Canonical Typicality and GAP Measures for Quantum States

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Joel Lebowitz, Christian Mastrodonato, Roderich Tumulka, and Nino Zanghì

Canonical Typicality

- Schrödinger (1927)
- Bocchieri and Loinger (1959)
- Lloyd (1988)
- Tasaki (1998)
- Gemmer and Mahler (2003)
- G, Lebowitz, Tumulka, and Zanghì (2006)
- Popescu, Short, and Winter (2006)

"It is time for this discovery to stay discovered." (Lloyd, Nature Physics, Nov. 2006)

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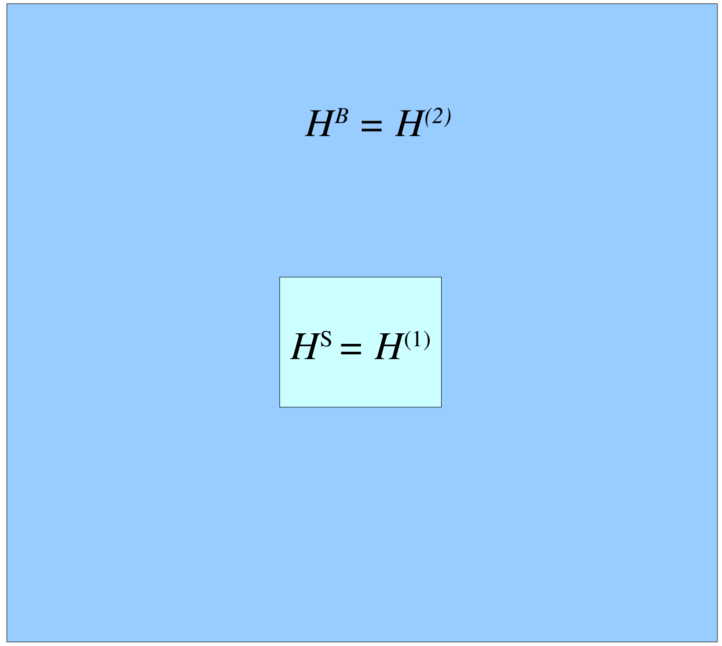
"When I discovered it, it stayed discovered." (Larry Shepp)



$$H^{(1+2)} = H^{(1)} + H^{(2)} = H^{(1)} \otimes I^{(2)} + H^{(2)} \otimes I^{(1)}$$

on

$$\mathcal{H}^{(1+2)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$$

Microcanonical(1+2[large]) $\Rightarrow$ Canonical(1)

$\rho_\beta = \exp(-\beta H^{(1)})/Z$ on $\mathcal{H}^{(1)}$

$\rho_{E,\delta} = P_{\mathcal{H}_{E,\delta}}/D = P_{E\leq H^{(1+2)}\leq E+\delta}/D$ on $\mathcal{H}^{(1+2)}$ $(D = \dim \mathcal{H}_{E,\delta})$

$$(\rho_{E,\delta})_1 = \mathrm{tr}^{(2)}(\rho_{E,\delta}) \simeq \rho_\beta \quad (\beta \leftrightarrow E)$$

$\rho_1 = \mathrm{tr}^{(2)}\rho$, on $\mathcal{H}^{(1)}$, reduced density matrix, etc.

Canonical Typicality

Fix $\Psi \in \mathcal{H}_{E,\delta}$.

$\rho^{\Psi} = |\Psi\rangle\langle\Psi|$ on $\mathcal{H}^{(1+2)}$

$$(\rho^{\Psi})_1 = \mathrm{tr}^{(2)} \rho^{\Psi} \simeq \rho_{\beta}$$

for typical Ψ, i.e., for the overwhelming majority of Ψ's in $\mathcal{H}_{E,\delta}$, defined in terms of the uniform distribution $u_{E,\delta}$ (the *micro-canonical measure*) on the unit sphere $\mathscr{S}(\mathcal{H}_{E,\delta})$ of $\mathcal{H}_{E,\delta}$.

Key: entanglement

Classically, randomness in yields randomness out. In quantum mechanics, we can have "randomness out, without randomness in." With quantum mechanics we have "spontaneous uncertainty." But the uncertainty is even greater than it seems, and that is the main issue I want to address.

