10 + u_{20}(t)(40 - 2t) - u_{25}(t)(50 - 2t)$$

3. (7 points) Use the Laplace transform technique to solve this initial value problem and find an expression for $u(t)$.

SOLUTION: Taking the Laplace transform of both sides of the above equation yields

$$\frac{1}{4}(s^2L(u) - s \cdot u(0) - u'(0)) + 24L(u) = \frac{10}{s} + e^{-20s}(-\frac{2}{s^2}) - e^{-25s}(-\frac{2}{s^2})$$

Note that in the last two terms on the right we use the fact that if $f(t) = -2t$ then $f(t-20) = 40 - 2t$ and $f(t-25) = 50 - 2t$. Since the mass begins at equilibrium, we have $u(0) = u'(0) = 0$. Solving for $L(u)$ therefore yields

$$\begin{aligned} L(u) &= \frac{1}{\frac{1}{4}s^2 + 24} \cdot \left(\frac{10}{s} - 2(e^{-20s} - e^{-25s})/s^2\right) \\ &= \frac{40}{s(s^2+96)} - 8\frac{(e^{-20s} - e^{-25s})}{s^2(s^2+96)} \end{aligned}$$

We invert these two terms one at a time. Using partial fractions we find that

$$\frac{40}{s(s^2+96)} = \frac{5}{12}\left(\frac{1}{s} - \frac{s}{s^2+96}\right)$$

and therefore, referring to our table of Laplace transforms,

$$L^{-1}\left(\frac{40}{s(s^2+96)}\right) = \frac{5}{12}(1 - \cos(4\sqrt{6}t))$$

For the second term, we have

$$L^{-1}\left(-8\frac{(e^{-20s} - e^{-25}s)}{s^2(s^2+96)}\right) = -8(u_{20}(t)h(t-20) - u_{25}(t)h(t-25))$$

where $h(t) = L^{-1}(\frac{1}{s^2(s^2+96)})$. By partial fractions we observe that

$$\frac{1}{s^2(s^2+96)} = \frac{1}{96}\left(\frac{1}{s^2} - \frac{1}{s^2+96}\right)$$

so $h(t) = \frac{1}{96}(t - \frac{1}{4\sqrt{6}}\sin(4\sqrt{6}t))$. Putting all this together, we conclude that

$$\begin{aligned} u(t) = \frac{5}{12}(1 - \cos(4\sqrt{6}t)) - \frac{1}{12}u_{20}(t)\left((t-20) - \frac{1}{4\sqrt{6}}\sin(4\sqrt{6}(t-20))\right) \\ + \frac{1}{12}u_{25}(t)\left((t-25) - \frac{1}{4\sqrt{6}}\sin(4\sqrt{6}(t-25))\right) \end{aligned}$$