**Vertex Menger's theorem, brief intro to Ramsey theory**

1. Menger's theorem– proof of the vertex version

The vertex version of Menger's theorem is Theorem 28.2 in the textbook:

Theorem: Let $G$ be a graph with vertices $v, w$. The maximum number of vertex-disjoint $vw$-paths in $G$ is equal to the minimum size of a $vw$-separating set (or $vw$-cut).

Proof: Given any $vw$-cut, we know that no set of vertex-disjoint $vw$-paths can be bigger than the size of the cut, since each path must use a different vertex from the cut. So it suffices to show that there is always a set of vertex-disjoint $vw$-paths of size equal to the minimum $vw$-cut size. We show this by induction on $|V(G)|$.

First, observe that the set of all neighbors $v$ is clearly a $vw$-cut, so the minimum size of such a cut is no more than $deg(v)$. By the same reasoning, the minimum size is also no more than $deg(w)$. We distinguish two cases:

Case 1. The only minimal $vw$-cuts are the neighborhoods of $v$ and $w$. Then if $v \cup w \cup N(v) \cup N(w)$ is not the whole $vw$-component of $G$, there is some other "intermediate" vertex $a$. Consider $G - a$; By inductive hypothesis, the maximum number of vertex-disjoint $vw$-paths in $G - a$ is the same as the minimum size of a $vw$-cut in $G - a$. But $G - a$ has the same minimal $vw$-cut size as $G$, since otherwise we could make a minimal cut in $G$ containing $a$, and $G - a$ has no more vertex-disjoint $vw$-paths than $G$, so equality for $G - a$ implies equality for $G$.

If there is no such vertex $a$, perhaps there is a vertex $u \in N(v) \cap N(w)$. Then the maximum number of vertex-disjoint $vw$-paths in $G - u$ is strictly smaller than that in $G$, so it's been reduced by at least one. We claim that the minimum size of a $vw$-cut in $G - u$ is reduced by no more than one compared to $G$; to see this, observe that if there were a smaller $vw$-cut in $G - u$, we could add $u$ to get a smaller $vw$-cut in $G$. So again, the inductive hypothesis for $G - u$ implies the desired conclusion for $G$.

Otherwise, we have disjoint nbhds $N(v), N(w)$ connected by some edges, with no intermediate vertices. Then we have reduced it to the Ḱonig-Egerváry Theorem: a set of vertex-disjoint $vw$-paths from $v$ to $w$ corresponds to a matching in the bipartite graph formed by $N(v), N(w)$, and a $vw$-cut in $G$ is a vertex cover in the bipartite graph.

Case 2. There is a minimal $vw$-cut different from $N(v), N(w)$. Call that cut $S$. Then there is a path to $v$ from each vertex in $S$, and a path to $w$ from each vertex in $S$ (otherwise $S$ is not minimal). If the $v$-paths are all disjoint and the $w$-paths are all disjoint, we can match up each $v$-path with its corresponding $w$-path to form the set of $vw$-paths we require. If an $v$-path meets a $w$-path, then we can go from $v$ to $w$ avoiding $S$, so $S$ isn't a $vw$-cut, contradicting the hypothesis.

Now let $G_1$ be the union of all the "simple" paths from $v$ to $S$, where "simple" here means that the paths meet $S$ only once at the end. Similarly let $G_2$ be the union of the paths from $w$ to $S$. Since the $v$-paths cannot meet the $w$-paths except at $S$, we know $G_1 \cap G_2 = S$. Let $|S| = m$; we want to show there are $m$ internally vertex-disjoint paths in $G_1$, and the same in $G_2$, so we can match up paths as described above.

To do this, let $G' = G_1 + b$ where $b$ is a new vertex connected directly to all points in $S$. This is a strictly smaller graph than $G$, since $S \neq N(w)$. We claim that the minimum $vb$-cut size in $G'$ is the same as the minimum $vw$-cut size in $G$: for if there were a smaller $vb$-cut in $G'$, it would also be a $vw$-cut in $G$. So by inductive hypothesis, there are $m$ internally disjoint paths from $v$ to $b$ in $G'$, i.e. $m$ internally disjoint paths from $v$ to $S$, one for each point in $S$. The same reasoning works for $G_2$, completing the proof.

2. Ramsey theory

For any $q$, the Ramsey number $R_q$ can be defined in several equivalent ways:

Definition 1. $R_q$ is the smallest number $n$ such that, in any set of $n$ people, either there are $q$ who are mutually acquainted or $q$ who are mutual strangers.

Definition 2. $R_q$ is the smallest number $n$ such that any graph on $n$ vertices contains either a clique of size $q$ or an independent set of size $q$.