Density matrix and random wave function

$$\mu \mapsto \rho_\mu = \int_{\mathscr{S}} \mu(d\psi)|\psi\rangle\langle\psi|$$

probability measure on the unit sphere $\mathscr{S}$ of a Hilbert space $\mathcal{H}$
$\mapsto$ density matrix on that Hilbert space

- onto
- many-to-one

Example: $\rho_{E,\delta} = \rho_\mu$ for $\mu = u_{E,\delta}$ and for $\mu = u^{eig}_{E,\delta}$
where $u^{eig}_{E,\delta}$ is the uniform distribution over the energy eigenstates in $\mathcal{H}_{E,\delta}$.

“In equilibrium the state of a system is random. In quantum mechanics the state of a system is given by its wave function. If the system is in equilibrium, with density matrix ρ_β, its wave function should be random, with some distribution μ on $\mathscr{S}$. What is μ?”

GAP Measures, $GAP(\rho)$

Gaussian

Adjusted

Projected

Density Matrix and Covariance

$$\rho = \rho_\mu = \int_{\mathcal{H}} \mu(d\psi)|\psi\rangle\langle\psi| \leftrightarrow \ \mu \text{ has covariance } \rho$$

.

(assuming, as we will more or less always do, that μ has mean 0.)

$$\langle\phi|\rho|\phi'\rangle = \int_{\mathcal{H}} \mu(d\psi)\langle\phi|\psi\rangle\langle\psi|\phi'\rangle$$

Remark: $\int_{\mathcal{H}} \mu(d\psi)\|\psi\|^2 = \int_{\mathcal{H}} \mu(d\psi)\text{tr}|\psi\rangle\langle\psi| = \text{tr}\rho = 1$.

$G(\rho)$

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$$\rho = \sum_n \rho_n |n\rangle\langle n|$$

$G(\rho)$ is the distribution of

$$\Psi^G = \sum_n Z_n |n\rangle$$

where the Z_n are independent complex-Gaussian with (mean 0 and) variance $E(|Z_n|^2) = \rho_n$.

$GA(\rho)$

$G(\rho)$ is not supported by $\mathscr{S}$. We would therefore like to project $G(\rho)$ onto $\mathscr{S}$. But doing so would alter the covariance—unless we first "adjust" $G(\rho)$:

$$GA(\rho)(d\psi) = \|\psi\|^2 G(\rho)(d\psi)$$

(Since $\int G(\rho)(d\psi)\|\psi\|^2 = 1$, $GA(\rho)$ is properly normalized; it is a probability measure on $\mathcal{H}$).

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$$\Psi^{GAP} = \Psi^{GA}/\|\Psi^{GA}\|$$

where Ψ^{GA} has distribution $GA(\rho)$.

$$E\left(|\Psi^{GAP}\rangle\langle\Psi^{GAP}|\right) = \int G(\rho)(d\psi)\|\psi\|^2\frac{|\psi\rangle\langle\psi|}{\|\psi\|^2} = \rho$$

Examples $(\rho = \sum_n \rho_n |n\rangle\langle n|)$

- $GAP(\rho_\beta) \leftrightarrow \Psi^G = \sum Z_n |n\rangle$ with $|n\rangle$ the energy eigenstates and $E(|Z_n|^2) = e^{-\beta E_n}/Z$ (for both G and for GAP).

- $GAP(\rho_{E,\delta}) = u_{E,\delta}$

- Not GAP: $\quad EIG(\rho)(|m\rangle) = \rho_m$

$$\longleftrightarrow \Psi = \sum_n Z_n |n\rangle, \quad Z_n = \delta_{nm} \text{ with probability } \rho_m$$

$EIG(\rho)$ is supported by the set of eigenvectors of ρ.
$GAP(\rho)(\{\text{eigenvectors of } \rho\}) = 0$.

Properties of $GAP(\rho)$

- $\rho_{GAP(\rho)} = \rho$

- $GAP(\rho)$ is covariant:

$$GAP(U\rho U^*) = U[GAP(\rho)]$$

  In particular, $GAP(\rho_\beta)$, $\rho_\beta \propto \exp(-\beta H)$, is a stationary measure for the Schrödinger dynamics generated by H.

