2 July 2007

Section 4.4, Problem 30

Our function is $f(x) = \frac{2x^2-5x+7}{x^2-9}$. First, we solve $f(x) = 0$:

$$\begin{aligned} 2x^2 - 5x + 7 &= 0 \\ x &= \frac{5 \pm \sqrt{5^2 - 4(2)(7)}}{4} \\ x &= \frac{5 \pm \sqrt{-31}}{4} \end{aligned}$$

Since the discriminant is negative, there are no solutions.

Now we need to find $f'(x)$. We use the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(2x^2 - 5x + 7)'(x^2 - 9) - (2x^2 - 5x + 7)(x^2 - 9)'}{(x^2 - 9)^2} \\ &= \frac{(4x - 5)(x^2 - 9) - (2x^2 - 5x + 7)(2x)}{(x^2 - 9)^2} \\ &= \frac{(4x^3 - 5x^2 - 36x + 45) - (4x^3 - 10x^2 + 14x)}{(x^2 - 9)^2} \\ &= \frac{(5x^2 - 50x + 45)}{(x^2 - 9)^2} \end{aligned}$$

We see that the numerator of $f'(x)$ factors into $5(x - 9)(x - 1)$, so we see that $f'(x) = 0$ for $x = 1$ and $x = 9$. We can then conclude that

$$\begin{aligned} f'(x) > 0 &\;:\; x < 1, \quad x > 9 \\ f'(x) < 0 &\;:\; 1 < x < 9 \end{aligned}$$

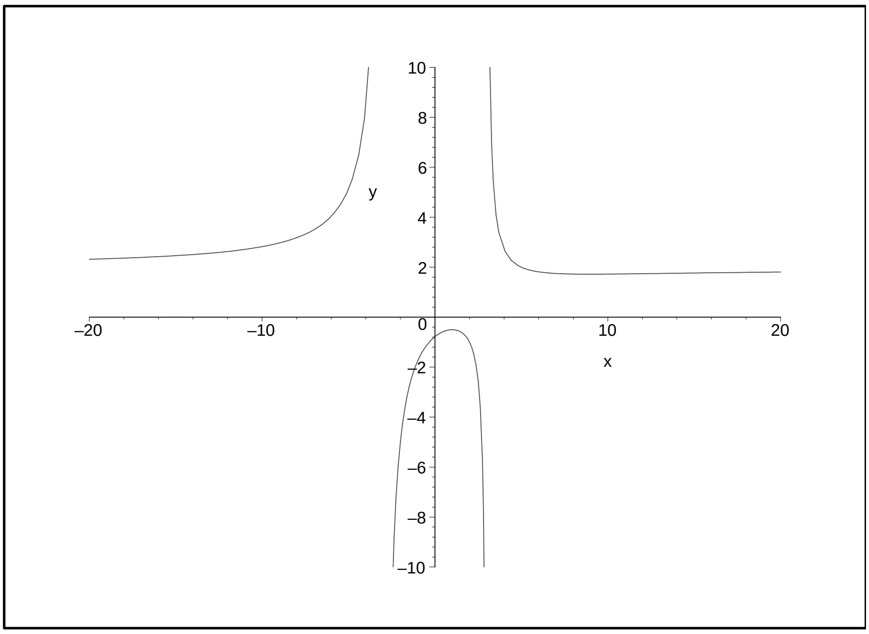
Now, for the second derivative:

$$
\begin{aligned}
f''(x) &= \frac{(5x^2-50x+45)'(x^2-9)^2-(5x^2-50x+45)((x^2-9)^2)'}{(x^2-9)^4} \\
&= \frac{(10x-50)(x^2-9)^2-(5x^2-50x+45)2(x^2-9)(2x)}{(x^2-9)^4} \\
&= \frac{(x^2-9)(10(x-5)(x^2-9)-20x(x-1)(x-9))}{(x^2-9)^4} \\
&= \frac{10(x^2-9)((x-5)(x^2-9)-2x(x-1)(x-9))}{(x^2-9)^4} \\
&= \frac{10(x^2-9)((x^3-5x^2-9x+45)-(2x(x^2-10x+9)))}{(x^2-9)^4} \\
&= \frac{10(x^2-9)((x^3-5x^2-9x+45)-(2x^3-20x^2+18x))}{(x^2-9)^4} \\
&= \frac{10(x^2-9)(-x^3+15x^2-27x+45)}{(x^2-9)^4}
\end{aligned}
$$

So when we solve the numerator for 0, we get three real roots; $x = 3$ and $x = -3$ from the $x^2 - 9$ term, and a calculator verifies that $x \approx 13.2144$ is the only real root of $-x^3 + 15x^2 - 27x + 45$. By checking signs, we see that

$$
\begin{aligned}
f''(x) > 0 &: \quad x < -3, 3 < x < 13.2144 \\
f''(x) < 0 &: \quad -3 < x < 3, x > 13.2144
\end{aligned}
$$

As for asymptotes, we see that vertical asymptotes occur at $x = -3$ and $x = 3$, and horizontal asymptotes occur at $y = 2$. Combining this all together, we have the following plot:



Note that the plot is indeed consistent with the information that we found out. There is a relative min at $x = 9$ and the function dips below the asymptote line $y = 2$ around that area and has asymptotic behavior below the line $y = 2$, whereas the asymptotic behavior as $x \to -\infty$ is above the line $y = 2$.