Miscellaneous terminology: Laplacian matrices, topological minors, etc.

Note that adjacency and incidence matrices are defined in section 2 of the book.

Definition: If G is a graph on n labelled vertices, the degree matrix B is an $n \times n$ matrix such that $B_{ii} = deg(v_i)$ and $B_{ij} = 0$ for $i \neq j$.

Definition: If G is a simple graph on n labelled vertices, then the Laplacian matrix L of G is equal to $B - M$, where B is the degree matrix and M is the adjacency matrix.

So the Matrix-Tree Theorem (Theorem 10.3 of the book) can be restated as follows:

Theorem: Let G be a connected simple graph and let L be its Laplacian matrix. Then the cofactor of any element of L is equal to the number of labelled spanning trees of G.

Definition: Let G, H be graphs. We say H is a subdivision of G if H can be obtained from G by replacing some edges of G by paths.

Definition: Two graphs G_1, G_2 are homeomorphic if there exists some graph G such that both G_1 and G_2 are subdivisions of G.

Definition: H is a topological minor of G if G contains a subgraph which is a subdivision of H.

Definition: H is a minor of G if G contains a subgraph contractible to H (note that contractibility is defined in the book in Section 12).

So Theorems 12.2 and 12.3 in the book can be restated as follows:

Theorem (Kuratowski's Theorem): A graph G is planar iff neither K_5 nor $K_{3,3}$ is a topological minor of G.

Theorem: A graph G is planar iff neither K_5 nor $K_{3,3}$ is a minor of G.

In fact the proof of Theorem 12.3 in the book actually proves:

Theorem: A graph G contains one of $K_5, K_{3,3}$ as a minor iff it contains one of them as a topological minor.

And one direction of this theorem is true in more generality:

Theorem: For any G and H, if H is a topological minor of G then H is a minor of G.

(Note that the converse is not true in this generality).