- GAP is hereditary: $\left[GAP(\rho^{(1)} \otimes \rho^{(2)})\right]_1 = GAP(\rho^{(1)})$

The wave function of a subsystem: the conditional wave function Ψ_1

For $\Psi \in \mathscr{S}(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)})$, and $\{\phi_n\}$ a basis of $\mathcal{H}^{(2)}$, Ψ_1 is random vector in $\mathcal{H}^{(1)}$, taking on the value

$$\Psi_1 = \langle \phi_n | \Psi \rangle / \| \langle \phi_n | \Psi \rangle \| \in \mathscr{S}(\mathcal{H}^{(1)})$$

with probability $\| \langle \phi_n | \Psi \rangle \|^2$. ($\Psi_1$ depends on the choice of basis.)

In other words, for $\Psi = \sum_n \psi_n \otimes \phi_n$

$$\text{Prob}(\Psi_1 = \psi_n / \|\psi_n\|) = \|\psi_n\|^2.$$

Ψ can be random: For any probability measure μ on $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$,

$$\mu_1 = \text{ the distribution of } \Psi_1$$

when Ψ has distribution μ.

Corollary of Property 3 (using equivalence of ensembles and and the continuity of $GAP(\rho)$):

$$(u_{E,\delta})_1 \simeq GAP(\rho_\beta)$$

$$\rho_{E,\delta} \simeq \rho_\beta^{(1+2)} = \rho_\beta^{(1)} \otimes \rho_\beta^{(2)}$$

$$u_{E,\delta} = GAP(\rho_{E,\delta}) \simeq GAP\left(\rho_\beta^{(1)} \otimes \rho_\beta^{(2)}\right)$$

$$\rho_\beta = \rho_\beta^{(1)}$$

GAP Typicality

The wave function of a subsystem of a large system is typically GAP-distributed ("large": $\dim \mathcal{H}^{(2)} \gg \dim \mathcal{H}^{(1)}$):

$$\begin{array}{ccccc} \Psi & \longrightarrow & \rho^{\Psi} & \longrightarrow & (\rho^{\Psi})_1 \\ \downarrow & & & & \downarrow \\ \Psi_1 & & \xrightarrow{dist} & & GAP[(\rho^{\Psi})_1] \end{array}$$

$$\mathit{dist}(\Psi_1) \simeq GAP[(\rho^{\Psi})_1]$$

for typical $\Psi \in \mathscr{S}(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)})$.

Too weak. $(\rho^{\Psi})_1$ is typically $\propto I^{(1)}$.) For given $\rho^{(1)}$, we need to condition on the event that Ψ is such that $(\rho^{\Psi})_1 = \rho^{(1)}$.

$$\begin{array}{ccccc} \Psi & \longrightarrow & \rho^{\Psi} & \stackrel{1}{\longrightarrow} & \rho^{(1)} \\ \downarrow & & & & \downarrow \\ \Psi_1 & & \stackrel{dist}{\longrightarrow} & & GAP(\rho^{(1)}) \end{array}$$

$$dist(\Psi_1) \simeq GAP(\rho^{(1)})$$

for typical $\Psi \in \mathscr{S}(\rho^{(1)}) = \left\{\Psi \in \mathscr{S}(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}) \,|\, (\rho^{\Psi})_1 = \rho^{(1)}\right\}$.

Corollary:

- For any $\Psi \in \mathscr{S}(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)})$

$$\mathit{dist}(\Psi_1) \simeq GAP[(\rho^{\Psi})_1]$$

for a typical choice of basis for $\mathcal{H}^{(2)}$.

- Thus, using canonical typicality: For a typical $\Psi \in \mathcal{H}_{E,\delta}$

$$\mathit{dist}(\Psi_1) \simeq GAP(\rho_\beta)$$

for a typical choice of basis for $\mathcal{H}^{(2)}$.

GAP typicality $\longrightarrow$
GAP is the “canonical” wave function distribution.

Sketch of Proofs

$$H^{(1)} = \sum_n E_n \, |n\rangle\langle n|$$

$$\rho_{E,\delta} = D^{-1} \sum_n P_{\mathcal{H}^{(2)}_{E-E_n,\,\delta}} \otimes |n\rangle\langle n|$$

$$\rho_1 = D^{-1} \sum_n \left(\text{dim}\, \mathcal{H}^{(2)}_{E-E_n,\,\delta} \right) |n\rangle\langle n| \simeq \rho_\beta$$

since $\text{dim}\, \mathcal{H}^{(2)}_{E-E_n,\,\delta} \sim e^{S(E-E_N)} \propto e^{-\frac{\partial S}{\partial E} E_n} = e^{-\beta E_n}$

