20 November 2008

Workshop 11, Problem 4(a)

We perform the ratio test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{\frac{(n+1)x^{n+1}}{(n+1)^2+1}}{\frac{nx^n}{n^2+1}}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{(n+1)^2+1}\,\frac{n^2+1}{nx^n}\right| \\
&= \lim_{n\to\infty}|x|\frac{(n+1)(n^2+1)}{n((n+1)^2+1)} \\
&= |x|\lim_{n\to\infty}\frac{n^3+n^2+n+1}{n^3+2n^2+2n} \\
&= |x|
\end{aligned}$$

Therefore, we are absolutely convergent when $|x| < 1$ and divergent when $|x| > 1$. We must check the endpoints. For $x = 1$ we have

$$\sum_{n=1}^{\infty}\frac{n}{n^2+1}.$$

We believe this behaves like $\sum\frac{1}{n}$ so we try the limit comparison test:

$$\lim_{n\to\infty}\frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$$

Therefore, by the limit comparison test, since $\sum\frac{1}{n}$ diverges then so does $\sum\frac{n}{n^2+1}$. For $x = -1$ we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}$$

This is an alternating series, and it satisfies the Leibniz Test because:

- $\lim_{n\to\infty}\frac{n}{n^2+1} = 0$, and
- $\frac{(n+1)}{(n+1)^2+1} \leq \frac{n}{n^2+1}$.

We know that the inequality holds because:

$$\begin{aligned}
\frac{(n+1)}{(n+1)^2+1} &\leq \frac{n}{n^2+1} \\
(n+1)(n^2+1) &\leq n((n+1)^2+1) \\
n^3+n^2+n+1 &\leq n^3+2n^2+2n \\
1 &\leq n^2+n
\end{aligned}$$

which is always true. Hence, we see that we converge in the interval $[-1, 1)$.