Definition 3. $R_q$ is the smallest number $n$ such that any 2-coloring of the edges of $K_n$ contains a monochromatic $K_q$ (a set of $q$ vertices such that the edges among those all get the same color).

The classic first result of Ramsey theory is:

Theorem: $R_3 = 6$.

Proof: To show that $R_3 \leq 6$, note that the 5-cycle is a graph on 5 vertices with no triangle (i.e. no 3-clique) and no independent set of size 3.

To show that $R_3 \geq 6$, take a 2-coloring of the edges of $K_6$ by, say, red and blue, and fix a particular vertex $v$. Now $v$ has 5 neighbors, so either there are at least three red edges from $v$ or there are at least three blue edges from $v$. Suppose there are three red edges; let the other endpoints of those edges be $v_1, v_2, v_3$. The triangle formed by those three vertices either has all its edges colored blue, in which case the coloring has a monochromatic blue $K_3$, or else has a red edge. If it has a red edge, say $v_1 v_2$, then the triangle formed by $v, v_1, v_2$ is a monochromatic red $K_3$. The reasoning is exactly the same in the case where $v$ has at least three blue edges from it.

By similar but slightly more complicated reasoning one may show that $R_4 = 18$. After that our knowledge fails. It is known that $43 \leq R_5 \leq 49$ and $102 \leq R_6 \leq 165$, but narrowing down the exact values of these is thought to be an intractable problem. About higher Ramsey numbers even less is known. However, we can at least give an upper bound:

Theorem: For any $q$ we have $R_q \leq 2^{2q}$.

Proof: Let $n \geq 2^{2q}$. Given a 2-coloring of the edges of $K_n$, fix a vertex $v_0$. $v_0$ has $n-1$ neighbors, so by the pigeonhole principle either $v_0$ has at least $n/2$ red edges from it or it has at least $n/2$ blue edges. In the former case, mark $v_0$ "red" and let $V_1$ be the set of "red neighbors" of $v$ (the other endpoints of the red edges); in the latter case mark $v_0$ "blue" and let $V_1$ be the set of "blue neighbors".

Now pick a vertex $v_1 \in V_1$ and consider its neighbors in $V_1$. Either $v_1$ has $n/4$ red edges from it to other vertices in $v_1$, or it has $n/4$ blue edges. Mark $v_1$ red or blue accordingly and let $V_2$ be the set of red neighbors in $V_1$ or blue neighbors in $V_1$ accordingly.

Repeat this process until you have vertices $v_0, v_1, ..., v_{2q-2}$ and sets $V_1, ..., V_{2q-2}$. We can do this since each $V_i$ is at least half the size of the previous one and we have at least $n/2 \geq 2^{2q-1}$ vertices in $V_1$. Now each of the vertices $v_0, ..., v_{2q-2}$ is marked red or blue. Since there are $2q-1$ of these vertices, at least $q$ must be marked with the same color. By construction, all edges among those $q$ vertices are colored the same, so they form a monochromatic $K_q$, as required.

These Ramsey numbers are sometimes called diagonal Ramsey numbers. To justify this term, we introduce a more general form of Ramsey numbers as follows:

Definition: $R(k,l)$ is the smallest $n$ such that any 2-coloring of the edges of $K_n$ contains either a monochromatic red $K_k$ or a monochromatic blue $K_l$.

Then $R_q = R(q,q)$. It is easy to see that $R(2,l) = l+1$ for any $l$. But apart from this very few exact values of "off-diagonal" Ramsey numbers (those for which $k \neq l$) are known: we know $R(3,4) = 9$ and $R(4,5) = 25$, for example, but $R(3,l)$ is not known exactly for $l \geq 10$ and $R(4,l)$ is not known exactly for $l \geq 6$. Arguments similar to those in the proof of the previous theorem may be used to show that $R(k, l)$ has some finite upper bound for any $k$ and $l$, but the upper bounds known are very large.

We may also generalize the Ramsey number definition to "multicolored" Ramsey numbers:

Definition: The multicolor Ramsey number $R(k_1, k_2, ..., k_s)$ is the smallest $n$ such that in any $s$-coloring of the edges of $K_n$, there is some $i$ such that the coloring contains a monochromatic $K_{k_i}$ of color $i$.

As before, it is known that these numbers are all finite, but only very large upper bounds can be found in general; the only nontrivial exact value of a multicolor Ramsey number known is $R(3, 3, 3) = 17$.

For those interested, a somewhat different and in some respects more extensive discussion of Ramsey theory can be found at the Wikipedia entry on Ramsey theory on the Web: http://en.wikipedia.org/wiki/Ramsey's_theorem. This includes a link to a survey paper giving known values for Ramsey numbers; that survey paper has a great many references for further reading.