Proof of CT

$$\Psi = \sum_n |n\rangle \otimes \Phi_n$$

$$\Phi_n = \sum_{E-E_n \leq E_m^{(2)} \leq E-E_n+\delta} X_{nm} |m\rangle^{(2)}$$

We may assume that the X_{nm} are i.i.d. complex-Gaussian, with $E(|X_{nm}|^2) = 1/D$. Then

$$\left(\rho^{\Psi}\right)_1 = \sum_{n,m} \langle \Phi_m | \Phi_n \rangle |n\rangle\langle m| \simeq \sum_n \rho_n |n\rangle\langle n|$$

$$\rho_n = \sum_{E-E_n \leq E_m^{(2)} \leq E-E_n+\delta} |X_{nm}|^2 \simeq \dim \mathcal{H}^{(2)}_{E-E_n,\,\delta}/D$$

Proof of GT

Schmidt decomposition (biorthonormal decomposition)

For $\Psi \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}, \quad \dim \mathcal{H}^{(1)} \leq \dim \mathcal{H}^{(2)}$,
there exists a basis ψ_n of $\mathcal{H}^{(1)}$ and an o.n. system χ_n for $\mathcal{H}^{(2)}$ such that

$$\Psi = \sum_n c_n \psi_n \otimes \chi_n, \qquad c_n \geq 0.$$

Schmidt decomposition and $\left(\rho^{\Psi}\right)_1$

$\left(\rho^{\Psi}\right)_1 = \rho^{(1)} = \sum_n \rho_n \, |n\rangle\langle n| \quad \leftrightarrow$ Ψ has SD of the form

$$\Psi = \sum_n \sqrt{\rho_n}\, |n\rangle \otimes \chi_n$$

$$\Rightarrow \mathscr{S}(\rho^{(1)})$$

$u_{\mathscr{S}(\rho^{(1)})} \leftrightarrow$ the o.n. system χ_n is random, with distribution invariant under the action of the unitary group of $\mathcal{H}^{(2)}$.

Lemma: *Fix a basis* ϕ_m *of* $\mathcal{H}^{(2)}$. *The empirical distribution of* $\sqrt{\dim \mathcal{H}^{(2)}}\Psi_1$, *assigning equal probabilities* $1/\dim \mathcal{H}^{(2)}$ *to the vectors* $\sqrt{\dim \mathcal{H}^{(2)}}\langle\phi_m|\Psi\rangle \in \mathcal{H}^{(1)}$, *is* $G(\rho^{(1)})$.

Proof:

$$\chi_n = \sum_m X_{nm}\,\phi_m/\sqrt{\dim \mathcal{H}^{(2)}}$$

$u_{\mathscr{S}(\rho^{(1)})} \rightarrow$ the X_{nm} are approximately i.i.d. complex-Gaussian, with $E(|X_{nm}|^2) = 1$.

With this approximation, the sequence of vectors

$$\sqrt{\dim \mathcal{H}^{(2)}}\langle \phi_m | \Psi \rangle = \sum_n \sqrt{\rho_n} X_{nm} \, |n\rangle = \sum_n Z_{nm} |n\rangle \quad m = 1, \ldots, \dim \mathcal{H}^{(2)}$$

is i.i.d., with common distribution that of $\sum_n Z_n \, |n\rangle$ where the Z_n are independent complex-Gaussian with $E(|Z_n|^2) = \rho_n$.

LLN $\Rightarrow$ the empirical distribution of the sequence is the theoretical distribution $G(\rho^{(1)})$.

Empirical distribution, improper normalization, G $\leftrightarrow$ quantum distribution, proper normalization, GAP.

In the new, post-1925 quantum theory the 'anarchist' position became dominant and modern quantum physics, in its 'Copenhagen interpretation', became one of the main standard bearers of philosophical obscurantism. In the *new* theory Bohr's notorious 'complementarity principle' enthroned [weak] inconsistency as a basic ultimate feature of nature, and merged subjectivist positivism and antilogical dialectic and even ordinary language philosophy into one unholy alliance. After 1925 Bohr and his associates introduced a new and unprecedented lowering of critical standards for scientific theories. This led to a defeat of reason within modern physics and to an anarchist cult of incomprehensible chaos. (Lakatos, Criticism and the Growth of Knowledge, p. 145, 1965)